Communications IV Exercise Solutions

Exercise 6.1

$$
\begin{aligned}
H(x) &= -\sum_{i=1}^{6} P(i)\ln P(i) \text{ nits} \\
&= -\{0.1\ln(0.1) + 0.2\ln(0.2) + 0.3\ln(0.3) + 0.05\ln(0.05) + 0.15\ln(0.15) + 0.1\ln(0.1)\} \\
&= 1.670 \text{ nits} = 2.409 \text{ bits}
\end{aligned}
$$

$$H(\text{uniform}) = \log_2(6) = 2.585 \text{ bits}$$

Exercise 6.7

The line x–1 is tangent to ln(x) at x = 1 and a simple sketch shows that ln(x) ≤ (x–1) for all x > 0.

$$
\begin{aligned}
H(x) &= -\sum_{i=1}^{N} P_i \ln P_i \\
H(y) &= \ln N = \sum_{i=1}^{N} P_i \ln N \\
H(x) - H(y) &= \sum_{i=1}^{N} P_i \ln(1/NP_i) \\
&\le \sum_{i=1}^{N} P_i \left[1/NP_i - 1\right] = 1 - 1 = 0 \\
H(x) &\le H(y)
\end{aligned}
$$

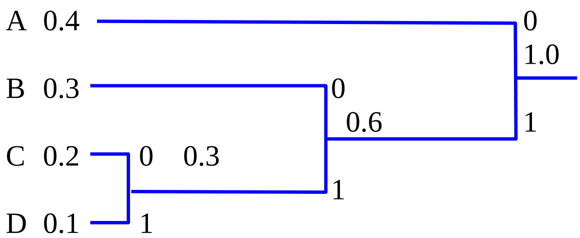
Exercise 6.22

1.

$$
\begin{aligned}
H(x) &= -\sum_{i=1}^{4} P(i)\ln P(i) \text{ nits} \\
&= -\{0.1\ln(0.1) + 0.2\ln(0.2) + 0.3\ln(0.3) + 0.4\ln(0.4)\} \\
&= 1.280 \text{ nits} = 1.847 \text{ bits}
\end{aligned}
$$

2. Average code length ≥ 1.847 bits.

3.



	Code	$n_i P_i$
A	0	0.40
B	10	0.60
C	110	0.60
D	111	0.30
		1.90

Average code word length = 1.90 bits (Efficiency = 97.2%).

Exercise 6.56

1. This is trivial.

2. If P_b is the probability of a bit error,

$$MSE_{chan} = P_b \Delta^2 \left(1 + 4 + 4^2 + \cdots + 4^{\upsilon-1}\right) = P_b \Delta^2 \frac{4^{\upsilon} - 1}{3}$$

3. With $MSE_{qant} = \Delta^2/12$ and defining $x_{max} = N\Delta/2$ gives the expression required.

$$MSE = \frac{x_{max}^2}{3N^2}\left[1 + 4P_b (4^{\upsilon} - 1)\right]$$

You can also deduce that the channel MSE and quantisation MSE are equal when

$$P_b = \frac{1}{4(4^{\upsilon} - 1)} = 3.81 \times 10^{-6} \text{ for } \upsilon = 8.$$

This corresponds to a SNR of 13.0 dB with gaussian noise.

4. Simply calculate the ratio of the signal and noise powers.

$$\text{Signal} = x(t), \quad \breve{x}(t) = \frac{x(t)}{x_{max}} \text{ [this would be m(t) in our notation].}$$

$$SNR = \frac{3N^2 < x^2 >}{x_{max}^2 [1 + 4P_b (4^{\upsilon} - 1)]} = \frac{3N^2 < \breve{x}^2 >}{1 + 4P_b (4^{\upsilon} - 1)}$$

Solutions to Exercises on Slides

Slide 6.07

For a source x with n symbols the probabilities are P_i, i = 1, 2, …, n. Since the sum of these must be unity, we have only n–1 independent variables.

$$\sum_{i=1}^{n} P_i = 1 \quad \Rightarrow \quad P_n = 1 - \sum_{i=1}^{n-1} P_i$$

$$H(x) = -\sum_{i=1}^{n} P_i \ln(P_i) \quad \text{nits}$$

$$\frac{\partial H(x)}{\partial P_i} = -[1 + \ln(P_i)] - [1 + \ln(P_n)]\frac{\partial P_n}{\partial P_i} \qquad ; i = 1, 2, \ldots, n-1$$

$$= -[1 + \ln(P_i)] + [1 + \ln(P_n)] \qquad \text{since } \frac{\partial P_n}{\partial P_i} = -1$$

$$= \ln\left(\frac{P_n}{P_i}\right) = 0 \quad \Rightarrow \quad P_i = P_n = \frac{1}{n} \quad ; i = 1, 2, \ldots, n-1$$

$$H_{max}(x) = -\sum_{i=1}^{n} \frac{1}{n} \ln\left(\frac{1}{n}\right) = \ln(n) \text{ nits} = \log_2(n) \text{ bits}$$