Communications IV Exercise Solutions

Exercise 4.10

$E\{x\} = 0$, $\sigma_x^2 = 10^{-8}$.

1. $P\{x > 10^{-4}\} = Q\{1\} \quad = 0.1581$.
   $P\{x > 4\times10^{-4}\} = Q\{4\} = 3.167\times10^{-5}$.
   $P\{-2\times10^{-4} < x < 10^{-4}\} = 1 - Q\{1\} - Q(2) = 0.8192$.

2. $P\{x > 10^{-4} \mid x > 0\} = P\{x > 10^{-4}\} / P\{x > 0\} = 2\,Q\{1\} = 0.3162$.

3.

$$P\{g < g_o\} = \begin{cases} 0 & ; g_o < 0 \\ 1 - Q\{g_o\} & ; g_o > 0 \end{cases}$$

$$p(g) = 0.5\,\delta(g) + u(g)\frac{1}{\sigma_x\sqrt{2\pi}}e^{-g^2/2\sigma_x^2}$$

The general expression fails because g'(x) = 0 for x < 0.

4.

$$E\{g\} = \int_{-\infty}^{\infty} g\,p(g)\,dg$$

$$= \frac{1}{\sigma_x\sqrt{2\pi}}\int_0^{\infty} g\,e^{-g^2/2\sigma_x^2}\,dg$$

$$= \frac{\sigma_x}{\sqrt{2\pi}}\left[-e^{-g^2/2\sigma_x^2}\right]_0^{\infty} = \frac{\sigma_x}{\sqrt{2\pi}} = 3.989\times10^{-5}.$$

5.

$$p(g) = 2\,u(g)\frac{1}{\sigma_x\sqrt{2\pi}}e^{-g^2/2\sigma_x^2}$$

$$E\{g\} = \frac{2\sigma_x}{\sqrt{2\pi}} = 7.979\times10^{-5}.$$

Exercise 4.44

$v(t) = x\cos(\omega_o t) + y\sin(\omega_o t)$, $E(x) = E(y) = 0$, $E\{x^2\} = E\{y^2\} = \sigma^2$, $E(xy) = 0$.

1. $E\{v\} = 0$

2.

$$\begin{aligned} R_{vv}(t+\tau, t) &= E\{[x\cos(\omega_o t + \omega_o\tau) + y\sin(\omega_o t + \omega_o\tau)][x\cos(\omega_o t) + y\sin(\omega_o t)]\} \\ &= \sigma^2\{\cos(\omega_o t + \omega_o\tau)\cos(\omega_o t) + \sin(\omega_o t + \omega_o\tau)\sin(\omega_o t)\} \\ &= \sigma^2\cos(\omega_o\tau) \end{aligned}$$

Hence v(t) is **wide sense stationary**. It is not cyclostationary because there is no dependence on t.

3. $S_{VV}(f) = 0.5\sigma^2\,\delta(f - f_o) + 0.5\sigma^2\,\delta(f + f_o)$.

4. If $\sigma_x \neq \sigma_y$ (there must be a misprint here) then the answers are:

E{v} = 0 as before.

$$\begin{aligned} R_{VV}(t+\tau,t) &= E\{[x\cos(\omega_o t + \omega_o\tau) + y\sin(\omega_o t + \omega_o\tau)][x\cos(\omega_o t) + y\sin(\omega_o t)]\} \\ &= \left\{\sigma_x^2\cos(\omega_o t + \omega_o\tau)\cos(\omega_o t) + \sigma_y^2\sin(\omega_o t + \omega_o\tau)\sin(\omega_o t)\right\} \\ &= 0.5\left(\sigma_x^2 + \sigma_y^2\right)\cos(\omega_o\tau) + 0.5\left(\sigma_x^2 - \sigma_y^2\right)\cos(2\omega_o t + \omega_o\tau) \end{aligned}$$

In this case v(t) is **not stationary**, but is **cyclostationary**.

$S_{VV}(f) = 0.25\left(\sigma_x^2 + \sigma_y^2\right)\delta(f - f_o) + 0.25\left(\sigma_x^2 + \sigma_y^2\right)\delta(f + f_o)$.

Exercise 4.48

Since X(t) is cyclostationary this means $E_X\{X(t)\}$ and $R_{XX}(t+\tau,t)$ are periodic in t (period T) and have time averages $\overline{X}$ and $\overline{R}_{XX}(\tau)$ respectively if averaged over the period T. The PDF of θ is p(θ) = 1/T, 0 < θ < T.

1.

$$Y(t) = X(t+\theta)$$

$$E\{Y(t)\} = E_X\left\{\frac{1}{T}\int_0^T X(t+\theta)\,d\theta\right\} = \frac{1}{T}\int_0^T E_X\{X(t+\theta)\}\,d\theta = \frac{1}{T}\int_t^{t+T} E_X\{X(t')\}\,dt' = \overline{X} \text{ (a constant)}$$

on substituting t' for t + θ

$$\begin{aligned} R_{YY}(t+\tau,t) &= E_X\left\{\frac{1}{T}\int_0^T X(t+\tau+\theta)\;X(t+\theta)\,d\theta\right\} \\ &= \frac{1}{T}\int_0^T R_{XX}(t+\tau+\theta,t+\theta)\,d\theta = \frac{1}{T}\int_t^{t+T} R_{XX}(t'+\tau,t')\,dt' = \overline{R}_{XX}(\tau) \end{aligned}$$

(Because $E_X\{X(t')\}$ and $R_{XX}(t'+\tau,t')$ are periodic in t', the integral over any interval of length T is the same).

Hence Y(t) is stationary.

2. Introducing a random time shift θ does not affect the power spectral density, so the power spectral density of X(t) is the same as that of Y(t).

3. Hence

$$S_{XX}(f) = S_{YY}(f) = FT\{R_{YY}(\tau)\} = FT\left\{\frac{1}{T}\int_0^T R_{XX}(t+\tau,t)\,dt\right\} = FT\{\overline{R}_{XX}(\tau)\}$$

Exercise 4.50

$$R_{xy}(\tau) = E\left\{x(t)y^*(t-\tau)\right\} = E\left\{y^*(t-\tau)x(t)\right\} = E\left\{y^*(t)x(t+\tau)\right\} = R^*_{yx}(-\tau)$$

$$S_{xy}(f) = \int_{-\infty}^{\infty} R_{xy}(\tau)e^{-j2\pi f\tau}\,d\tau = \int_{-\infty}^{\infty} R^*_{yx}(-\tau)e^{-j2\pi f\tau}\,d\tau = \int_{-\infty}^{\infty} R^*_{yx}(\tau)e^{+j2\pi f\tau}\,d\tau = S^*_{yx}(f)$$

Note this result is true for x(t) and y(t) complex.

Exercise 4.56

$$h(t) = \delta'(t) + \delta'(t-T)$$
$$H(f) = j2\pi f\left(1 + e^{-j2\pi fT}\right) = j4\pi f e^{-j\pi f}\ \cos(\pi fT)$$

1. y(t) is stationary because it is the response of a stationary signal applied to a linear time invariant filter.

2. The power spectral density of y(t) is:

$$S_{yy}(f) = S_{xx}(f)|H(f)|^2 = S_{xx}(f)\ \left(8\pi^2 f^2(1+\cos[2\pi fT])\right) = 16\pi^2 f^2\cos^2(\pi fT)$$

3. Since H(f) is zero at f = 0, ±1/2T, ±3/2T, ±5/2T, etc, these frequencies cannot be present in y(t).

Solutions to Exercises on Slides

Slide 4.06

Most of these results are "obvious" and do not require the integrations shown.

$$x_1(t) = \cos(\omega t)$$

$$\left\langle x_1(t)\right\rangle = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}\cos(\omega t)\,dt = \lim_{T\to\infty}\frac{2}{\omega T}\sin(\omega T/2) = 0$$

$$x_2(t) = \cos^2(\omega t)$$

$$\left\langle x_2(t)\right\rangle >= \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}\cos^2(\omega t)\,dt = \lim_{T\to\infty}\frac{1}{2T}\int_{-T/2}^{T/2}[1+\cos(2\omega t)]\,dt$$

$$= \lim_{T\to\infty}\frac{1}{2T}\left[T + \frac{\sin(\omega T)}{\omega}\right] = \frac{1}{2}$$

$$x_3(t) = \cos(\omega t)\sin(\omega t) = 0.5\sin(2\omega t)$$

$$\left\langle x_3(t)\right\rangle >= \lim_{T\to\infty}\frac{1}{2T}\int_{-T/2}^{T/2}\sin(2\omega t)\,dt = \lim_{T\to\infty}\frac{1}{2\omega T}\sin(\omega T) = 0$$

$$x_4(t) = \cos(\omega_1 t)\cos(\omega_2 t) = 0.5\cos[(\omega_1+\omega_2)t] + 0.5\cos[(\omega_1-\omega_2)t]$$

$$\left\langle x_4(t)\right\rangle >= \lim_{T\to\infty}\frac{1}{2T}\int_{-T/2}^{T/2}\left\{\cos[(\omega_1+\omega_2)t] + \cos[(\omega_1-\omega_2)t]\right\}dt$$

$$= \lim_{T\to\infty}\frac{1}{T}\left\{\frac{\sin[(\omega_1+\omega_2)T/2]}{\omega_1+\omega_2} + \frac{\sin[(\omega_1-\omega_2)T/2]}{\omega_1-\omega_2}\right\} = 0\ ;\omega_1 \neq \omega_2$$

Slide 4.13

$$z(t) = ax(t) + by(t)$$

$$R_{zz}(\tau) = E\{z(t)z^*(t-\tau)\} = E\{[ax(t)+by(t)][a^*x^*(t-\tau)+b^*y^*(t-\tau)]\}$$

$$= |a|^2 R_{xx}(\tau) + ab^* R_{xy}(\tau) + a^* b R_{yx}(\tau) + |b|^2 R_{yy}(\tau)$$

$$S_{zz}(f) = |a|^2 S_{xx}(f) + ab^* S_{xy}(f) + a^* b S_{yx}(f) + |b|^2 S_{yy}(f)$$

$$= S_{xx}(f)\left|a + \frac{bS_{yx}(f)}{S_{xx}(f)}\right|^2 + \left(S_{yy}(f) - \frac{|S_{yx}(f)|^2}{S_{xx}(f)}\right)$$

Because $S_{zz}(f) \geq 0$, then because the first term could be zero by choice of a and b at some frequency f, the second term ≥ 0 which gives the required result. Note that $S_{yx}(f) = S_{xy}^*(f)$.

Slide 4.24

$$y(t) = \int_{-\infty}^{t} [x(t') - x(t'-T)]dt' = \int_{-\infty}^{t} x(t')\,dt' - \int_{-\infty}^{t-T} x(t')\,dt'$$

$$\text{If } x(t) = \delta(t),\ y(t) = h(t) = u(t) - u(t-T) = \text{rect}\left(\frac{t - T/2}{T}\right)$$

$$Y(f) = \left[\frac{X(f)}{j2\pi f} + \frac{1}{2}X(0)\delta(f)\right]\left[1 - e^{-j2\pi fT}\right] = X(f)\left[\frac{1 - e^{-j2\pi fT}}{j2\pi f}\right] = T\,X(f)\,e^{-j\pi fT}\,\text{sinc}(fT)$$

$$H(f) = \frac{1 - e^{-j2\pi fT}}{j2\pi f} = T\,e^{-j\pi fT}\,\text{sinc}(fT)$$

Make sure you understand why the delta function disappears, and note the conversion of h(t) into a rect function and the conversion of H(f) into a sinc function.

Slide 4.27

$$Z(f) = H_1(f)X(f) + H_2(f)Y(f)$$

$$|Z(f)|^2 = |H_1(f)|^2|X(f)|^2 + H_1(f)H_2^*(f)X(f)Y^*(f) + H_2(f)H_1^*(f)Y(f)X^*(f) + |H_2(f)|^2|Y(f)|^2$$

$$S_{zz}(f) = |H_1(f)|^2 S_{xx}(f) + H_1(f)H_2^*(f)S_{xy}(f) + H_2(f)H_1^*(f)S_{yx}(f) + |H_2(f)|^2 S_{yy}(f)$$

$$= |H_1(f)|^2 S_{xx}(f) + 2\,\text{Re}\{H_1(f)H_2^*(f)S_{xy}(f)\} + |H_2(f)|^2 S_{yy}(f)$$

Slide 4.40

$$B_n = \frac{1}{|H_o|^2}\int_0^\infty |H(f)|^2\,df = \frac{1}{1}\int_0^\infty \frac{1}{1+4\pi^2 f^2 R^2 C^2}\,df = \frac{1}{2\pi RC}\left[\arctan(2\pi fRC)\right]_0^\infty = \frac{1}{2\pi RC}\frac{\pi}{2} = \frac{1}{4RC}$$

Slide 4.46

$$n_c(t) = \text{Re}\left\{n^+(t)\,e^{-j2\pi f_o t}\right\} = \frac{1}{2}\left\{n^+(t)\,e^{-j2\pi f_o t} + n^-(t)\,e^{+j2\pi f_o t}\right\}$$

$$n_s(t) = \text{Im}\left\{n^+(t)\,e^{-j2\pi f_o t}\right\} = \frac{1}{2j}\left\{n^+(t)\,e^{-j2\pi f_o t} - n^-(t)\,e^{+j2\pi f_o t}\right\}$$

$$R_{n_c n_s}(\tau) = E\left\{n_c(t)\,n_s^*(t-\tau)\right\}$$

$$= \frac{j}{4}E\left\{\left[n^+(t)\,e^{-j2\pi f_o t} + n^-(t)\,e^{+j2\pi f_o t}\right]\left[n^+(t-\tau)\,e^{-j2\pi f_o (t-\tau)} - n^-(t-\tau)\,e^{+j2\pi f_o (t-\tau)}\right]^*\right\}$$

$$= \frac{j}{4}\left\{R_{n^+n^+}(\tau)\,e^{-j2\pi f_o \tau} - R_{n^-n^-}(\tau)\,e^{+j2\pi f_o \tau}\right\} \qquad \text{since } n^+(t) \text{ and } n^-(t) \text{ are uncorrelated}$$

$$S_{n_c n_s}(f) = \frac{j}{4}\left\{S_{n^+n^+}(f+f_o) - S_{n^-n^-}(f-f_o)\right\} = j\left\{S_{nn}^{(+)}(f+f_o) - S_{nn}^{(-)}(f-f_o)\right\}$$

$$\text{since} \quad S_{n^+n^+}(f) = 4\,u(f)\,S_{nn}(f) = 4\,S_{nn}^{(+)}(f) \quad \text{and} \quad S_{n^-n^-}(f) = 4\,u(-f)\,S_{nn}(f) = 4\,S_{nn}^{(-)}(f)$$

NB. Take particular note of complex conjugation and its effect on the exponential terms. Also note that $S_{nn}^{(+)}(f)$= u(f) $S_{nn}(f)$ is the positive frequency part of $S_{nn}(f)$ and is not the power spectrum of $n^+(t)$ which is 4 u(f) $S_{nn}(f)$ and similarly $S_{nn}^{(-)}(f)$ is the negative frequency part of $S_{nn}(f)$ = u(−f) $S_{nn}(f)$.