Communications IV Exercise Solutions

Exercise 6.1

\[ H(x) = -\sum_{i=1}^{6} P(i) \ln P(i) \text{ nits} \]
\[ = -\{0.1\ln(0.1) + 0.2\ln(0.2) + 0.3\ln(0.3) + 0.05\ln(0.05) + 0.15\ln(0.15) + 0.1\ln(0.1)\} \]
\[ = 1.670 \text{ nits} = 2.409 \text{ bits} \]
\[ H(\text{uniform}) = \log_2(6) = 2.585 \text{ bits} \]

Exercise 6.7

The line \( x-1 \) is tangent to \( \ln(x) \) at \( x = 1 \) and a simple sketch shows that \( \ln(x) \leq (x-1) \) for all \( x > 0 \).

\[ H(x) = -\sum_{i=1}^{N} P_i \ln P_i \]
\[ H(y) = \ln N = \sum_{i=1}^{N} P_i \ln N \]
\[ H(x) - H(y) = \sum_{i=1}^{N} P_i \ln \left( \frac{1}{NP_i} \right) \]
\[ \leq \sum_{i=1}^{N} P_i \left[ \frac{1}{NP_i} - 1 \right] = 1 - 1 = 0 \]
\[ H(x) \leq H(y) \]

Exercise 6.22

1.

\[ H(x) = -\sum_{i=1}^{4} P(i) \ln P(i) \text{ nits} \]
\[ = -\{0.1\ln(0.1) + 0.2\ln(0.2) + 0.3\ln(0.3) + 0.4\ln(0.4)\} \]
\[ = 1.280 \text{ nits} = 1.847 \text{ bits} \]

2. Average code length \( \geq 1.847 \) bits.

3.

<table>
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<tbody>
<tr>
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<td></td>
<td>1</td>
<td>111</td>
</tr>
<tr>
<td>C</td>
<td>0.2</td>
<td>0</td>
<td>0.3</td>
<td>1</td>
<td>110</td>
</tr>
<tr>
<td>D</td>
<td>0.1</td>
<td>1</td>
<td></td>
<td></td>
<td>111</td>
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</tbody>
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<table>
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<tr>
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<tr>
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<tr>
<td>111</td>
<td>0.30</td>
</tr>
<tr>
<td>1.90</td>
<td></td>
</tr>
</tbody>
</table>

Average code word length = 1.90 bits (Efficiency = 97.2%).
Exercise 6.56

1. This is trivial.

2. If \( P_b \) is the probability of a bit error,
   \[
   \text{MSE}_{\text{chan}} = P_b \Delta^2 \left( 1 + 4 + 4^2 + \ldots + 4^{u-1} \right) = P_b \Delta^2 \frac{4^u - 1}{3}
   \]

3. With MSE_{\text{qu}} = \Delta^2/12 and defining \( x_{\max} = N\Delta/2 \) gives the expression required.
   \[
   \text{MSE} = \frac{x_{\max}^2}{3N^2} \left[ 1 + 4P_b (4^u - 1) \right]
   \]

You can also deduce that the channel MSE and quantisation MSE are equal when
   \[
   P_b = \frac{1}{4(4^u - 1)} = 3.81 \times 10^{-6} \text{ for } u = 8.
   \]

   This corresponds to a SNR of 13.0 dB with gaussian noise.

4. Simply calculate the ratio of the signal and noise powers.
   \[
   \text{SNR} = \frac{3N^2 < x^2 >}{x_{\max}^2 [1 + 4P_b (4^u - 1)]} = \frac{3N^2 < x_{\max}^2 >}{1 + 4P_b (4^u - 1)}
   \]

[Signal = x(t), \( \bar{x}(t) = \frac{x(t)}{x_{\max}} \) [this would be m(t) in our notation].]
Solutions to Exercises on Slides

Slide 6.07

For a source $x$ with $n$ symbols the probabilities are $P_i$, $i = 1, 2, \ldots, n$. Since the sum of these must be unity, we have only $n-1$ independent variables.

$$\sum_{i=1}^{n} P_i = 1 \quad \Rightarrow \quad P_n = 1 - \sum_{i=1}^{n-1} P_i$$

$$H(x) = -\sum_{i=1}^{n} P_i \ln(P_i) \text{ nits}$$

$$\frac{\partial H(x)}{\partial P_i} = -[1 + \ln(P_i)] - [1 + \ln(P_n)] \frac{\partial P_n}{\partial P_i} ; \quad i = 1, 2, \ldots, n - 1$$

$$= -[1 + \ln(P_i)] + [1 + \ln(P_n)] \quad \text{since} \quad \frac{\partial P_n}{\partial P_i} = -1$$

$$= \ln\left(\frac{P_n}{P_i}\right) = 0 \quad \Rightarrow \quad P_i = P_n = \frac{1}{n} \quad ; \quad i = 1, 2, \ldots, n - 1$$

$$H_{\text{max}}(x) = -\sum_{i=1}^{n} \frac{1}{n} \ln\left(\frac{1}{n}\right) = \ln(n) \text{ nits} = \log_2(n) \text{ bits}$$