The Problem of Detailed Balance for the Feynman-Smoluchowski Engine (FSE) and the Multiple Pawl Paradox

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Abstract. The Feynman-Smoluchowski Engine (FSE) is simply a ratchet and pawl device, connected by a shaft to a vane, that is small enough to rectify the effect of random bombardment of gas molecules on the vane. The significance of the FSE is that it has inspired much activity in the area of Brownian ratchets. There is considerable interest in both the FSE and Brownian ratchets as these models are finding increasing use in a wide range of multidisciplinary applications from stochastic signal processing to econophysics to biology and even sociological processes. In this paper we examine the problem of detailed balance of the FSE and we introduce a new problem that considers a ratchet wheel with multiple pawls.

INTRODUCTION

The Feynman-Smoluchowski Engine (FSE) consists of a ratchet connected to a set of vanes via an axle. As the air molecules randomly bombard the vanes, the ratchet oscillates. It would appear that the action of the ratchet & pawl 'rectifies' these oscillations and the system rotates in one direction, thus being able to perform useful work, in violation of the Second Law of Thermodynamics. In 1912, Smoluchowski was the first to correctly suggest that there is no net motion, at equillibrium, as fluctuations in the spring loaded pawl will occasionally allow the ratchet wheel to rotate in the opposite direction – thus preserving detailed balance.

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Of course, for the non-equillibrium case, when energy is supplied into the system, there is net motion without violation of the Second Law.

The ratchet & pawl device was revisited, in 1963, by Feynman [2] in greater detail – detailed balance probabilities are given and engine efficiency calculations are explored. It is now well-known that Feynman's treatment was flawed, as he incorrectly applied the quasi-static assumption to the FSE, leading to an incorrect calculation of engine efficiency [3]. This paper now also questions Feynman's treatment of the detailed balance. Although Smoluchowski & Feynman only saw the FSE as a 'thought experiment,' the FSE is no longer hypothetical as the so-called Feynman-micromotor [4] has been fabricated using MEMS technology and has inspired the 'Brownian ratchet' concept. Hence there is renewed interest in the FSE, and correct analysis is now of importance.

Consideration of the case of a ratchet wheel with multiple pawls, inspired us to scrutinize the treatment of detailed balance in [2]. Hence, we begin our discussion by performing a detailed balance, using Feynman's method based on energy probabilities, to highlight the problems. Then we offer a solution by adopting a different approach, based on crossing rates. Resulting further open questions are then identified.

FEYNMAN'S APPROACH

Before we discuss the multiple pawl scenario, we briefly recall Feynman's approach. Feynman begins by calling the threshold energy that the ratchet wheel needs to rotate clockwise (CW) one notch passed the pawl, ϵ. He then states the probability of the ratchet wheel attaining ϵ is $e^{-\epsilon/kT}$. Also he states that this is the same probability required for the pawl to fluctuate enough to disengage, thus allowing the ratchet to rotate counterclockwise (CCW).

Without discussion, Feyman implicitely identifies these probabilities as the *same* probabilities required for CW and CCW rotation. Thus he concludes that the system is balanced and there is no net rotation on average. Of course, his final conclusion is correct, as we cannot allow a violation of the Second Law of thermodynamics. However, one question is the leap in logic from probabilities to do with pawl and ratchet states, to probabilities of CW and CCW rotation.

The real situation is much more complex. For instance, when the pawl is disengaged, the ratchet wheel can rotate in either direction! Also when pawl is engaged, the ratchet wheel may attain the energy ϵ, but in the wrong (CCW) direction, and thus will be dissipated as heat.

These arguments demonstrate that in order to fully understand the FSE, a detailed balance from first principles is required. But let us now use Feynman's approach, to examine the detailed balance of multiple and single pawl systems, to further hightlight the difficulties.

THE MANY PAWL PARADOX

An interesting question is to ask what happens if the ratchet has more than one pawl? It would appear *prima facie* that as the fluctuations in all the pawl springs are not totally correlated, then the chance of disengagement is reduced and therefore the wheel will rotate in one direction. This cannot be correct as it would then be possible to construct a machine that would disobey the Second Law.

Firstly, for the case of one pawl let the spring stiffness constant be λ. If the ratchet tooth height is Y then the energy to disengage, by Feynman, is $\epsilon = \frac{1}{2}\lambda Y^2$, to produce CCW movement. By symmetry this is also equal to the energy required to move the ratchet clockwise, against the pawl spring. The probability of clockwise movement is $e^{-\epsilon/kT}$, which is also the probability of an counterclockwise movement. Notice that we are ignoring any constant premultipliers to the probabilities, as these will balance out anyhow.

Now, let there be i pawls each with a different spring constant λ_i. For simplicity let the spring constants be normalized by λ, thus $\lambda_i = n_i\lambda$, where n_i is a unitless multiplier. Assuming worst case of no correlation between the pawls, the probability that the ratchet moves clockwise is $P(\mathrm{CW}) = e^{-\sum n_i\epsilon/kT}$, whereas the probability that the ratchet will move counterclockwise is $P(\mathrm{CCW}) = \prod e^{-n_i\epsilon/kT} = e^{-\sum n_i\epsilon/kT} = P(\mathrm{CW})$.

Hence a clockwise movement is equally probable to a counterclockwise movement and therefore useful work cannot be done (at equillibrium). Although we will later show the use of Feynman's approach is not correct, it produces the correct answer. It even produces the right heuristic interpretation: although many pawls in parallel reduce the probability of disengagement, due to uncorrelated fluctuation, the probability of clockwise movement is equally reduced by the parallel effect of the spring constants!

THE SINGLE PAWL CASE

Now, for the single pawl case, let us consider the CW and CCW directions separately. **CW Rotation:** As before, let the required energy threshold for the ratchet wheel to rotate one notch passed the pawl be ϵ. In general we can say that $\epsilon = \epsilon_r + \epsilon_p$, where ϵ_r is supplied by the ratchet wheel fluctuation trying to move passed the pawl, and ϵ_p is supplied by the pawl fluctuation trying to (partially) disengage. Now the probability of attaining ϵ_r is $e^{-\epsilon_r/kT}$ and attaining ϵ_p is $e^{-\epsilon_p/kT}$. But note that when the ratchet wheel gets a 'kick' of energy equal to ϵ_r there is a chance of $\frac{1}{2}$ that the kick would be in the CW direction. Similarly, the pawl can fluctuate upwards (to escape the ratchet teeth) or downwards (to dig into the ratchet teeth) and the chance of attaining ϵ_p in the upwards direction will be $\frac{1}{2}e^{-\epsilon_p/kT}$. Therefore, the probability of CW rotation is, $P(\mathrm{CW}) = \frac{1}{2}e^{-\epsilon_p/kT}\frac{1}{2}e^{-\epsilon_r/kT} = \frac{1}{4}e^{(-\epsilon_p-\epsilon_r)/kT} = \frac{1}{4}e^{-\epsilon/kT}$. **CCW Rotation:** In this case, we require an energy ϵ from the pawl alone to disengage from the ratchet wheel. When the the pawl is disengaged, there is a chance of $\frac{1}{2}$ that the ratchet wheel will rotate in the CCW direction. Hence, $P(\text{CCW}) = \frac{1}{2}e^{-\epsilon/kT}\frac{1}{2} = \frac{1}{4}e^{-\epsilon/kT}$.

Therefore, $P(\text{CW}) = P(\text{CCW})$ and we have detailed balance. But do we? When calculating $P(\text{CW})$, we ignored the case when ϵ_p acts in the direction to dig the pawl deeper into the ratchet teeth – in this case the ratchet must attain $\epsilon_p + \epsilon$ for CW rotation. However, if we alter the probabilities to reflect this, we apparently lose detailed balance.

An even more serious flaw is as follows. For CW rotation, the requirement is for $\epsilon_r + \epsilon_p > \epsilon$. However, it is easily shown that, if ϵ_r and ϵ_p are independent and have exponential distributions then, the probability of this is

$$\int_{\epsilon}^{\infty} dE\, p(E_p) \otimes p(E_r) = (1 + \epsilon/kT)e^{-\epsilon/kT}.$$

Hence we have that $P(\text{CW})$ is always unequal to $P(\text{CCW})$, which is clearly not allowed. The question then arises, where is the flaw and what is the correct approach?

BALANCE OF CROSSING RATES

In seeking an alternative method for analytically obtaining detailed balance, we turned to the idea of calculating crossing rates [5] of the pawl over the top of a ratchet tooth. For simplicity, a linear ratchet is considered as in Figure 1 – equivalent results apply to the more familiar rotary ratchet. We define y = position of pawl above bottom of ratchet teeth, x = position of ratchet. X = horizontal pitch of ratchet tooth, Y = vertical height of ratchet tooth, y_o = rest position of pawl, x_o = rest position of the ratchet, m_p = mass of pawl, m_r = mass of ratchet, λ_p = spring constant of spring connected to pawl, λ_r = spring constant of spring connected to ratchet, d_p = damper constant for damper connected to pawl, d_r = damper constant for damper connected to ratchet.

We take $x = 0$ to correspond to when the bottom of the ratchet tooth is opposite the pawl. There is a constraint that $y/Y \geq x/X$, for $0 < x < X$ since the pawl cannot be below the ratchet tooth. Now, the ratchet slips one tooth to the right (normal ratchet action) if x crosses the value X in the positive direction. Similarly, a slip to the left occurs when x crosses the value 0 in the negative direction (abnormal ratchet action).

The Hamiltonian of the system is: $H = \frac{1}{2}\lambda_p (x - x_o)^2 + \frac{1}{2} m_p \dot{x}^2 + \frac{1}{2}\lambda_r (y - y_o)^2 + \frac{1}{2} m_r \dot{y}^2$ where $\dot{x} = dx/dt$ and $\dot{y} = dy/dt$. Notice this does not correspond exactly to Figure 1 – we have added a second spring to the ratchet that can be discarded at the end of the analysis. This is necessary to provide a constraint in x, to enable the integrals that follow.

The steady state joint probability density function of these variables is given by the Gibbs relation: $p(x, \dot{x}, y, \dot{y}) = \frac{1}{Z_o} e^{-H/kT}$ where Z_o is a normalising constant called the partition function.

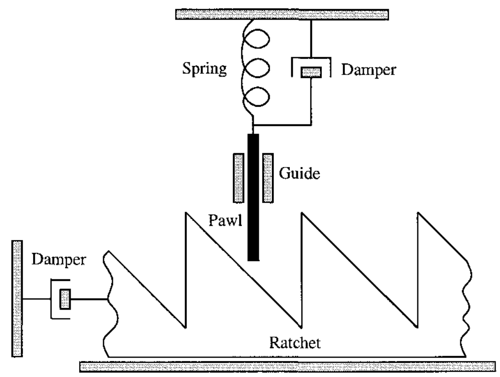

FIGURE 1. Ratchet and pawl system

The crossing rates of x at some level x_1, in Transition State Theory [5], are:

$$\nu^+ = \int_0^\infty d\dot{x} \int_{y_1}^\infty dy \int_{-\infty}^\infty d\dot{y}\, \dot{x}\, p(x, \dot{x}, y, \dot{y}) \tag{1}$$

$$\nu^- = \int_{-\infty}^0 d\dot{x} \int_{y_1}^\infty dy \int_{-\infty}^\infty d\dot{y}\, \dot{x}\, p(x, \dot{x}, y, \dot{y}) \tag{2}$$

where y_1 is the ratchet height when $x = x_1$.

For the above probability density function $\nu^+ = \nu^- = \nu$ and is given by:

$$\nu = \frac{2\pi}{Z_o} \sqrt{\frac{(kT)^3}{\lambda_p\, m_p\, m_r}}\, Q\left(\frac{(y_1 - y_o)\sqrt{\lambda_p}}{\sqrt{kT}}\right) e^{-\lambda_r\,(x_1 - x_o)^2/kT}$$

and $Q(x)$ is the Gaussian error probability function.

Now for a right slip across $x = X$ we must have $y_1 = Y$ and hence:

$$\nu_{right} = \frac{2\pi}{Z_o} \sqrt{\frac{(kT)^3}{\lambda_p\, m_p\, m_r}}\, Q\left(\frac{(Y - y_o)\sqrt{\lambda_p}}{\sqrt{kT}}\right) e^{-\lambda_r\,(X - x_o)^2/kT}$$

whereas for a left slip across $x = 0$ we have:

$$\nu_{left} = \frac{2\pi}{Z_o} \sqrt{\frac{(kT)^3}{\lambda_p\, m_p\, m_r}}\, Q\left(\frac{(Y - y_o)\sqrt{\lambda_p}}{\sqrt{kT}}\right) e^{-\lambda_r\,(-x_o)^2/kT}$$

since we must still have $y_1 = Y$ for this slip to occur.

If we remove the extraneous ratchet spring, by letting $\lambda_r \to 0$, then we have: $\nu_{left} = \nu_{right}$ – thus detailed balance is preserved. It can be shown that this treatment also works for the multiple pawl case. Note that even with the ratchet spring included, balance occurs if $x_o = X/2$ as might be expected.

The dampers do not explicitly appear in any of the above analysis, but will affect the nature of the fluctuations.

CONCLUSIONS AND OPEN QUESTIONS

The analysis presented shows that there is no paradox associated with the FSE, but questions Feynman's approach. Note that it was necessary to initially include the spring connected to the ratchet in order to properly treat the constraint between the ratchet and pawl positions, but this spring was later discarded. The analysis opens up a number of other questions: (a) Is there a deeper significance to including the extraneous spring? Is there a thermodynamical principle that can be invoked here? For instance, did the spring function as load to thermodynamically complete the system (cf. [6])? (b) Is there a way Feynman's type of approach can be corrected and be made to work? (c) If we calculate the crossing rate for a case where the Hamiltonian is the energy of a simple harmonic oscillator with a natural frequency ω_0, we find that $\nu^+ = \frac{\omega_0}{2\pi} e^{-\epsilon/kT}$. By inspection, $\frac{\omega_0}{2\pi}$ is the frequency of attempts to cross the barrier, and $e^{-\epsilon/kT}$ is the fraction of successful attempts. How do we reconcile this interpretation of the exponential term, with the idea that it is the probability of attaining an energy greater than ϵ? (d) Suppose the ratchet had teeth with irregular heights, what happens then? (e) If a weight is connected to the ratchet which tends to move it to the left, the ratchet will slip backwards more often than forwards. Since the weight is doing work on the system, it would be a useful pedagogical exercise to discuss where this energy is going. (f) Is there any fundamental difference between the non-equillibrium mode of the FSE and a Brownian ratchet? Both are ratchets, both give directed motion under non-equillibrium conditions and both require an input of external energy to operate. Is the difference semantics or fundamental?

REFERENCES

1. M. von Smoluchowski, "Experimentell nachweisbare, der ublichen Thermodynamik widersprechende Molekularphanomene," *Physik. Zeitschr.,* vol. XIII, pp. 1069-1080, (1912).
2. R.P. Feynman, R.B. Leighton and M. Sands, "Ratchet and pawl," *The Feynman Lectures on Physics,* 1963, Chapter 46, vol. 1, Addison-Wesley, pp. 46-1ff.
3. J.M.R. Parrondo and P. Espanol, "Criticism of Feynman's analysis of the ratchet as an engine," *Am. J. Phys.*, vol. 64, no. 9, pp. 1125-1130, (1996).
4. M. Tabib Azar, *et al.*, "Novel highly scalable and very high torque micromotor for MEMS and MOEMS applications using the mechanical rectification of oscillatory motion," *Proc. SPIE,* **3276,** pp.271-279, (1998).
5. P. Hänggi *et al., Rev. Mod. Phys.,* **62**, 250 (1990).
6. N. Wax, "A resolution of a circuit paradox," *Proc. IEEE,* **56,** 2370 (1968)