Communications IV Exercise Solutions

Exercise 2.23

$$v(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \qquad \text{is periodic of period } T_s$$

$$= \sum_{k=-\infty}^{\infty} V_k \, e^{j2\pi kt/T_s} \quad \text{where} \quad V_k = \frac{1}{T_s}\int_{-T_s/2}^{T_s/2} v(t)\, e^{j2\pi kt/T_s} dt = \frac{1}{T_s}$$

$$x(t) \otimes v(t) = \int_{-\infty}^{\infty} x(\lambda)\, v(t-\lambda)\, d\lambda = \sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} x(\lambda)\,\delta(t-\lambda-nT_s)\, dt = \sum_{n=-\infty}^{\infty} x(t - nT_s)$$

$$= \frac{1}{T_s}\sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty} x(\lambda)\, e^{j2\pi k(t-\lambda)/T_s}\, d\lambda = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X\left(\frac{k}{T_s}\right) e^{j2\pi kt/T_s}$$

Hence $\displaystyle\sum_{n=-\infty}^{\infty} x(t - nT_s) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X\left(\frac{k}{T_s}\right) e^{j2\pi kt/T_s}$ where $\displaystyle X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi ft}\, dt$

Putting t = 0 gives **Poisson's sum formula** $\displaystyle\sum_{n=-\infty}^{\infty} x(nT_s) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X\left(\frac{k}{T_s}\right)$.

Exercise 2.24

1. Apply Poisson's sum formula to $x(t) = e^{-\alpha|t|}$, $X(f) = 2\alpha/(\alpha^2+4\pi^2 f^2)$ with $T_s = 1$.

$$\sum_{n=-\infty}^{\infty} e^{-\alpha|n|} = \sum_{k=-\infty}^{\infty} \frac{2\alpha}{\alpha^2 + 4\pi^2 n^2}$$

$$\text{LHS} = \frac{1 + e^{-\alpha}}{1 - e^{-\alpha}} = \coth\left(\frac{\alpha}{2}\right)$$

2. Apply Poisson's sum formula to x(t) = sinc(t), X(f) = rect(f) with $T_s = 1/K$.

$$\sum_{n=-\infty}^{\infty} \text{sinc}\left(\frac{n}{K}\right) = K\sum_{k=-\infty}^{\infty} \text{rect}(kK) = K \quad ; K = 1, 2, \ldots$$

For enthusiasts, consider non-integer values of K, and in particular $0 < K \leq 1$.

3 Apply Poisson's sum formula to x(t) = sinc^2(t), X(f) = Δ(f) with $T_s = 1/K$.

$$\sum_{n=-\infty}^{\infty} \text{sinc}^2\left(\frac{n}{K}\right) = K\sum_{k=-\infty}^{\infty} \Delta(kK) = K \quad ; K = 1, 2, \ldots$$

Also, consider non-integer values of K, and in particular $0 < K \leq 1$.

Exercise 2.25

$$x_1(t) = u(t)e^{-\alpha t}, \qquad X_1(f) = \frac{1}{\alpha + j\omega}$$

$$y_1(t) = \delta(t), \qquad Y_1(f) = 1$$

$$H(f) = \frac{Y_1(f)}{X_1(f)} = \alpha + j\omega$$

$$x(t) = u(t)e^{-\alpha t}\cos(\beta t), \qquad X(f) = \frac{0.5}{\alpha + j(\omega - \beta)} + \frac{0.5}{\alpha + j(\omega + \beta)} = \frac{\alpha + j\omega}{(\alpha + j\omega)^2 + \beta^2}$$

$$Y(f) = H(f)X(f) = \frac{0.5[\alpha + j\omega]}{[\alpha + j(\omega - \beta)]} + \frac{0.5[\alpha + j\omega]}{[\alpha + j(\omega + \beta)]} = \frac{[\alpha + j\omega]^2}{(\alpha + j\omega)^2 + \beta^2}$$

$$= A + \frac{B}{\alpha + j(\omega - \beta)} + \frac{C}{\alpha + j(\omega + \beta)}$$

$$A = 1$$

$$B = j0.5\beta$$

$$C = -j0.5\beta$$

$$y(t) = \delta(t) + j\,u(t)\,0.5\beta e^{-\alpha t}\left\{e^{j\beta t} - e^{-j\beta t}\right\} = \delta(t) - u(t)\beta e^{-\alpha t}\sin(\beta t)$$

To find the partial fraction coefficients A, B and C, it is easiest to use coefficient matching, and it helps to put $s = j\omega$.

$$[\alpha + s]^2 = A[\alpha + s - j\beta][\alpha + s + j\beta] + B[\alpha + s + j\beta] + C[\alpha + s - j\beta]$$

$$\text{Coeff } s^2 \Rightarrow 1 = A$$

$$\text{Coeff } s \Rightarrow 2\alpha = 2A\alpha + B + C \Rightarrow B + C = 0$$

$$\text{Const} \Rightarrow \alpha^2 = A[\alpha^2 + \beta^2] + B[\alpha + j\beta] + C[\alpha - j\beta] \Rightarrow B = +j0.5\beta,\ C = -j0.5\beta$$

Alternatively, use the fact that multiplication by $j\omega$ is equivalent to differentiation in the time domain.

$$y(t) = \alpha u(t)e^{-\alpha t}\cos(\beta t) + \frac{d}{dt}\left\{u(t)e^{-\alpha t}\cos(\beta t)\right\}$$

$$= \alpha u(t)e^{-\alpha t}\cos(\beta t) + \delta(t)e^{-\alpha t}\cos(\beta t) - \alpha u(t)e^{-\alpha t}\cos(\beta t) - \beta u(t)e^{-\alpha t}\sin(\beta t)$$

$$= \delta(t) - \beta u(t)e^{-\alpha t}\sin(\beta t)$$

Exercise 2.28

$$x(t) = \text{rect}(t), \quad X(f) = \text{sinc}(f), \quad y(t) = \Delta(t), \quad Y(f) = \text{sinc}^2(f)$$

$$H(f) = \frac{Y(f)}{X(f)} = \text{sinc}(f) + \sum_{k \neq 0} a_k \, \delta(f - k)$$

where a_k is arbitrary. The δ functions arise from dividing $\text{sinc}^2(f)$ by sinc(f) which gives 0/0 (undefined) for f = non-zero integer and so can be a delta function at those frequencies. However, note that Y(f) = H(f) X(f). For a real filter $a_k = a^*_{-k}$, but such a system would be marginally stable (find h(t) and you will see why).

1. If x(t) = cos(2πt) then X(f) = 0.5 δ(f−1)+0.5 δ(f+1) so Y(f) will not be defined in general (the product of delta functions is not defined). However if all $a_k = 0$, then Y(f) = 0.

2. If $a_1 = a_{-1} = 0.5$ then h(t) = rect(t) + cos(2πt). Convolving cos(2πt) with cos(2πt) gives an indeterminate result as we discovered in part (1).

3. To uniquely determine the system, x(t) must be such that X(f) is not zero at some frequency, so we get an unambiguous result when we divide Y(f) by X(f). With $x(t) = u(t)\,e^{-\alpha t}$, we have

   $$X(f) = \frac{1}{\alpha + j2\pi f}$$

   which is not zero at any frequency. The same is true if x(t) = u(t), so in either case the system response can be determined from a measurement of the response to this excitation.

Exercise 2.49

Let X(f) and M(f) be the Fourier transforms of x(t) and m(t) respectively. The Fourier transform of c(t) = m(t) x(t) is $C(f) = M(f) \otimes X(f) = 0.5M(f) \otimes X^+(f) + 0.5M(f) \otimes X^-(f)$, where $X^+(f) = 2\,u(f)\,X(f)$ and $X^-(f) = 2\,u(-f)\,X(f)$.

Because the spectra M(f) and X(f) do not overlap, then $M(f) \otimes X^+(f)$ is zero for $f \leq 0$ and similarly $M(f) \otimes X^-(f)$ is zero for $f \geq 0$. (Sketch the convolutions and this will be obvious, but note that the result is only true because m(t) is lowpass and x(t) is bandpass. It would not be true otherwise.)

The Hilbert transform of c(t) has a transform:

$$\begin{aligned}
\hat{C}(f) &= -j\,\text{sgn}(f)\,C(f) = -j0.5u(f)M(f) \otimes X^+(f) + j0.5u(-f)M(f) \otimes X^-(f) \\
&= -j0.5\,M(f) \otimes X^+(f) + j0.5M(f) \otimes X^-(f) \\
&= M(f) \otimes \{-\,j\,\text{sgn}(f)X(f)\} \\
\hat{c}(t) &= m(t)\,\hat{x}(t)
\end{aligned}$$

where in the second line we can omit the u(f) and u(−f) because the quantities they multiply are already zero for f < 0 and f > 0 respectively.

Exercise 2.55

This is trivial. Because taking the Hilbert transform is equivalent to a linear filtering operation, if it is combined with any other filtering operation, the operations may be done in any order.

$$x(t) \Leftrightarrow X(f)$$
$$\frac{dx(t)}{dt} \Leftrightarrow j2\pi f\, X(f)$$
$$HT\{x(t)\} = \hat{x}(t) \Leftrightarrow -j\,\mathrm{sgn}(f)\,X(f)$$
$$HT\left\{\frac{dx(t)}{dt}\right\} \Leftrightarrow -j\,\mathrm{sgn}(f)\{j2\pi f\, X(f)\} = j2\pi f\{-j\,\mathrm{sgn}(f)\,X(f)\} \Leftrightarrow \frac{d\hat{x}(t)}{dt}$$

Exercise 2.58

Note that f = 0 should be f > 0 to make any sense.

$$H_\theta(f) = \begin{cases} e^{-j\theta} & ; f > 0 \\ e^{+j\theta} & ; f < 0 \end{cases} = \cos(\theta) - j\,\mathrm{sgn}(f)\sin(\theta)$$
$$h_\theta(t) = \cos(\theta)\,\delta(t) + \frac{\sin(\theta)}{\pi t}$$
$$x_\theta(t) = x(t)\cos(\theta) + \hat{x}(t)\sin(\theta)$$
$$\text{Energy}\{x_\theta(t)\} = \int_{-\infty}^{\infty} x_\theta^2(t)\,dt = \int_{-\infty}^{\infty} \left|X_\theta(f)\right|^2 df = \int_{-\infty}^{\infty} \left|X(f)\,H_\theta(f)\right|^2 df = \int_{-\infty}^{\infty} \left|X(f)\right|^2 df = \int_{-\infty}^{\infty} x^2(t)\,dt$$

Hence $x_\theta(t)$ has the same energy as x(t).

Solutions to Exercises on Slides

Slide 0.23

F is a real quadratic expression in the complex variables x and y and the object is to 'complete the square' as a quadratic in x. Completing the square means having x only in a term which is a magnitude squared. Hence we concentrate on the $|x|^2$ and x*y and xy* terms and add a $|y|^2$ term which will make this a 'perfect square'. Note that A, C, F are real and B is complex.

$$
\begin{aligned}
F &= A|x|^2 + Bx^*y + B^*xy^* + C|y|^2 \\
&= A\left(x + \frac{By}{A}\right)\left(x^* + \frac{B^*y^*}{A}\right) + \left(C - \frac{|B|^2}{A}\right)|y|^2 \\
&= A\left|x + \frac{By}{A}\right|^2 + \left(C - \frac{|B|^2}{A}\right)|y|^2
\end{aligned}
$$

Slide 2.16

$$
x(t) = \sum_{k=-\infty}^{\infty} g(t - kT) = \sum_{n=-\infty}^{\infty} X_n \, e^{j2\pi nt/T}
$$

$$
\begin{aligned}
X_n &= \frac{1}{T}\int_0^T x(t)\, e^{-j2\pi nt/T}\, dt = \frac{1}{T}\int_0^T \sum_{k=-\infty}^{\infty} g(t - kT)\, e^{-j2\pi nt/T}\, dt \\
&= \frac{1}{T}\sum_{k=-\infty}^{\infty}\int_0^T g(t - kT)\, e^{-j2\pi nt/T}\, dt = \frac{1}{T}\sum_{k=-\infty}^{\infty}\int_{-kT}^{T-kT} g(t')\, e^{-j2\pi nt'/T}\, dt' \\
&= \frac{1}{T}\int_{-\infty}^{\infty} g(t')\, e^{-j2\pi nt'/T}\, dt' = \frac{1}{T} G(n/T)
\end{aligned}
$$

Slide 2.18

$$
\begin{aligned}
&X(f) \leftrightarrow x(t) \\
&X^*(f) \leftrightarrow x^*(-t) \\
&|X(f)|^2 \leftrightarrow \int_{-\infty}^{\infty} x(\lambda)\, x^*(\lambda - t)\, dt
\end{aligned}
$$

If x(t) is real, the complex conjugate can be ignored.

Slide 2.37

$$x(t) = 4\,\mathrm{sinc}(2t)$$

$$X(f) = 2\,\mathrm{rect}(f/2) = 2\ \mathrm{rect}(f-0.5) + 2\ \mathrm{rect}(f+0.5)$$

$$y_1(t) = x(t)\cos(20\pi t) = 0.5\,x(t)\,e^{j20\pi t} + 0.5\,x(t)\,e^{-j20\pi t}$$

$$Y_1(f) = 0.5\,X(f-10) + 0.5\,X(f+10) = \mathrm{rect}\left(\frac{f-10}{2}\right) + \mathrm{rect}\left(\frac{f+10}{2}\right)$$

$$= \mathrm{rect}(f-10.5) + \mathrm{rect}(f-9.5) + \mathrm{rect}(f+9.5) + \mathrm{rect}(f+10.5)$$

$$\hat{X}(f) = -j\,\mathrm{sgn}(f)\,X(f) = -2j\,\mathrm{rect}(f-0.5) + 2j\,\mathrm{rect}(f+0.5)$$

$$y_2(t) = \hat{x}(t)\sin(20\pi t) = -0.5j\,\hat{x}(t)\,e^{j20\pi t} + 0.5j\,\hat{x}(t)\,e^{-j20\pi t}$$

$$Y_2(f) = -0.5j\,\hat{X}(f-10) + 0.5j\,\hat{X}(f+10)$$

$$= -\mathrm{rect}(f-10.5) + \mathrm{rect}(f-9.5) - \mathrm{rect}(f+10.5) + \mathrm{rect}(f+9.5)$$

$$Y(f) = Y_1(f) - Y_2(f) = 2\,\mathrm{rect}(f-10.5) + 2\,\mathrm{rect}(f+10.5)$$

Note how it is convenient to split the spectrum X(f) into two rectangle functions, one for positive f and one for negative f, since these are treated differently when we form the Hilbert transform.

Note how the 'lower sideband' components of $y_1(t)$ and $y_2(t)$ cancel and the 'upper sideband' components add together to make y(t) an 'upper sideband' signal.

This problem can also be done graphically (recommended method).