Communications IV Exercise Solutions

Exercise 7.10

Assuming a matched filter is used:

\[ S_{n_0}(f) = \frac{N_o}{2} = 5 \times 10^{-3} \]

\[ \text{SNR} = \frac{E_b}{\alpha} = \frac{A^2 T_b}{\alpha} = \frac{A^2 \times 10^{-5}}{5 \times 10^{-3}} = 0.002 A^2 = 4.753^2 \text{ for } P_e = 10^{-6} \]

\[ A = 106.3 \]

Exercise 7.15

\[ H(f) = \frac{1 - e^{-j2\pi T}}{j2\pi f} \]

\[ h(t) = u(t) - u(t - T) \]

\[ p(t) = h(T - t) = u(T - t) - u(-t) = u(t) - u(t - T) \]

Exercise 7.17

1.

\[ s_0(T) = T \sqrt{E_b / T} \]

\[ n_0(T) = \int_0^T n(\lambda) \, d\lambda \]

\[ E[n_o^2(T)] = \int_0^T \int_0^T \delta(\lambda - \mu) \, d\lambda \, d\mu = \frac{N_o T}{2} \]

\[ \text{SNR} = \frac{2E_b}{N_o} \]

By hindsight we see that this is a matched filter. But note that the integrator must be “dumped” (reset to zero) after each pulse.

2.

\[ s_0(t) = \pm \sqrt{E_b / T} \left( 1 - e^{-t/RC} \right) \quad ; 0 \leq t \leq T \]

\[ s_0(T) = \pm \sqrt{E_b / T} \left( 1 - e^{-T/RC} \right) \]

\[ h(t) = u(t) \frac{1}{RC} e^{-t/RC} \]

\[ n_o(T) = \int_0^T \frac{1}{RC} e^{-\lambda/RC} \, n(T - \lambda) \, d\lambda \]
\[ s_o(t) = \pm \sqrt{E_b/T} \left(1 - e^{-t/RC}\right) \quad ; 0 \leq t \leq T \]
\[ s_o(T) = \pm \sqrt{E_b/T} \left(1 - e^{-T/RC}\right) \]
\[ h(t) = u(t) \frac{1}{RC} e^{-t/RC} \]
\[ n_o(T) = \frac{T}{RC} e^{-\lambda/RC} n(T - \lambda) d\lambda \]
\[ E[n_o^2] = \int_0^T \int_0^T \frac{1}{R^2C^2} e^{-\lambda/RC} e^{-\mu/RC} \delta(\lambda - \mu) d\lambda d\mu \]
\[ = \frac{N_o}{2R^2C^2} \int_0^T e^{-\lambda/RC} e^{-\mu/RC} \delta(\lambda - \mu) d\lambda d\mu \]
\[ = \frac{N_o}{2R^2C^2} \int_0^T e^{-2\lambda/RC} d\lambda \]
\[ = \frac{N_o}{4RC} \left(1 - e^{-2T/RC}\right) \]
\[ \text{SNR} = \frac{4RE_b}{N_oT} \left(1 - e^{-T/RC}\right)^2 = \frac{2E_b}{N_o} \quad \text{if} \quad T \ll RC \]

The circuit shows that the capacitor must be “dumped” before the current symbol is applied, otherwise there will be significant ISI. If you plot the SNR a function of \( x = T/RC \), it has a maximum at \( x = 0 \) and is monotonically decreasing. Hence the optimum RC is a value large compared with T, but this is equivalent to an integrator (ie. the same as in the first part).

**Exercise 7.34**

Assuming a matched filter,
\[ N_o = 2 \times 10^{-10} \text{ W/Hz} \]
\[ \text{SNR} = \frac{2E_b}{N_o} \frac{A^2T}{N_o} = 4.753^2 \quad \text{if} \quad P_b = 10^{-6} \]
\[ A = 4.753 \sqrt{N_o/T} = \begin{cases} \frac{6.722 \times 10^{-3}}{2.126 \times 10^{-2}} & 10 \text{kb/s} \\ \frac{6.722 \times 10^{-2}}{1 \text{Mb/s}} & 100 \text{kb/s} \end{cases} \]

If the impedance of the system is R, the values of A must be multiplied by \( \sqrt{R} \).
**Exercise 7.43**

For constellation (a) the outer symbols have coordinates \((\pm 2A, \pm 2A)\). The average energy per symbol is:

\[
E_s = \frac{1}{8} \left(4 \times 4A^2 + 4 \times 8A^2\right) = 6.0A^2
\]

For constellation (b) the symbols on the X axis have coordinates \((\pm A, 0)\), the two on the Y axis have coordinates \((0, \pm A\sqrt{3})\) and the outer symbols have coordinates \((\pm 2A, \pm A\sqrt{3})\). The average energy per symbol is:

\[
E_s = \frac{1}{8} \left(2 \times A^2 + 2 \times 3A^2 + 4 \times (4A^2 + 3A^2)\right) = 4.5A^2
\]

These would have the same error performance, so clearly constellation (b) is more power efficient.

**Solutions to Exercises on Slides**

**Slide 7.52**

In MPSK the symbols are on a circle of radius \(r = \sqrt{E_s}\) and are spaced \(d = 2r \sin(\pi/M)\) and \(E_b = E_s/K\). Hence

\[
d^2 = 4KE_b \sin^2(\pi/M)
\]

\[
P_e = Q\left(\sqrt{\frac{d^2}{2N_0}}\right) = Q\left(\sqrt{\frac{2KE_b \sin^2(\pi/M)}{N_0}}\right)
\]

To get the same error probability for 4PSK and 8PSK requires

\[
\frac{E_{4PSK}}{E_{4PSK}} \sin^2(\pi/4) = \frac{E_{8PSK}}{E_{8PSK}} \sin^2(\pi/8)
\]

\[
\frac{E_{8PSK}}{E_{4PSK}} = 3.4142 \quad (5.3\text{dB})
\]

\[
\frac{E_{8PSK}}{E_{4PSK}} = \frac{2}{3} \times 3.4142 = 2.2761 \quad (3.6\text{dB})
\]
The inner circle has radius \( r_1 = d/\sqrt{2} = 0.7071d \). The symbol in the first quadrant on the outer circle has coordinates \((x, x)\) and we must have \( d^2 = x^2 + (x-r_1)^2 \) giving

\[
2x^2 - 2r_1x + r_1^2 - d^2 = 0 \\
x^2 - 0.7071dx - 0.25d^2 = 0 \\
x = 0.96593d
\]

\( r_2 = x\sqrt{2} = 1.3660d = 0.5(1 + \sqrt{3})d \)

\[
E_b(\text{av}) = 0.5(r_1^2 + r_2^2) = 1.1830d^2 \\
E_b(\text{av}) = E_s(\text{av})/3 = 0.39432d^2 = 19.953 \, N_o \\
\]

\[
P_{\text{sym}} = Q\left(\frac{d^2}{2N_o}\right) = Q\left(\frac{19.953}{\sqrt{2 \times 0.39432}}\right) = Q\left(\frac{5.030}{5.030}\right) = 2.45 \times 10^{-7}
\]

since we have \( E_b/\text{N}_o = 13 \, \text{dB} = 19.953. \)