4035, Communications IV

Lecture Notes

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4035 Communications IV

Lectures: 18
Tutorials: 4

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Lectures

• **Slides** - These contain the salient points, but are not a complete coverage of topics. You must have the latest version and you need to:

* take notes on any additional material presented.
* promptly complete any missing steps in derivations.
* promptly do any exercises set in lectures.

• **Attendance** at lectures and tutorials is expected, and all material in lectures, tutorials and exercises is examinable.

• **Podcasts** are there for revising something you missed. They do not replace lectures.

Questions

- If you have questions, put up your hand in the lecture.
- I will try to hang around at the end of lectures for individual longer questions.
- Contact me by e-mail for questions that occur to you later.
- I have a wiki you can interact on.
- I have an open door policy. If my office door is open come in. If it is shut, it is a subtle hint that I don’t want to be disturbed.

Tutorials

• **How Many** - There will be 4 tutorials.

• **Benefits** - To get any benefit from tutorials, you must attempt the exercises **before** the tutorial session, and use the session to ask questions on parts that you had difficulty with. Complete any unfinished parts as soon as possible.

• **Solutions** will be discussed at the tutorial session, written solutions may be provided later.

Exercises

• **When** - As the course progresses a number of exercises will be set. There is no assessment for these, but it is expected that you do these as soon after the lecture as possible.

• **Solutions** - Worked solutions to most exercises will be provided, but looking at these before attempting the problems, or simply working through the solutions without attempting the exercises means you will learn very little.

Exercises (cont)

• **Purpose** - The purpose of the exercises is to enable you to explore the different ways of solving problems, and become adept at finding the most efficient method. You need to find out these things for yourself, and not just believe that the supplied solution is the best way.

• **Exam Questions** - The exercises and tutorials are not necessarily sample examination questions. These often involve more calculation than would be required in an examination, but they do provide a guide to the sorts of things you might be asked.

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Assignment

• **Type** - There will be one Matlab exercise on material related to the course.

• **When** - This will be issued about 3 to 4 weeks prior to the due date.

• **Assessment** - This will count 10% towards the final assessment. Note that there will be no opportunity to redeem a poor performance in an assignment.

• **Help** - The student branch of the IEEE (EEESAU) runs Matlab courses.

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Assessment

Closed Book Examination (2 hours) - 90%

Matlab Assignment - 10%

Competence in the use of **Matlab®** is expected, and the assignment set will require its use. You will find that it provides very useful tools for processing and displaying communication signals, and many of the diagrams in the lecture slides were produced using it. It is available on most school computers.

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Reference Books

***Proakis & Salehi** : "Communication Systems Engineering" 2nd Ed (Prentice Hall).

Ziemer & Tranter : "Principles of Communications" 5th Ed (Wiley)

* You must purchase Proakis & Salehi. The lectures will follow this book fairly closely and exercises and tutorials will be set from this, so it is necessary to buy a copy.

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My Advice

If you choose to study by:

* not attending lectures or not paying attention
* not attempting any exercises or tutorials during the course
* working through solutions to exercises and tutorials before attempting them

then do not be disappointed if you do not do well in this course.

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How to Pass this Course

- Come to all lectures
- Do all exercises
- Read the text book
- Work through text book examples
- Study an hour per day (on this subject)
- Focus and good study habits
- Do everything I say ☺

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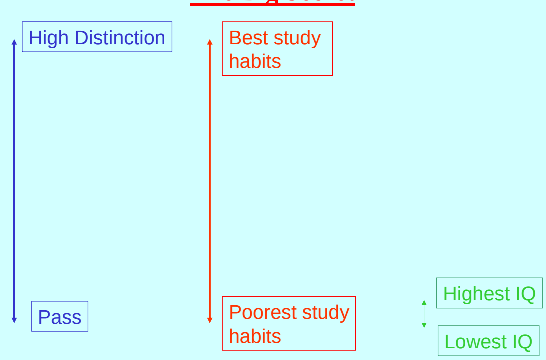
The Big Secret



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The Dream & The Reality

"Leadership is the ability to redefine reality"
— Warren Bennis

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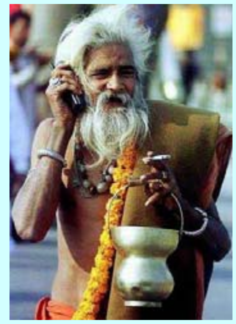
Redefining Reality



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Special Functions

In this course we will use functions with which you may not be familiar. These functions are defined on the Fourier Transform sheet, but a summary is provided on the following pages as well.

They are usually functions of either *t* or *f*, but are defined in terms of an arbitrary variable *x*.

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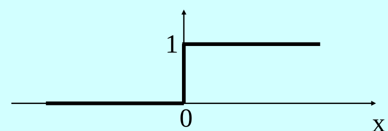
1. Unit Step Function

Notation: u(x)   [Proakis uses $u_{-1}(x)$]

$$u(x) = \begin{cases} 0 & ; x < 0 \\ 1 & ; x > 0 \end{cases}$$



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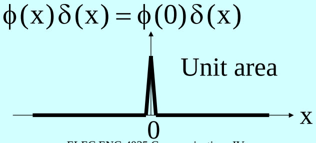
2. Delta Function

Notation: δ(x). [or $u_0(x)$]

This is defined by the integral:

$$\int_{-\infty}^{\infty} \phi(x)\delta(x)\,dx = \phi(0)$$

$$\phi(x)\delta(x) = \phi(0)\delta(x)$$



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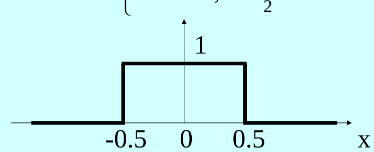
3. Rectangle Function

Notation: rect(x) or Π(x).

$$\text{rect}(x) = \begin{cases} 0 & ; x < -\frac{1}{2} \\ 1 & ; -\frac{1}{2} < x < \frac{1}{2} \\ 0 & ; x > \frac{1}{2} \end{cases}$$



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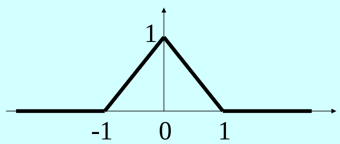
4. Triangle Function

Notation: $\Delta(x)$ or $\Lambda(x)$.

$$\Delta(x) = \begin{cases} 0 & ; x < -1 \\ 1 - |x| & ; -1 < x < 1 \\ 0 & ; x > 1 \end{cases}$$



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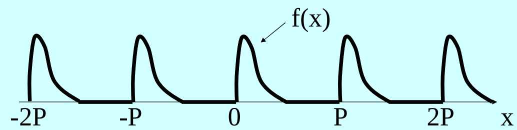
5. Repeat Function

Notation: $\text{rep}_P\{f(x)\}$

$$\text{rep}_P\{f(x)\} = \sum_{n=-\infty}^{\infty} f(x - nP)$$



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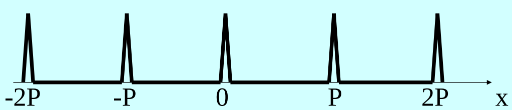
6. Comb Function

Notation: $\text{comb}_P(x)$

$$\text{comb}_P(x) = \sum_{n=-\infty}^{\infty} \delta(x - nP) = \text{rep}_P\{\delta(x)\}$$



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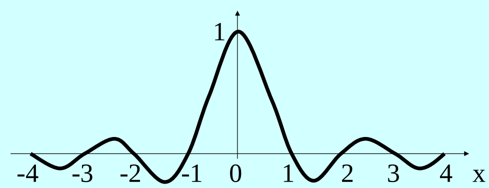
7. Sinc Function

Notation: sinc(x)

$$\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$



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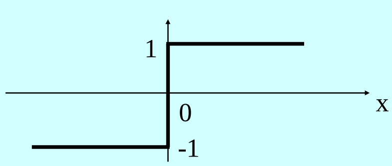
8. Sign Function

Notation: sgn(x)

$$\text{sgn}(x) = \begin{cases} -1 & ; x < 0 \\ +1 & ; x > 0 \end{cases} = 2u(x) - 1$$



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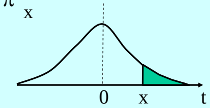
9. The Gaussian Error Function

There are two functions in use. The most common is Q(x) which is the one we will use.

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-t^2/2}\, dt = \frac{1}{2}\text{erfc}\left(\frac{x}{\sqrt{2}}\right)$$

$$\text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2}\, dt = 2Q\left(x\sqrt{2}\right)$$



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10. Completing the Square

A general complex quadratic expression, with **F**, **A** and **C** real, but **B**, **x** and **y** complex is shown below. We often need to "complete the square".

$$F = A|x|^2 + Bx^*y + B^*xy^* + C|y|^2$$
$$= A\left|x + \frac{By}{A}\right|^2 + \left(C - \frac{|B|^2}{A}\right)|y|^2$$

Exercise: Derive this result, you need to be able to do this.

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11. Integrals

Consider an integral of the form:

$$\int_{-\infty}^{\infty} v^2(t)\cos^2(\omega_o t)\,dt = \tfrac{1}{2}\int_{-\infty}^{\infty} v^2(t)\,dt + \tfrac{1}{2}\int_{-\infty}^{\infty} v^2(t)\cos(2\omega_o t)\,dt$$

If v(t) has no frequency components for $|f| \geq f_o$, the second integral is zero, since if $x(t) = v^2(t)$ and $y(t) = \cos(2\omega_o t)$:

$$\int_{-\infty}^{\infty} x(t)\,y^*(t)\,dt = \int_{-\infty}^{\infty} X(f)\,Y^*(f)\,df = 0$$

because the spectra of x(t) and y(t) do not overlap.

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Section 1: Introduction

Contents

1. Introduction

1.1 What is Communication?

Communication involves the transmission of information from one point to another.

Communication Theory is the study of communication systems and the signals associated with them.

Various aspects include:

- Signal representation in time & frequency
- Bandwidths required for various signals
- Modulation and demodulation methods
- Filtering of signals
- Random signals and noise
- Effects of noise on communication systems
- Errors in digital systems
- Coding and error correction
- Information theory

By necessity, the transmission of information from one point to another requires that the signals be random (unknown), since if they were deterministic (known), there would be no need to transmit them.

The basic message signal is called the ***baseband*** signal, and this is usually converted into another form suitable for the transmission medium. This conversion process is called ***modulation***, with the reverse process called ***demodulation***.

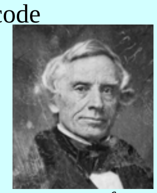
Accompanying most transmissions are random perturbations not related to the wanted signal. These perturbations are called ***noise***, and may originate in the transmission medium or in the receiving apparatus.

Hence both the signal and noise in communication systems will be random, so probabilistic methods of describing the properties of such signals will be required.

1.2 Telegraphy & Telephony

The first electric communication system was the ***telegraph***. The first telegraph line linked Washington and Baltimore in 1844, and encoded letters of the alphabet, numerals and punctuation marks using a variable length binary code invented by Samuel Morse.

A = • –	J = • – – –
E = •	Q = – – • –
T = –	Z = – – • •



In 1875 Emile Baudot invented a code which encoded each letter with a fixed length binary code, the forerunner to the **ASCII code** we use today.

The first **transatlantic telegraph cable** was laid in 1858 but failed after 4 weeks. A second cable became operational in 1866.

The **telephone** was patented by Alexander Graham Bell in 1876, and in 1877 the Bell Telephone Company was formed.

In 1906, Lee DeForest invented the **triode valve** which made it possible to amplify signals and allow telegraph and telephone communication over larger distances. Transcontinental telephone transmissions became operational in 1915.

The first **transatlantic telephone cable** was not laid until 1953.

Automatic switching of telephone calls was developed by Almon Strowger in 1897.

Electronic switching became economically feasible with the invention of the transistor, and the first digital switch was placed in service in Illinois in 1960.

Today, fibre optic cables are rapidly replacing copper cables, and all telephone switching is carried out electronically.

1.3 Wireless Communication Systems

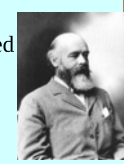
Wireless communications stem from the work of Oersted, Faraday, Gauss, Maxwell and Hertz in the 19th century.

In 1831 Faraday showed that a moving magnet induced a voltage in a nearby conductor.

In 1864 Maxwell developed the basic theory of electromagnetic radiation.

In 1887 Maxwell's theory was verified experimentally by Hertz.

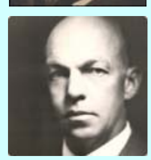
In 1894 Oliver Lodge demonstrated wireless communication over a distance of 150 metres.



In 1897 Guglielmo Marconi transmitted radio signals over a distance of 2 km, and in 1901 a transatlantic communication of 2700 km was achieved - and this was before vacuum tubes.

John Fleming invented the vacuum diode in 1904, followed by the triode invented by Lee DeForest in 1906.

The first ***amplitude modulation*** broadcast occurred in 1920 in Pittsburg.



The ***superheterodyne receiver*** was invented by Edwin Armstrong during World War I, and in 1933 Armstrong demonstrated the first ***frequency modulation*** system.

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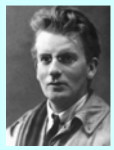
The first television system using electronic scanning was built by Vladimir Zworykin in 1929, although there were many earlier attempts using mechanical scanning by people such as John Logie Baird.

Commercial television broadcasting began in London in 1936 and 5 years later in the United States of America.



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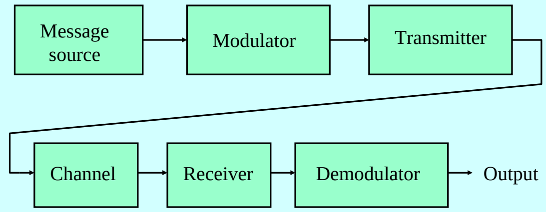
1.4 Communication Systems



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In what follows we will usually assume an additive noise channel. The simplest of these is the ***additive white Gaussian noise*** (AWGN) channel, where the noise has a uniform power spectral density and is added to the wanted signal.

We can also have multiplicative noise, the most common of which is fading, a very important feature of mobile and high frequency communication channels.

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1.5 Decibels

Many quantities in communication theory, and in particular signal to noise ratios, are expressed in decibels.

$$\mathrm{SNR(dB)} = 10\log_{10}(\mathrm{SNR})$$

Where SNR is the actual **power ratio**. In all of the various formulae used in this course, the actual power ratio must be used. **Values in decibels must never be used.**

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Section 2: Frequency Domain Analysis

Contents

2. Frequency Domain Analysis

2.1 Fourier Series

A signal x(t) which has period T can be expressed as:

$$x(t) = \sum_{n=-\infty}^{\infty} X_n \, e^{jn2\pi t/T}$$

$$X_n = \frac{1}{T} \int_{(T)} x(t) \, e^{-jn2\pi t/T} dt$$

Notes:

• (T) means integration over any time interval of length T.

• We will use f in Hz as the frequency variable. The symbol ω will only be used to represent 2πf.

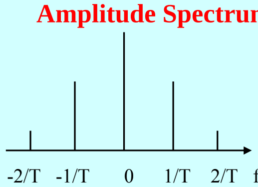
• Frequency f is a property of the complex exponential $e^{j2\pi ft}$ and may be positive or negative. Real signals contain both positive and negative frequencies. eg. $\cos(2\pi ft) = 0.5e^{j2\pi ft} + 0.5e^{-j2\pi ft}$.

• The number X_n is a complex number (dimensions V) and represents a component of frequency n/T Hz, and for a real signal $X_{-n} = X_n^*$. This is called the ***Hermitian*** property.

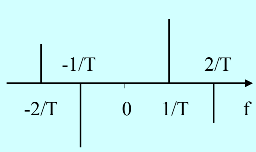
• The ***fundamental frequency*** is $f_o = 1/T$ corresponding to n = 1. All other frequency components are multiples of this frequency.

• The ***amplitude spectrum*** $|X_n|$ of a real signal is an even function of frequency, whereas the ***phase spectrum*** arg X_n is an odd function.

Amplitude Spectrum



Phase Spectrum



• The average ***power*** is obtained by averaging over one period.

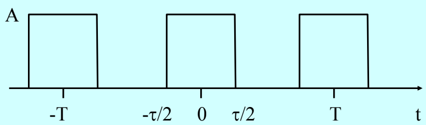
$$P = \frac{1}{T}\int_{(T)} |x(t)|^2 dt = \frac{1}{T}\int_{(T)} x(t)\, x^*(t)\, dt$$

$$= \frac{1}{T}\int_{(T)} x(t) \left[\sum_{n=-\infty}^{\infty} X_n^* \, e^{-jn2\pi t/T} \right] dt$$

$$= \sum_{n=-\infty}^{\infty} X_n X_n^* = \sum_{n=-\infty}^{\infty} |X_n|^2$$

This is ***Parseval's Theorem***. Note that P is actually the mean square value.

Example: Rectangular pulse train.



$$x(t) = \text{rep}_T\{A\,\text{rect}(t/\tau)\}$$

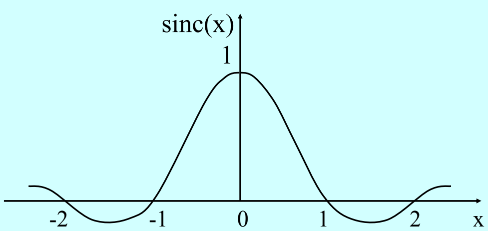
$$X_n = \frac{1}{T}\int_{-\tau/2}^{\tau/2} A\,e^{-jn2\pi t/T}dt$$

$$= \frac{A}{j2\pi n}\left(e^{jn\pi\tau/T} - e^{-jn\pi\tau/T}\right)$$

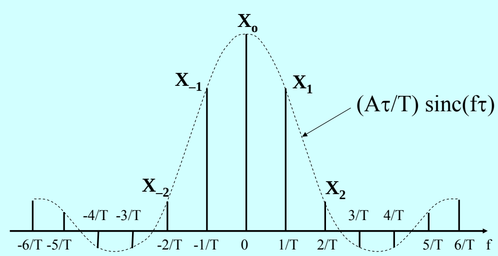
$$= \frac{A\tau}{T}\,\frac{\sin(n\pi\tau/T)}{n\pi\tau/T}$$

$$= \frac{A\tau}{T}\,\text{sinc}(n\tau/T)$$

where $\text{sinc}(x) = \sin(\pi x)/(\pi x)$.



The Fourier series spectrum is:



2.2 Fourier Transforms

The ***Fourier transform*** is an extension of Fourier series to non-periodic signals.

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi f t}dt$$

$$x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi f t}df$$

We will use the notation $x(t) \leftrightarrow X(f)$. If $x(t)$ is in volts, the dimensions of $X(f)$ are volt-sec or V/Hz.

Fourier transforms exist for signals of ***finite energy***.

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} x(t)\,x^*(t)\,dt$$

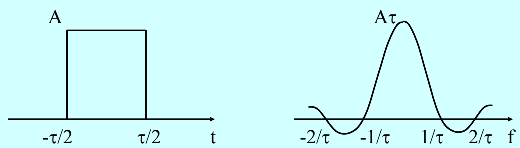
$$= \int_{-\infty}^{\infty} x(t)\left\{\int_{-\infty}^{\infty} X^*(f)e^{-j2\pi f t}df\right\}dt$$

$$= \int_{-\infty}^{\infty} X(f)X^*(f)\,df = \int_{-\infty}^{\infty} |X(f)|^2 df$$

This is ***Rayleigh's Energy Theorem***.

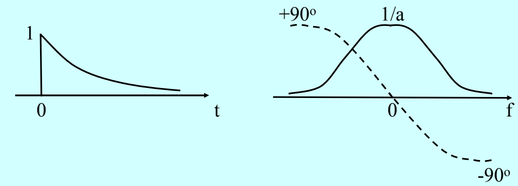
Note that $G_{xx}(f) = |X(f)|^2$ is also called the ***energy spectral density*** of the finite energy signal.

Example: Rectangular pulse



$$x(t) = A\,\mathrm{rect}(t/\tau) \qquad X(f) = A\tau\,\mathrm{sinc}(f\tau)$$

Example: Exponential pulse



$$x(t) = u(t)\,e^{-at} \qquad X(f) = \frac{1}{a + j2\pi f}$$

Make sure that you know how to deal with:

- Time scaling
- Frequency scaling
- Time shifts
- Frequency shifts
- Time inversion
- Differentiation
- Integration
- Multiplication by t
- See the Fourier transform sheet provided

For periodic signals we have:

$$x(t) = \sum_{k=-\infty}^{\infty} g(t - kT) = \mathrm{rep}_T\{g(t)\} = \sum_{n=-\infty}^{\infty} X_n\, e^{jn2\pi t/T}$$

$$X(f) = \sum_{n=-\infty}^{\infty} X_n\, \delta(f - n/T)$$

$$X_n = \frac{1}{T}\int_{(T)} x(t)\, e^{-j2\pi nt/T}\, dt = \frac{1}{T} G(n/T)$$

$$G(f) = \int_{-\infty}^{\infty} g(t)\, e^{-j2\pi f t}\, dt$$

Exercise: Prove that $X_n = (1/T)\, G(n/T)$.

2.3 Convolution

If we have $V(f) = X(f)Y(f)$, what is $v(t)$?

$$v(t) = \int_{-\infty}^{\infty} X(f)\,Y(f)\,e^{j2\pi ft}\, df$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(\lambda)\,e^{-j2\pi f\lambda}\, d\lambda\, Y(f)\,e^{j2\pi ft}\, df$$

$$= \int_{-\infty}^{\infty} x(\lambda)\,y(t-\lambda)\, d\lambda = x(t) \otimes y(t)$$

This is called the ***convolution*** of x(t) and y(t). We can also define convolution in the frequency domain.

$$x(t)\,y(t) \leftrightarrow X(f) \otimes Y(f) = \int_{-\infty}^{\infty} X(\lambda)\,Y(f-\lambda)\, d\lambda$$

$$X(f)\,Y(f) \leftrightarrow x(t) \otimes y(t) = \int_{-\infty}^{\infty} x(\lambda)\,y(f-\lambda)\, d\lambda$$

Exercise: Prove $\int_{-\infty}^{\infty} x(\lambda)\,x^*(\lambda - t)\, d\lambda \leftrightarrow |X(f)|^2 = G_{XX}(f)$

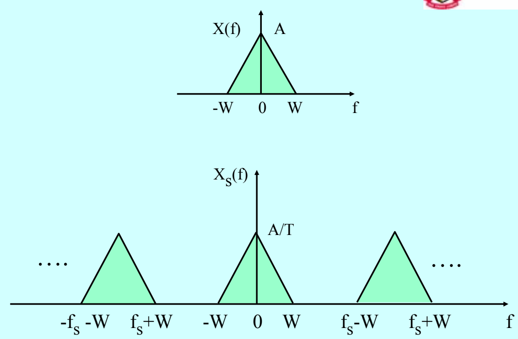
2.4 The Sampling Theorem

If a signal is sampled at intervals of $T = 1/f_s$ we have:

$$x_s(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT) = x(t)\,\mathrm{comb}_T(t)$$

$$X_s(f) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(f - k/T) = \frac{1}{T}\,\mathrm{rep}_{1/T}\{X(f)\}$$

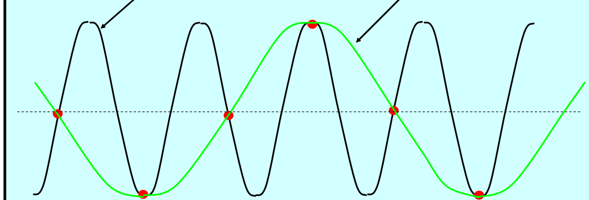
We can recover $x(t)$ from $x_s(t)$ by low pass filtering if the bandwidth of $x(t)$ is less than $1/2T$ (ie. less than one half the sampling rate).



If we sample at lower than the required rate of $f_s = 2W$, we get ***aliasing.*** This is where signals at frequencies greater than the ***Nyquist frequency*** $f_s/2$, reappear at frequencies mirror imaged about $f_s/2$. For instance if the sampling rate is 10 kHz, a 6.5 kHz signal will appear as a 3.5 kHz signal when we try to recover the signal.

To avoid this, frequencies higher than $f_s/2$ must be removed by an anti-aliasing filter before sampling.

Frequency 7500 Hz　　Frequency 2500 Hz



Sampling frequency 10000 Hz

2.5 The Analytic Signal

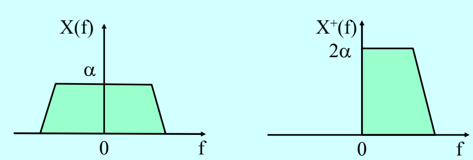
With a <u>real</u> signal $x(t)$ we have $X(-f) = X^*(f)$, so the negative frequency part is redundant. As for sinewaves, it is convenient to deal with signals which contain only positive frequencies.

$$A\cos(2\pi f_o t + \theta) = \mathrm{Re}\{A\,e^{j(2\pi f_o t + \theta)}\}$$

(Real signal)　　(Analytic signal)

The analytic signal is also called the ***pre-envelope.***

To obtain the analytic signal, we simply discard the negative frequency part and double the positive frequency part. For a real signal $x(t)$, we designate the analytic signal as $x^+(t)$.



$$X^+(f) = 2u(f)X(f)$$

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Example:

$$x(t) = \frac{a}{a^2 + t^2}$$

$$X(f) = \pi e^{-2\pi|f|a}$$

$$X^+(f) = 2\pi u(f) e^{-2\pi f a}$$

$$x^+(t) = \frac{1}{a - jt} = \left\{\frac{a}{a^2 + t^2}\right\} + j\left\{\frac{t}{a^2 + t^2}\right\}$$

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$$X^+(f) = 2\pi u(f) e^{-2\pi fa}$$

$$u(t) e^{-at} \leftrightarrow \frac{1}{a + j2\pi f}$$

$$u(-f) e^{af} \leftrightarrow \frac{1}{a + j2\pi t} \quad \text{using} \quad X(t) \leftrightarrow x(-f)$$

$$u(f) e^{-af} \leftrightarrow \frac{1}{a - j2\pi t} \quad \text{using} \quad x(-t) \leftrightarrow X(-f)$$

$$2\pi u(f) e^{-2\pi af} \leftrightarrow \frac{2\pi}{2\pi a - j2\pi t} = \frac{1}{a - jt} = x^+(t)$$

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The real part of the analytic signal is the original signal, the imaginary part is called the ***Hilbert transform*** of x(t) and is denoted $\hat{x}(t)$.

$$Ae^{j(2\pi f_o t+\theta)} = A\cos(2\pi f_o t + \theta) + jA\sin(2\pi f_o t + \theta)$$

$$x(t) = A\cos(2\pi f_o t + \theta) = \tfrac{1}{2}Ae^{j(2\pi f_o t+\theta)} + \tfrac{1}{2}Ae^{-j(2\pi f_o t+\theta)}$$

$$\hat{x}(t) = A\sin(2\pi f_o t + \theta) = -\tfrac{j}{2}Ae^{j(2\pi f_o t+\theta)} + \tfrac{j}{2}Ae^{-j(2\pi f_o t+\theta)}$$

$$\hat{X}(f) = -j\,\mathrm{sgn}(f) X(f)$$

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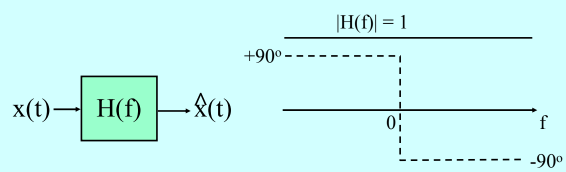
We recognise this as a filtering operation with a filter:

$$H(f) = -j\ \mathrm{sgn}(f).$$



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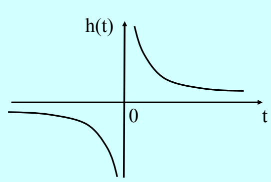
The impulse response of the Hilbert transformer is:

$$h(t) = \frac{1}{\pi t} \quad \Rightarrow \quad \hat{x}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x(\lambda)}{t - \lambda} d\lambda$$



The Hilbert transformer is ***non-causal***.
Bandlimiting removes the infinity at t = 0.

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2.6 Applications of the Analytic Signal

(i) Phasors

With a sinusoid $x(t) = A\cos(2\pi f_o t + \theta)$ the analytic signal is $x^+(t) = A\,e^{j(2\pi f_o t + \theta)}$. If we factor out the $e^{j2\pi f_o t}$ term, the remaining factor $Ae^{j\theta}$ is the ***phasor*** representing x(t).

An important aspect of the phasor is the reference frequency f_o (specified separately).

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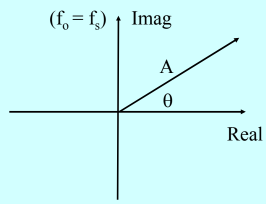
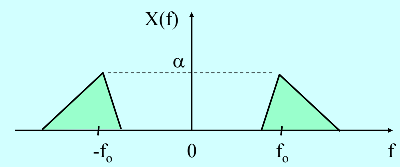
The reference frequency f_o is arbitrary, but for a sinewave A $\cos(2\pi f_s t+\theta)$ we usually choose it as the frequency of the sinewave, since then the phasor is a constant.

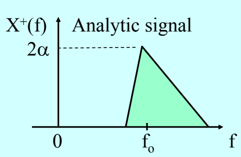


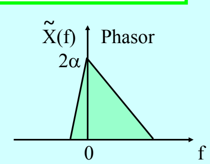
Phasors are of most use when we consider ***narrowband*** signals. These are signals which have their frequency components concentrated near some frequency f_o.



The phasor is denoted $\tilde{x}(t)$ and is simply a frequency down-shifted version of the analytic signal. It contains all the information about x(t) **except** the carrier frequency f_o.

$$\tilde{x}(t) = x_c(t) + jx_s(t) = x^+(t)e^{-j2\pi f_o t}$$





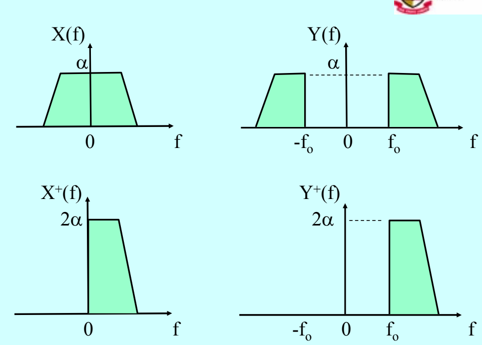
$$\tilde{x}(t) = x^+(t)e^{-j2\pi f_o t}$$
$$= x_c(t) + jx_s(t) = r(t)e^{j\theta(t)}$$
$$x(t) = \mathrm{Re}\left\{x^+(t)\right\} = \mathrm{Re}\left\{\tilde{x}(t)e^{j2\pi f_o t}\right\}$$
$$= x_c(t)\cos(2\pi f_o t) - x_s(t)\sin(2\pi f_o t)$$
$$= r(t)\cos\{2\pi f_o t + \theta(t)\}$$

- **Envelope** $r(t) = |\tilde{x}(t)|$
- **Relative phase** $\theta(t) = \arg \tilde{x}(t)$
- **Frequency deviation** $= d\theta(t)/dt$; rad/sec

(ii) Single Sideband Signals

A single sideband (SSB) signal is one in which the positive frequency components of a baseband signal x(t) are translated up by f_o and the negative frequency components down by f_o.

To determine an expression describing how the SSB signal y(t) is related to the baseband signal x(t), we first consider the relation between the respective analytic signals.



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$$Y^+(f) = X^+(f - f_o)$$

$$y^+(t) = x^+(t)e^{j2\pi f_o t}$$

$$y(t) + j\hat{y}(t) = \{x(t) + j\hat{x}(t)\}e^{j2\pi f_o t}$$

$$\boxed{y(t) = x(t)\cos(2\pi f_o t) - \hat{x}(t)\sin(2\pi f_o t)}$$

Exercise: For $x(t) = 4\,\text{sinc}(2t)$ and $f_o = 10$ Hz, calculate the spectrum of $\hat{x}(t)$ and hence that of $y(t)$ from the expression above. Sketch the spectra obtained in each case.

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Exercises: You are expected to attempt the following exercises in Proakis & Salehi. Completion of these exercises is part of the course. Solutions will be available later.

- **2.23**
- **2.24**
- **2.25**
- **2.28**
- **2.49**
- **2.55**
- **2.58 (f = 0 should be f > 0)**

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Section 3: Analog Modulation Systems

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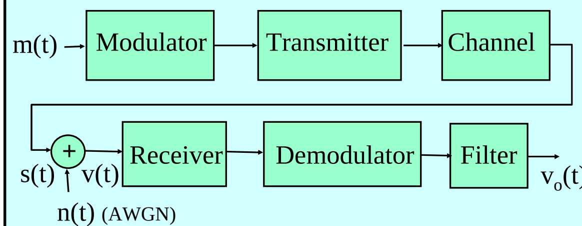
3. Analog Modulation Systems

The general structure of an analog communication system is shown below.



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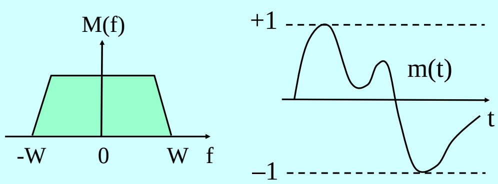
The message or baseband signal m(t) is assumed to be a bandlimited low pass signal of bandwidth W, scaled so that it lies in the range $-1 \le m(t) \le 1$.



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The modulator varies some parameter of a sinewave at frequency $f_o >> W$ in sympathy with the baseband signal m(t).

The most common parameters used are the ***amplitude*** or ***frequency***, although sometimes phase is used.

The signal which arrives at the receiver is s(t), accompanied by noise n(t) which we assume is additive white Gaussian noise (AWGN). We will consider the effects of noise later.

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3.1 Amplitude Modulation

In the absence of noise, an amplitude modulation (AM) system receives a signal s(t) given by:

$$s(t) = A[1 + a\, m(t)]\cos(2\pi f_o t)$$

where a is the ***modulation index*** $0 \le a \le 1$, and f_o is the carrier frequency. It should be noted that the term multiplying the carrier is always positive, so the carrier phase is always 0^o.

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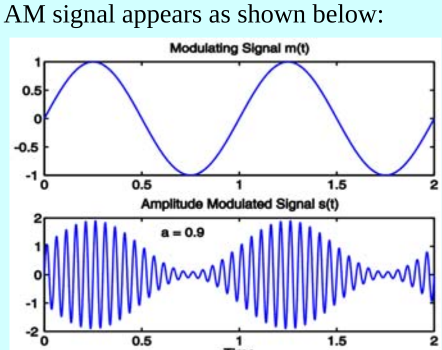
The AM signal appears as shown below:



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Now the AM signal is:

$$s(t) = A\{1 + am(t)\}\cos 2\pi f_o t$$
$$= A\cos 2\pi f_o t + Aam(t)\cos 2\pi f_o t$$

The first term is called the ***carrier component*** and it carries no message information. The second term consists of an ***upper and lower sideband***, and is the part that carries the message.

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For a modulation signal m(t) with a spectrum (Fourier transform) M(f), the spectrum of the AM signal s(t) is:

$$S(f) = \frac{A}{2}\delta(f - f_o) + \frac{A}{2}\delta(f + f_o) + \frac{Aa}{2}M(f - f_o) + \frac{Aa}{2}M(f + f_o)$$

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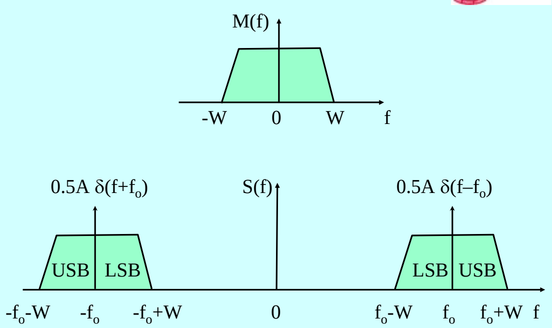

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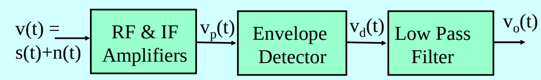
An AM receiver has the structure shown below:



The RF and IF amplifiers amplify the signal and have a bandwidth of $B \geq 2W$ in order to reduce the noise reaching the demodulator, but have negligible effect on the signal components. We will assume the filtering has no effect on s(t).

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We note that the bandwidth required is $B \geq 2W$, and that the signal consists of a carrier component of power $0.5A^2$, and an upper and lower sideband each of power $0.25A^2a^2\langle m^2(t)\rangle$ where

$$\langle m^2(t)\rangle = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} m^2(t)\,dt = \text{mean square value}$$

For sinewave modulation m(t) = cos(pt) we have $\langle m^2(t)\rangle = 0.5$, but for signals such as speech or music $\langle m^2(t)\rangle$ can be 0.1 or less so that peak clipping does not occur.

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For a signal with a Gaussian probability density function, $\langle m^2(t)\rangle = 0.1$ will give a clipping probability of 1.6×10^{-3} (can you derive that?).

The maximum efficiency in AM is obtained when a = 1, but in practice the modulation index is less than 1 to prevent nonlinear effects in transmitters and simple demodulators when the amplitude of the AM signal is close to zero. A value of 0.95 is a reasonable compromise.

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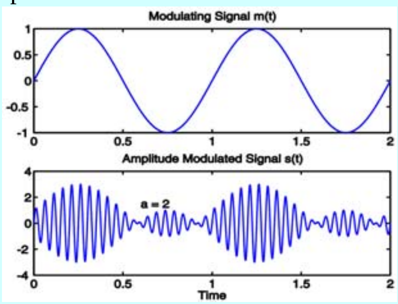
The diagram below shows that if a > 1, the envelope detector will not demodulate correctly.

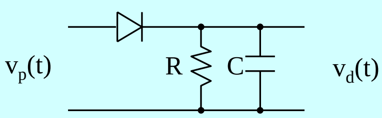


In analysing communication systems, in most cases the gain of the various amplifiers is not of interest, particularly if we are calculating signal to noise ratios (see later).

Hence for the purposes of analysis we will often omit these gain terms, but ***if the actual signal levels are of interest then they must be included***.

Also the carrier frequency f_o may be changed to an intermediate frequency, but as this has no effect on the modulation it too will be ignored.

The simplest demodulator is an ***envelope detector*** using a peak rectifier circuit:



$$v_p(t) = A\{1 + am(t)\}\cos 2\pi f_o t$$

$$v_d(t) = A + Aam(t)$$

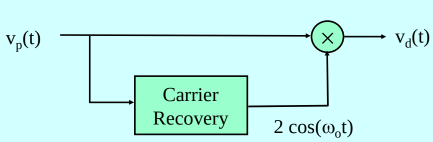
The RC time constant must be small compared to 1/W so that it can follow the modulation, but large compared with $1/f_o$ so that it peak rectifies.

Properties of an AM signal

- Carrier power = $0.5A^2$ (no modulation)
- Average power = $\langle s^2(t)\rangle = 0.5A^2(1+a^2\langle m^2(t)\rangle)$
- Sideband power = $0.5A^2a^2\langle m^2(t)\rangle$
- Bandwidth B = 2W
- Can use a simple demodulator

3.2 Synchronous Demodulation

Another method of demodulation is ***synchronous demodulation***. In this the pre-detection signal is multiplied by a locally generated carrier signal.



Suppose that in general we have an RF signal of the form:

$$v(t) = x(t)\cos(\omega_o t) - y(t)\sin(\omega_o t)$$

where x(t) and y(t) are slowly varying compared to the carrier frequency f_o. The signals x(t) and y(t) can be recovered by the process of ***synchronous demodulation***. This involves multiplying the signal v(t) by either $2\cos(\omega_o t)$ to recover x(t), or by $-2\sin(\omega_o t)$ to recover y(t), and then low pass filtering.

$$2v(t)\cos(\omega_o t) = 2x(t)\cos^2(\omega_o t) - 2y(t)\sin(\omega_o t)\cos(\omega_o t)$$
$$= \mathbf{x(t)} + x(t)\cos(2\omega_o t) - y(t)\sin(2\omega_o t)$$
$$-2v(t)\sin(\omega_o t) = -2x(t)\cos(\omega_o t)\sin(\omega_o t) + 2y(t)\sin^2(\omega_o t)$$
$$= \mathbf{y(t)} - y(t)\cos(2\omega_o t) - x(t)\sin(2\omega_o t)$$

Low pass filtering will remove the components multiplied by cos $(2\omega_o t)$ and sin $(2\omega_o t)$ (because these will have frequency components in the vicinity of $2f_o$) and the first one will therefore give x(t) and the second one y(t).

In the AM case we have (ignoring noise components at this stage):

$$v_p(t) = A\{1 + am(t)\}\cos(\omega_o t)$$

The synchronous demodulator will extract the components which multiply $\cos(\omega_o t)$.

$$v_d(t) = 2v_p(t)\cos(2\pi f_o t)$$
$$= A + A\,a\,m(t) + \text{high frequency terms}$$

The high frequency terms are at a frequency near $2f_o$ and are removed by the post-detection filter, and need concern us no further.

This detector is ***linear***, and produces a better result than the envelope detector when there are noise components present (see later).

It also correctly demodulates if a > 1. The envelope detector is non-linear for large noise, and produces distortion of the signal and also extra noise.

3.3 Double Sideband Suppressed Carrier

AM is inefficient due to the carrier component which carries no useful information and is only transmitted to simplify the demodulation process.

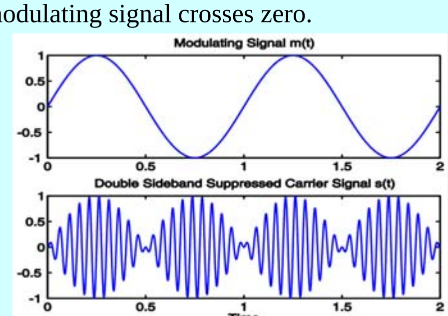
If we omit the carrier term, we have ***double sideband suppressed carrier*** (DSBSC).

$$s(t) = A\,m(t)\cos(2\pi f_o t)$$
$$v_p(t) = A\,m(t)\cos(2\pi f_o t)$$

The modulation index "a" now has no meaning and is omitted. To demodulate this signal we must use synchronous demodulation, because an envelope detector will not retrieve the modulation m(t). The bandwidth required is 2W.

$$B = 2W$$

In DSBSC, the carrier reverses in phase as the modulating signal crosses zero.



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Properties of a DSBSC signal

- Carrier power = 0 (no modulation)
- Average power = $\langle s^2(t)\rangle = 0.5A^2\langle m^2(t)\rangle$
- Sideband power = $0.5A^2\langle m^2(t)\rangle$
- Bandwidth B = 2W
- More efficient
- Must use a synchronous detector

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3.4 Single Sideband Suppressed Carrier

In DSBSC (and also in AM) the upper and lower sidebands are hermitian images of each other, so one of them may be omitted. This leads to ***single sideband suppressed carrier*** (SSBSC).

For an upper sideband system:

$$s(t) = A\{m(t)\cos(2\pi f_o t) - \hat{m}(t)\sin(2\pi f_o t)\}$$

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This SSBSC signal only has frequency components in the range f_o to f_o + W, so the RF and IF amplifiers now have a bandwidth B $\approx$ W rather than 2W.

$$B = W$$

This is important, since if the filters pass noise components at frequencies below f_o, then extra noise will appear at the demodulator output.

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Properties of a SSBSC signal

- Carrier power = 0 (no modulation)
- Average power = $A^2\langle m^2(t)\rangle$
- Sideband power = $A^2\langle m^2(t)\rangle$
- Bandwidth B = W
- Bandwidth efficient
- Must use a synchronous detector
- No good for DC or low frequencies
- No good for pulse signals

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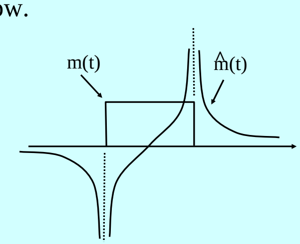
The reason it is not satisfactory for pulse signals is because of the nature of the Hilbert transform. For a rectangular pulse, the Hilbert transform is shown below.



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3.5 Vestigial Sideband Modulation

The video baseband signal used in TV has a bandwidth of 6 MHz. To conserve bandwidth, SSB should be used, but the video signal has significant low frequency content (average brightness) and has rectangular synchronising pulses.

The compromise is ***vestigial sideband*** modulation.

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In vestigial sideband the full upper sideband of bandwidth W_2 = 6 MHz is transmitted, but only W_1 = 1.25 MHz of the lower sideband is transmitted, along with a carrier. This effectively makes the system AM at low modulation frequencies and SSB at high modulation frequencies.

The absence of the lower sideband components at high frequencies must be compensated for, and this is done by the RF and IF filters.

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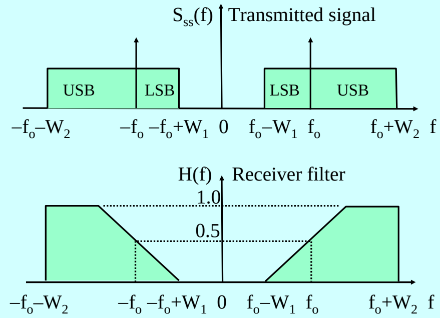

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Exercise: Determine an expression for the vestigial sideband signal before and after receiver filtering, and hence show that a synchronous demodulator gives the required baseband signal.

[Hint: Use the analytic signal and express $m(t) = m_1(t) + m_2(t)$, where $m_1(t)$ contains frequencies ±(0 to W_1) and $m_2(t)$ contains frequencies ±(W_1 to W_2), and note that the ramp part of the receiver filter is $H(f) = (f-f_o+W_1)/2W_1$ in the vicinity of $f = f_o$].

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3.6 Frequency Modulation

The modulated signal in a ***frequency modulation*** (FM) system has the form:

$$s(t) = A\cos\left(2\pi f_o t + 2\pi f_d \int_{-\infty}^{t} m(\lambda)\,d\lambda\right)$$

$$= A\cos\left(2\pi f_o t + \phi(t)\right)$$

$$\phi(t) = 2\pi f_d \int_{-\infty}^{t} m(\lambda)\,d\lambda$$

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The information is carried as the ***instantaneous frequency*** $f_i(t)$ of the signal.

$$f_i(t) = f_o + \frac{1}{2\pi}\frac{d\phi(t)}{dt} = f_o + f_d\, m(t)$$

Since $|m(t)| \le 1$, the parameter f_d is called the ***peak frequency deviation*** in Hz. For sinusoidal modulation $m(t) = \cos(2\pi f_m t)$ we have:

$$s(t) = A\cos\left(2\pi f_o t + \frac{f_d}{f_m}\sin(2\pi f_m t)\right)$$

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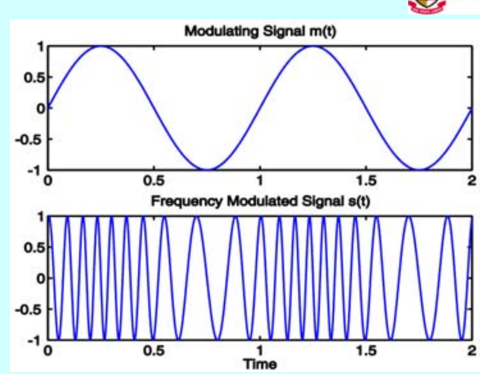

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We can expand s(t) as a Fourier series:

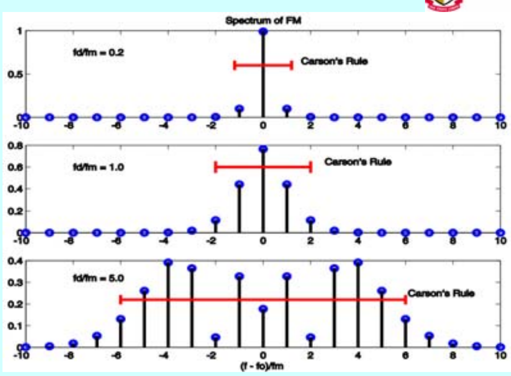
$$s(t) = A\sum_{n=-\infty}^{\infty} J_n\left(\frac{f_d}{f_m}\right)\cos\left(2\pi(f_o + nf_m)t\right)$$

where $J_n(x)$ is the Bessel function of the first kind. This tells us that the FM signal is not band-limited, but in practice the Bessel function is small for $n > f_d/f_m$. This means that if $f_d << f_m$, the bandwidth required is slightly greater than $2f_m$, whereas if $f_d >> f_m$ then the bandwidth required is slightly greater than $2f_d$.

A "rule of thumb" known as ***Carson's Rule*** gives the bandwidth required as $B = 2(f_d+f_m)$. This bandwidth contains >98% of the power for all values of f_d and f_m.

In a frequency modulation system with a frequency deviation f_d and a baseband bandwidth W, the bandwidth required by Carson's Rule is:

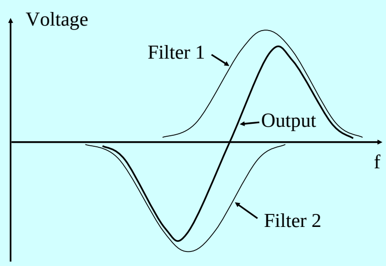
$$B = 2(f_d + W)$$



The ratio $\beta = f_d/W$ is called the ***modulation index***.

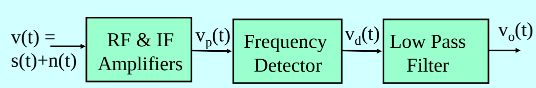
Demodulation of FM can be achieved using a ***frequency discriminator***. In its simplest form, this consists of two offset bandpass filters each with an envelope detector, and the output is taken as the difference of the two envelope detectors.

To prevent amplitude variations affecting the demodulator, it is usual to ***limit*** (clip) the signal before applying it to the discriminator.



Frequency Discriminator

The structure of an FM receiver is shown below.



$$s(t) = A\cos\left(2\pi f_o t + 2\pi f_d \int m(\tau)\,d\tau\right)$$

$$v_p(t) = A\cos\left(2\pi f_o t + 2\pi f_d \int m(\tau)\,d\tau\right)$$

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Properties of an FM signal

- Average power = $0.5A^2$
- Bandwidth $B = 2(f_d + W)$
- High output SNR (see later)
- Usually better quality than AM

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3.7 Radio and Television Broadcasting

1. AM Radio Broadcasting

• Frequency range	526.5 –1605.5 kHz
• Channel separation	9 kHz
• Modulation Bandwidth	9 kHz
• RF Bandwidth	18 kHz
• Intermediate frequency	455 kHz

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The superhetodyne receiver is the one most commonly used. This changes the received RF frequency to a fixed IF frequency of 455 kHz. The has the advantage that it is much easier to design the gain and bandwidth characteristics of a fixed frequency amplifier compared with a tunable amplifier.

Most stations broadcast a stereo signal which is achieved by modulating the L+R signal in the normal way and the L-R signal as a phase modulation of the carrier.

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2. FM Radio Broadcasting

• Frequency range	88–108 MHz
• Channel separation	100 kHz
• Modulation Bandwidth	15 kHz
• RF Bandwidth	180 kHz
• Intermediate frequency	10.7 MHz

Most stations broadcast a stereo signal in which the L-R signal is DSBSC modulated onto a 38 kHz subcarrier and added to the L+R signal before being applied to the frequency modulator.

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The L and R signals are pre-emphasised at the transmitter with a 50 μs time constant. This boosts the gain at frequencies above about 3 kHz and results in an improved noise performance. A complementary de-emphasis is done at the receiver.

Also added to the modulating signal is a 19 kHz sub-carrier which is doubled and used to demodulate the L-R signal.

For more details consult Proakis.

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3. Television Broadcasting

• Frequency range VHF	46.25–216.25 MHz
• Frequency range UHF	639.25-814.25 MHz
• Channel separation	7 MHz (min)
• Modulation Bandwidth	6 MHz
• RF Bandwidth	7 MHz
• Intermediate frequency	30-37 MHz
• Luminance signal (Y)	Vestigial sideband AM
• Chrominance (DSBSC)	4.43 MHz subcarrier
• Sound signal (FM 50kHz)	5.5 MHz subcarrier

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The ***chrominance signal*** is the B-Y signal DSBSC modulated onto a 4.43 MHz cosine carrier plus the R-Y signal DSBSC modulated onto a sine carrier. It is demodulated using a subcarrier reference signal which is transmitted during the horizontal blanking period. In the PAL system, the sine carrier is reversed in phase every line (PAL = "phase alternation line") as this helps reduce the effect of phase errors when demodulating the chrominance signal.

Exercises: You are expected to attempt the following exercises in Proakis & Salehi. Completion of these exercises is part of the course. Solutions will be available later.

3.3
3.8
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3.20

Section 4: Random Processes and Linear Systems

Contents

4. Random Processes & Linear Systems

Most of the signals in communication signals are random signals, since if they were deterministic (ie. known) there would be no point in transmitting them over a communication channel.

It is assumed that you have had an introduction to random processes previously, but the important relations will be revisited in this chapter.

A discrete random variable x_i will have probabilities $P\{x_i\}$. A continuous random variable x will have a probability density function p(x).

Discrete random variable

$$\sum_{i=1}^{n} P\{x_i\} = 1$$

$$E\{g(x)\} = \sum_{i=1}^{n} g\{x_i\} P\{x_i\}$$

Continuous random variable

$$\int_{-\infty}^{\infty} p(x)\,dx = 1$$

$$E\{g(x)\} = \int_{-\infty}^{\infty} g(x)\,p(x)\,dx$$

4.1 Correlation Functions

The ***autocorrelation function*** of a signal x(t), and the ***crosscorrelation function*** of x(t) and y(t) are:

$$R_{xx}(t_1, t_2) = E\left\{x(t_1)\,x^*(t_2)\right\}$$
$$R_{xy}(t_1, t_2) = E\left\{x(t_1)\,y^*(t_2)\right\}$$

E{.} is the ***expectation operator***, which means taking the ensemble average.

If x(t) and y(t) are ***stationary***, then these are functions only of $\tau = t_1 - t_2$.

$$R_{xx}(\tau) = E\left\{x(t)\,x^*(t-\tau)\right\}$$
$$R_{xy}(\tau) = E\left\{x(t)\,y^*(t-\tau)\right\}$$

If the signals are real, we can ignore the complex conjugate on the second term. If x(t) is in volts, the correlation function has dimensions of volt^2.

If the signals are ***ergodic***, we can also find the correlation functions by a ***time average***.

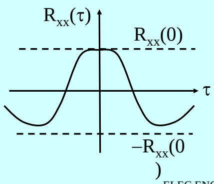
$$R_{xx}(\tau) = \left\langle x(t)\,x^*(t-\tau)\right\rangle$$
$$R_{xy}(\tau) = \left\langle x(t)\,y^*(t-\tau)\right\rangle$$

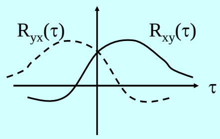
$$\left\langle f(t)\right\rangle = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} f(t)\,dt$$

Exercise: Find the time averages of $\cos(\omega t)$, $\cos^2(\omega t)$, $\cos(\omega t)\sin(\omega t)$, $\cos(\omega_1 t)\cos(\omega_2 t)$.

Properties of correlation functions:

- $R_{xx}(0) = E\{|x(t)|^2\} = \langle |x(t)|^2 \rangle = P$
- $|R_{xx}(\tau)| \le R_{xx}(0)$
- $R_{xx}(-\tau) = R^*_{xx}(\tau)$ ie. an even function if real
- $R_{yx}(\tau) = R^*_{xy}(-\tau)$





4.2 Power Spectral Density

With signals of ***finite energy*** (pulses) we have the energy spectral density $G_{xx}(f)= |X(f)|^2$. For signals of ***finite power*** (random signals, speech, noise) we have a ***power spectral density*** $S_{xx}(f)$.

Suppose x(t) is a finite power process. A time truncated version of this is:

$$x_T(t) = x(t)\,\mathrm{rect}(t/T)$$

The energy density of this signal is $|X_T(f)|^2$, so we can define the power spectral density as the limit as $T \to \infty$ of $|X_T(f)|^2/T$. With random signals, a consistent limit is not reached unless we form the ensemble average over all possible realisations.

$$S_{xx}(f) = \lim_{T\to\infty} \frac{E\left\{ |X_T(f)|^2 \right\}}{T}$$

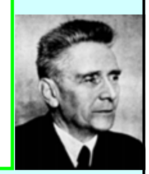
However, this is usually not a satisfactory way to compute the power spectral density.

Wiener-Khinchin Theorem

This theorem states that the power spectral density of an ergodic signal is the Fourier transform of its autocorrelation function.

$$S_{xx}(f) = \int_{-\infty}^{\infty} R_{xx}(\tau) e^{-j2\pi f\tau} d\tau$$

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} S_{xx}(f) e^{j2\pi f\tau} df$$



The power in the signal x(t) at frequencies in the range $f_1 \le f \le f_2$ is given by:

$$P_{12} = \int_{f_1}^{f_2} S_{xx}(f)\, df$$

Note that the power spectral density is always ***real*** and ***non-negative*** for all signals, real or complex. For real signals, it is also an ***even function*** of frequency.

We can also define the ***cross power spectral density*** as the Fourier transform of the cross correlation function.

$$S_{xy}(f) = \int_{-\infty}^{\infty} R_{xy}(\tau) e^{-j2\pi f\tau} d\tau$$

$$R_{xy}(\tau) = \int_{-\infty}^{\infty} S_{xy}(f) e^{j2\pi f\tau} df$$

$$S_{yx}(f) = S^*_{xy}(f)$$

It can be shown that:

$$|S_{xy}(f)|^2 \le S_{xx}(f)S_{yy}(f)$$

This implies that if x(t) and y(t) have no common frequency components (ie. $S_{xx}(f)S_{yy}(f) = 0$), then x(t) and y(t) are uncorrelated.

Exercise: By considering $z(t) = a\,x(t) + b\,y(t)$, (a,b complex), and using $S_{zz}(f) \ge 0$, prove the above relation.

4.3 Cyclostationary Processes

Many of the processes in communication systems are not strictly stationary, but are ***cyclostationary***. This means the underlying process has a periodic structure, and as a result statistics such as the ***mean*** and ***correlation function*** are periodic.

Hence when we form $E\{x(t)\,x^*(t-\tau)\}$ we find the result is a function of both t and τ but is periodic in t. If we average over t, then we get a result which only depends on τ.

Example: The simplest cyclostationary process is a sinewave.

$$x(t) = A\cos(\omega_o t + \theta)$$

$$E\{x(t)x(t-\tau)\} = A^2\cos(\omega_o t + \theta)\cos(\omega_o t - \omega_o\tau + \theta)$$

$$= \tfrac{1}{2}A^2\cos(\omega_o\tau) + \tfrac{1}{2}A^2\cos(2\omega_o t - \omega_o\tau + 2\theta)$$

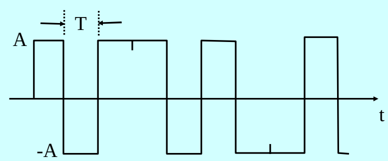
We see that the second term is periodic in t and has an average value of zero.

$$\overline{R}_{xx}(\tau) = \tfrac{1}{2}A^2\cos(\omega_o\tau)$$

$$S_{xx}(f) = \tfrac{1}{4}A^2\delta(f - f_o) + \tfrac{1}{4}A^2\delta(f + f_o)$$

Example: Random Binary Waveform

This consists of rectangular pulses of duration T and amplitude ±A with equal probability and uncorrelated with each other.



We will consider the general case where the pulse shape is p(t), so we can write:

$$x(t) = \sum_{k=-\infty}^{\infty} A\,a_k\,p(t-kT)$$

where p(t) is the pulse shape and $a_k = \pm 1$ is the digital data. For the previous slide we have p(t) = rect(t/T). With the a_k equally likely and uncorrelated, we have $E\{a_k a_r\} = 1$ if k = r and zero otherwise.

The autocorrelation function of x(t) is:

$$R_{xx}(t, t-\tau) = E\{x(t)x(t-\tau)\}$$

$$= \sum_{k=-\infty}^{\infty}\sum_{r=-\infty}^{\infty} A^2 E\{a_k a_r\}p(t-kT)p(t-\tau-rT)$$

$$= A^2\sum_{k=-\infty}^{\infty} p(t-kT)p(t-\tau-kT)$$

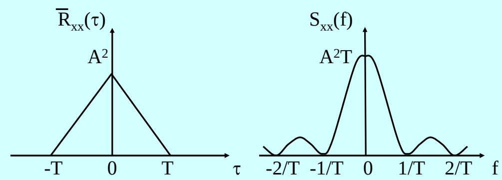
We note that this is periodic in t, so this is a ***cyclo-stationary process***. Hence we must first average the correlation function over one period.

$$\overline{R}_{xx}(\tau) = \frac{A^2}{T}\sum_{k=-\infty}^{\infty}\int_{-T/2}^{T/2} p(t-kT)\,p(t-\tau-kT)\,dt$$

$$= \frac{A^2}{T}\int_{-\infty}^{\infty} p(t)\,p(t-\tau)\,dt = \frac{A^2}{T}\,p(t)\otimes p(-t)$$

$$S_{xx}(f) = \frac{A^2}{T}\left|P(f)\right|^2$$

For a rectangular pulse we have $\overline{R}_{xx}(\tau) = A^2\,\Delta(\tau/T)$ and hence $S_{xx}(f) = A^2T\,\mathrm{sinc}^2(fT)$. The bandwidth required is therefore approximately W = 1/T Hz.



4.4 Dimensions of Power

The dimensions of the autocorrelation function of x(t) is V^2 if x(t) is a voltage and A^2 if it is a current. It is common practice in communication theory to define the ***"power"*** of a signal x(t) as the average value of $x^2(t)$, which is of course actually the ***mean square value***.

The ***power spectral density*** of x(t) then has the dimensions V^2/Hz or A^2/Hz.

If the power spectral density is expressed in terms of W/Hz, then if x(t) is the voltage or current in a resistance R, we must multiply by R to get the power spectral density in V^2/Hz, or divide by R to get it in A^2/Hz.

Alternatively, the power spectral density in V^2/Hz or A^2/Hz is sometimes called the power spectral density of the signal in 1 ohm.

When we calculate power ratios, the resistance R cancels out, so it is not usually of interest.

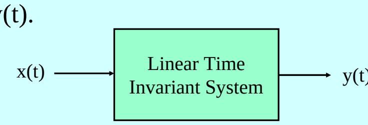
4.5 Linear Time Invariant Systems

A linear time invariant (LTIV) system can be described either by its ***impulse response*** h(t) or its ***frequency response*** H(f) and these are a Fourier transform pair.

$$H(f) = \int_{-\infty}^{\infty} h(t)\,e^{-j2\pi f t}\,dt$$

$$h(t) = \int_{-\infty}^{\infty} H(f)\,e^{j2\pi f t}\,df$$

Consider a LTIV system with an input x(t) and an output y(t).



$$Y(f) = H(f)X(f)$$

$$y(t) = h(t)\otimes x(t) = \int_{-\infty}^{\infty} h(\lambda)\,x(t-\lambda)\,d\lambda$$

Exercise: If $y(t) = \int_{-\infty}^{t}\left[x(t') - x(t'-T)\right]dt'$ find h(t) & H(f).

The time domain convolution is valid for all signals, but the frequency domain relation is only true if the Fourier transforms exist, which they may not (eg. random signals usually do not have Fourier transforms). However for random signals we have:

$$S_{xy}(f) = S_{xx}(f)H^*(f)$$
$$S_{yx}(f) = S_{xx}(f)H(f)$$
$$S_{yy}(f) = S_{xx}(f)|H(f)|^2$$

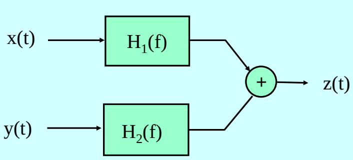
To calculate power spectral density relations.

1. Y(f) = X(f) H(f) for finite energy signals.
2. $|Y(f)|^2 = X(f)H(f)X^*(f)H^*(f) = |X(f)|^2|H(f)|^2$
   $X(f)Y^*(f) = X(f)X^*(f)H^*(f) = |X(f)|^2H^*(f)$
   $Y(f)X^*(f) = X(f)H(f)X^*(f) = |X(f)|^2H(f)$
3. Replace $|X(f)|^2$ by $S_{xx}(f)$, $|Y(f)|^2$ by $S_{yy}(f)$
   $X(f)Y^*(f)$ by $S_{xy}(f)$, $Y(f)X^*(f)$ by $S_{yx}(f)$

Power spectral densities satisfy the same relations as ***energy spectral densities***.

The results for more complicated situations can be derived in a similar way.

Exercise: Calculate $S_{zz}(f)$.



Answer:

$$S_{zz}(f) = S_{xx}(f)|H_1(f)|^2 + 2\,\mathrm{Re}\{S_{xy}(f)H_1(f)H_2^*(f)\} + S_{yy}(f)|H_2(f)|^2$$

You should verify this result. Note that this also shows that if signals are ***uncorrelated***, we can add their power spectral densities. Unless stated otherwise, ***power spectral densities are always two-sided*** (ie. includes negative frequencies).

4.6 Gaussian Noise

Many of the random signals we will consider, and in particular noise, will have a Gaussian probability density function.

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\eta)^2/2\sigma^2}$$

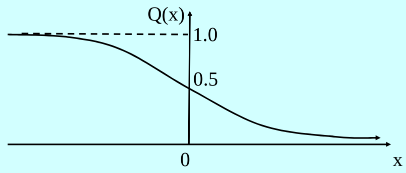
where $\eta = E\{x\}$ is the mean value and $\sigma^2 = E\{(x-\eta)^2\}$ is the variance.

In digital systems we will be interested in the probability that x exceeds some value x_o, and this is given by:

$$P\{x > x_o\} = \frac{1}{\sigma\sqrt{2\pi}} \int_{x_o}^{\infty} e^{-(x-\eta)^2/2\sigma^2} dx$$
$$= \frac{1}{\sqrt{2\pi}} \int_{(x_o-\eta)/\sigma}^{\infty} e^{-t^2/2} dt \quad ; t = \frac{x-\eta}{\sigma}$$
$$= Q\left\{\frac{x_o-\eta}{\sigma}\right\}$$

Q(x) is the Gaussian error function:

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-t^2/2}\, dt$$



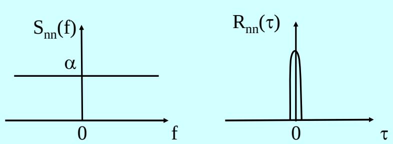
A table of the Q function is provided, but some special values are listed below.

$Q(0) = 0.500$
$Q(1.645) = 0.050$
$Q(1.960) = 0.025$

x	1.282	2.326	3.090	3.719	4.265	4.753	5.199
Q(x)	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}

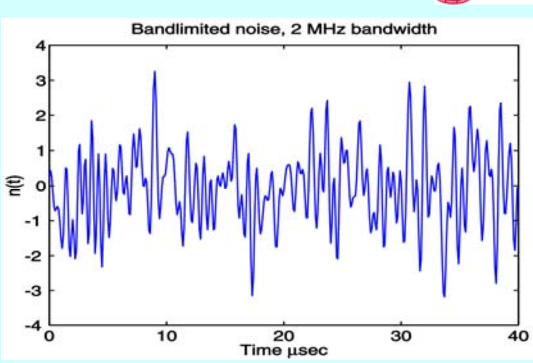
4.7 White Noise

A signal which has $S_{nn}(f) = \alpha = N_o/2$ (a constant) for all frequencies is called ***white noise***. Its autocorrelation function is $R_{nn}(\tau) = \alpha\, \delta(\tau)$.



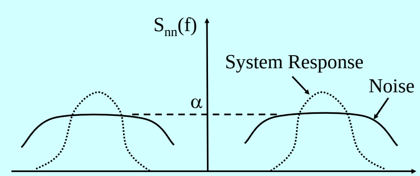
It is usual notation to use $S_{nn}(f) = N_o/2$, where N_o is the ***single-sided power spectral density*** of the noise. In these notes I will often use α instead of $N_o/2$.

True white noise is an idealisation since it has infinite bandwidth and hence infinite power. In practice we encounter ***bandlimited white noise***, but if its bandwidth is greater than that of the system to which it is applied, we can assume it to be white noise without error.



Bandlimited white noise

If the noise spectral density is constant over the range of frequencies to which the system responds, then we get the same result if we assumed it was constant at all frequencies.



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4.8 Noise Bandwidth

If white noise is applied to a filter H(f), the mean square output noise is:

$$\langle n_o^2(t)\rangle = \int_{-\infty}^{\infty} \alpha\,|H(f)|^2\,df$$
$$= 2\alpha \int_0^{\infty} |H(f)|^2\,df$$
$$= 2\alpha\,|H_o|^2 B_n$$

$$B_n = \frac{1}{|H_o|^2}\int_0^{\infty} |H(f)|^2\,df$$

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B_n is called the ***noise bandwidth*** in Hz, and is the bandwidth of the rectangular filter which has the same mean square noise at its output.

Note that ***bandwidth is a positive frequency concept*** (and does not include negative frequencies), so the integration is from 0 to ∞.

H_o is the maximum passband gain.

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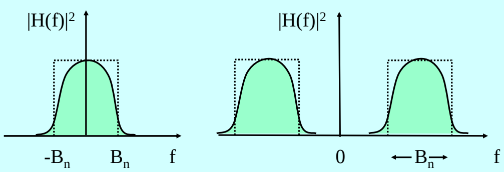
Noise bandwidth of low pass and band pass filters. Note that bandwidth is always measured at positive frequencies.



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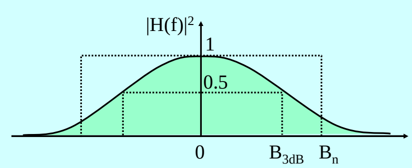
Exercise: Show that for a low pass RC filter:

$$H(f) = \frac{1}{1 + j2\pi f\,RC}$$

$$B_n = \frac{1}{4RC} = \frac{\pi}{2}B_{3dB}$$



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4.9 Narrowband Noise

This looks like a sinewave with random varying amplitude and phase.

$$n(t) = n_c(t)\cos(2\pi f_o t) - n_s(t)\sin(2\pi f_o t)$$
$$= r(t)\cos[2\pi f_o t + \theta(t)]$$

The envelope r(t) and phase θ(t) [and hence the in-phase and quadrature components $n_c(t)$ and $n_s(t)$] vary at a rate comparable to the **bandwidth**.

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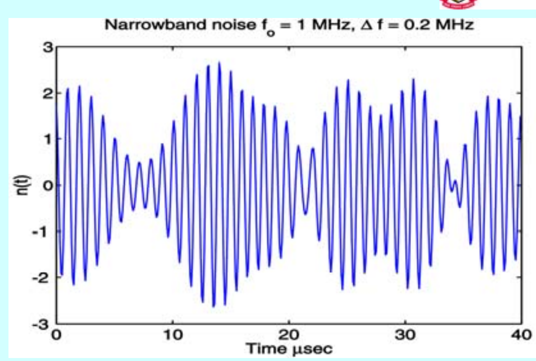

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The **analytic signal** for narrowband noise is

$$n^{+}(t) = [n_c(t) + j n_s(t)] e^{j2\pi f_o t}$$

and note that $n(t) = \mathrm{Re}\{n^{+}(t)\}$. The **noise phasor** is

$$\tilde{n}(t) = n_c(t) + j n_s(t)$$

When we demodulate signals, it will be $n_c(t)$ or $n_s(t)$ which will be of interest, so we need to find their power spectral densities.

Since $n^{+}(t)$ is obtained from $n(t)$ by filtering with $H(f) = 2u(f)$, the power spectrum of $n^{+}(t)$ is:

$$S_{n^{+}n^{+}}(f) = 4u(f) S_{nn}(f) = 4 S_{nn}^{(+)}(f)$$

(Don't confuse $S_{nn}^{(+)}(f)$ with the analytic signal). The complex conjugate of $n^{+}(t)$ is denoted $n^{-}(t)$ and has only negative frequencies, so its power spectrum is:

$$S_{n^{-}n^{-}}(f) = 4u(-f) S_{nn}(f) = 4 S_{nn}^{(-)}(f)$$

Also $n^{+}(t)$ and $n^{-}(t)$ are uncorrelated, because they have no common frequency components.

Now $\tilde{n}(t) = n_c(t) + j n_s(t) = n^{+}(t) e^{-j2\pi f_o t}$ so we have:

$$n_c(t) = \mathrm{Re}\left\{ n^{+}(t) e^{-j2\pi f_o t} \right\}$$

$$= \frac{1}{2}\left\{ n^{+}(t) e^{-j2\pi f_o t} + n^{-}(t) e^{j2\pi f_o t} \right\}$$

Similarly :

$$n_s(t) = \mathrm{Im}\left\{ n^{+}(t) e^{-j2\pi f_o t} \right\}$$

$$= \frac{1}{2j}\left\{ n^{+}(t) e^{-j2\pi f_o t} - n^{-}(t) e^{j2\pi f_o t} \right\}$$

Since $n^{+}(t)$ and $n^{-}(t)$ are uncorrelated, we can add their power spectral densities:

$$S_{n_c n_c}(f) = \frac{1}{4}\left\{ S_{n^{+}n^{+}}(f + f_o) + S_{n^{-}n^{-}}(f - f_o) \right\}$$

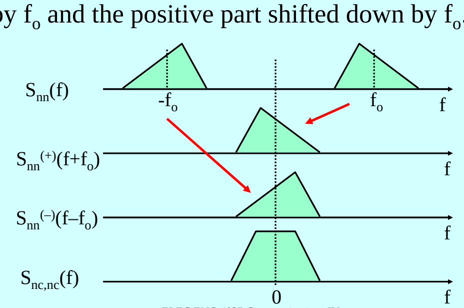
$$S_{n_c n_c}(f) = S_{nn}^{(+)}(f + f_o) + S_{nn}^{(-)}(f - f_o)$$

$$S_{n_s n_s}(f) = S_{nn}^{(+)}(f + f_o) + S_{nn}^{(-)}(f - f_o)$$

$$S_{n_c n_s}(f) = j\left\{ S_{nn}^{(+)}(f + f_o) - S_{nn}^{(-)}(f - f_o) \right\}$$

Exercise: Prove the last relation.

While this looks messy, it is simply the sum of the negative frequency part of $S_{nn}(f)$ shifted up by f_o and the positive part shifted down by f_o.



If the power spectrum $S_{nn}(f)$ is symmetrical about f_o, then the cross power spectrum of $n_c(t)$ and $n_s(t)$ disappears. It is usually not of interest anyway.

If the $S_{nn}(f) = N_o/2$, then the power spectral densities of $n_c(t)$ and $n_s(t)$ are both N_o (at all frequencies of interest $|f| < f_o$). This will be important when we deal with the effect of noise on modulated signals.

$$S_{nn}(f) = \frac{N_o}{2}\left\{ \mathrm{rect}\left(\frac{f - f_o}{B}\right) + \mathrm{rect}\left(\frac{f + f_o}{B}\right)\right\}$$

$$S_{n_c n_c}(f) = S_{n_s n_s}(f) = N_o \,\mathrm{rect}\left(\frac{f}{B}\right)$$

Note that the RF noise n(t) has a power spectral density $N_o/2$ and bandwidth B, but $n_c(t)$ has a power spectral density N_o [which is double that of n(t)], and a bandwidth B/2 [which is half that of n(t)] and similarly for $n_s(t)$.

Exercises: You are expected to attempt the following exercises in Proakis & Salehi. Completion of these exercises is part of the course. Solutions will be available later.

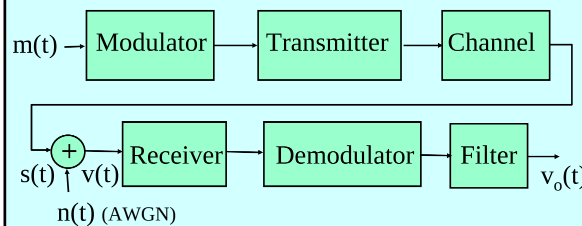
4.10
4.44 (Part 4 is $\sigma_x^2 \neq \sigma_y^2$)
4.48
4.50
4.56

Section 5: Effect of Noise on Analog Systems

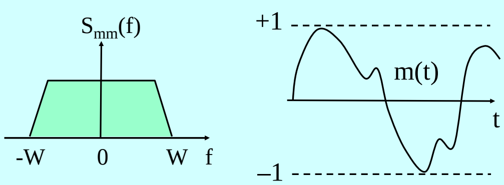
Contents

5. Effect of Noise on Analog Systems

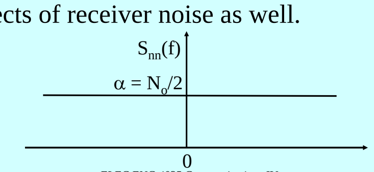
An analog communication system is subject to additive noise as shown below.



In practice the message signal m(t) will be a low pass ***random signal*** of bandwidth W, so we need to consider its power spectral density $S_{mm}(f)$. As before, we require $-1 \leq m(t) \leq 1$.



The noise n(t) will be assumed to be **additive white Gaussian noise** (AWGN). Its spectral density is $S_{nn}(f) = \alpha = N_o/2$. Note that although N_o is the 'single sided' noise spectral density, in all our working we will use double sided power spectral densities. We will assume n(t) includes the effects of receiver noise as well.



Signal to noise ratio

In assessing the performance of analog communication systems, we will be concerned with ***signal to noise ratio*** (SNR). Since amplifier gains do not affect SNR, they will usually be ignored. However if the actual signal levels are of interest, then the amplifier gains must be included.

Note that the SNR is often expressed in **decibels (dB) = 10 $\log_{10}$(power ratio)** and **(power ratio) = $10^{(0.1*dB)}$**. Do not use dB in any formulae.

We will compare the performance of various systems on the basis of the **output SNR** obtained compared with that of a baseband system for the same ***average received signal power*** and the same ***noise power spectral density***. (In practice it might be better to compare on the basis of ***peak power***, since this is the limiting factor in transmitter design).

Of course, in many situations a baseband system is not a viable alternative (eg. radio broadcasting), but it serves as a useful comparison basis.

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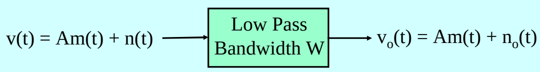
5.1 Baseband Transmission

No modulator is used, so the receiver simply consists of a low pass filter of bandwidth **W** to remove extraneous noise components, while not affecting the signal $s(t) = Am(t)$.



We will assume the low pass filter is ideal (which is not true in practice), but we can approximate this very closely.

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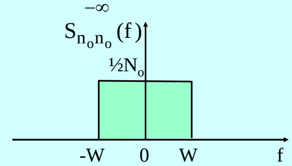
The (average) received signal power is $P_r = A^2\langle m^2 \rangle$. The output noise $n_o(t)$ has a power spectral density as shown below, and the output noise power is:

$$\left\langle n_o^2(t) \right\rangle = \int_{-\infty}^{\infty} S_{n_o n_o}(f)\,df = N_o W$$



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Hence for a baseband system we have:

$$v(t) = s(t) + n(t) = Am(t) + n(t)$$

$$v_o(t) = s_o(t) + n_o(t) = Am(t) + n_o(t)$$

$$SNR_o = \frac{\left\langle s_o^2(t) \right\rangle}{\left\langle n_o^2(t) \right\rangle}$$

$$SNR_o = \frac{A^2\left\langle m^2(t) \right\rangle}{N_o W} = \frac{P_r}{N_o W}$$

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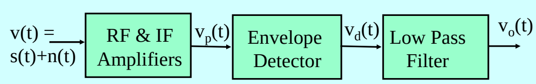
5.2 Amplitude Modulation

In an amplitude modulation (AM) system the received signal $v(t) = s(t) + n(t)$ is given by:

$$v(t) = A[1 + a\,m(t)]\cos(2\pi f_o t) + n(t)$$

where A is the carrier amplitude and 'a' is the ***modulation index*** $0 \le a \le 1$.



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The RF and IF amplifiers amplify the signal and have a bandwidth of $B \ge 2W$. It will be assumed that they have no effect on the signal components, but bandlimit the noise reaching the demodulator.

Just prior to the demodulator we have the ***pre-detection signal*** $v_p(t)$ given by (ignoring gains):

$$v_p(t) = A[1 + am(t)]\cos(2\pi f_o t) + n_p(t)$$

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where $n_p(t)$ is ***bandlimited white noise***. To draw a phasor diagram, we need to express this in phasor form.

$$n_p(t) = n_c(t)\cos(2\pi f_o t) - n_s(t)\sin(2\pi f_o t)$$
$$= \mathrm{Re}\left\{ \left(n_c(t) + jn_s(t)\right) e^{j2\pi f_o t} \right\}$$

$$S_{n_c n_c}(f) = S_{n_s n_s}(f)$$
$$= S_{n_p n_p}(f + f_o) + S_{n_p n_p}(f - f_o)$$
$$= N_o \quad ; |f| < B/2$$

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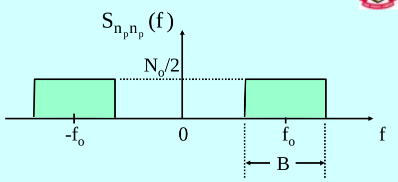
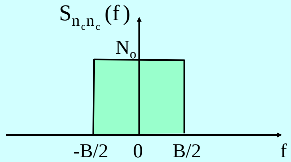
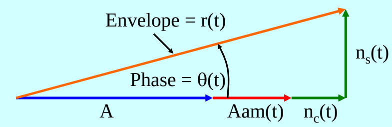




The phasor diagram of the pre-detection signal $v_p(t)$ is:



Note that the noise causes phase modulation, and the envelope is not linearly related to the message signal.

The ***envelope detector*** measures r(t).

$$\begin{aligned} v_d(t) &= r(t) \\ &= \sqrt{(A + Aa\,m(t) + n_c(t))^2 + n_s^2(t)} \\ &\approx A + Aa\,m(t) + n_c(t) \end{aligned}$$

The approximation is valid if $n_c(t)$ and $n_s(t)$ are small compared to A. We ignore the DC component and pass the AC components to the post-detection low pass filter.

The output of the post-detection filter (which we assume is an ideal low pass filter of bandwidth W except that it does not pass DC) is:

$$v_o(t) = s_o(t) + n_o(t)$$

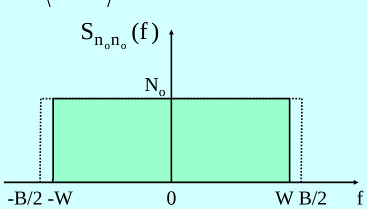
$$\text{where } s_o(t) = Aa\,m(t)$$

The DC component A is often used for automatic gain control (AGC) of the RF and IF amplifiers, so that the voltage presented to the demodulator is relatively independent of the received signal level.

The output signal and noise powers are:

$$\langle s_o^2(t)\rangle = A^2a^2\langle m^2(t)\rangle$$

$$\langle n_o^2(t)\rangle = 2N_oW$$



Hence the output signal to noise ratio for AM is:

$$SNR_{am} = \frac{\langle s_o^2(t)\rangle}{\langle n_o^2(t)\rangle} = \frac{A^2a^2\langle m^2\rangle}{2N_oW}$$

For comparison purposes, we need to express this in terms of the **average received power P_r.**

$$\begin{aligned} P_r &= \langle s^2(t)\rangle = \langle [A(1+am(t))\cos(\omega_o t)]^2\rangle \\ &= \frac{1}{2}A^2\left(1 + a^2\langle m^2(t)\rangle\right) \end{aligned}$$

Hence we obtain:

$$SNR_{am} = \frac{A^2 a^2 \langle m^2 \rangle}{2N_o W} = \left(\frac{a^2 \langle m^2 \rangle}{1 + a^2 \langle m^2 \rangle} \right) \left(\frac{P_r}{N_o W} \right)$$

where $P_r = \frac{A^2}{2}\left(1 + a^2 \langle m^2 \rangle\right)$

and we note that P_r/N_oW is the output SNR for a baseband system. AM does not perform as well in comparison.

Comparing AM with a baseband system on the basis of average power, its performance is rather poor, since the factor multiplying P_r/N_oW can be quite small. While for a sinewave with 100% modulation this factor is 0.3333, for $\langle m^2 \rangle = 0.1$ we have 0.0909.

Exercise: Do the comparison of these systems on the basis of peak power. (Answer: SNR_{am} is 0.25 of that for a baseband system for 100% sinewave modulation and the same peak power).

The ***pre-detection signal to noise ratio*** is given by:

$$SNR_p = \frac{P_r}{\langle n_p^2 \rangle} = \frac{P_r}{N_o B} = \frac{A^2(1 + a^2 < m^2 >)}{2N_o B}$$

The envelope detector linear approximation gives the correct result provided this is greater than about 5 dB.

A synchronous demodulator will extract the message signal **linearly** even with large noise. The SNR calculations are the same as before, except they are valid even for small values of SNR_p.

$$v_p(t) = A\{1 + am(t)\}\cos(\omega_o t) + n_c(t)\cos(\omega_o t) - n_s(t)\sin(\omega_o t)$$

$$v_d(t) = 2 v_p(t)\cos(\omega_o t) = A + A\,a\,m(t) + n_c(t) + \text{high frequency terms}$$

The high frequency terms are at a frequency near $2f_o$ and are removed by the post-detection filter, and need concern us no further.

This detector is ***linear***, and produces a better result than the envelope detector if $n_c(t)$ and $n_s(t)$ are large, and also correctly demodulates if $a > 1$ (as in colour TV for instance). The envelope detector is non-linear for large noise, and produces distortion of the signal and also extra noise.

5.3 Double Sideband Suppressed Carrier

If we omit the carrier component of AM, we obtain the DSBSC signal below. The modulation index “a” now has no meaning and is omitted. To demodulate this signal we must use synchronous demodulation, because an envelope detector will not retrieve the modulation m(t). The bandwidth required is the same as for AM, viz. $\mathbf{B \geq 2W}$.

$$v(t) = A\,m(t)\cos(2\pi f_o\, t) + n(t)$$

$$v_p(t) = A\,m(t)\cos(2\pi f_o\, t) + n_p(t)$$

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$$v_d(t) = A\,m(t) + n_c(t)$$
$$v_o(t) = A\,m(t) + n_o(t)$$
$$P_r = \tfrac{1}{2} A^2 \left\langle m^2(t) \right\rangle$$
$$SNR_{dsbsc} = \frac{A^2 \left\langle m^2 \right\rangle}{2 N_o W} = \frac{P_r}{N_o W}$$

Hence this gives the same SNR as a baseband system for the same received power (and noise spectral density).

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The ***pre-detection signal to noise ratio*** is given by:

$$SNR_p = \frac{P_r}{\left\langle n_p^2 \right\rangle} = \frac{P_r}{N_o B} = \frac{A^2 < m^2 >}{2 N_o B}$$

The synchronous detector is linear and the output SNR result is correct even if $SNR_p < 0$ dB.

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5.4 Single Sideband Suppressed Carrier

For an upper sideband system:

$$v(t) = A\left\{ m(t)\cos(2\pi f_o t) - \hat{m}(t)\sin(2\pi f_o t) \right\} + n(t)$$
$$v_p(t) = A\left\{ m(t)\cos(2\pi f_o t) - \hat{m}(t)\sin(2\pi f_o t) \right\} + n_p(t)$$

The RF and IF filters only pass frequencies from f_o to $f_o + W$, so the bandwidth $\mathbf{B \geq W}$. Note that as a consequence, the power spectrum of $n_p(t)$ is **not** the same as for AM and DSBSC.

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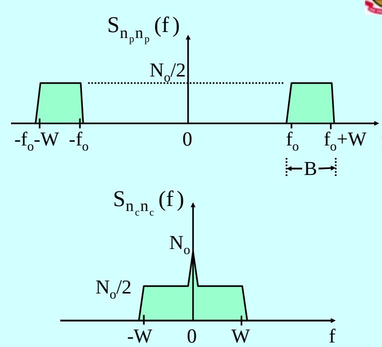

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Note that $n_c(t)$ has a power spectral density of $N_o/2$. Synchronous demodulation gives:

$$v_d(t) = A\,m(t) + n_c(t)$$
$$v_o(t) = A\,m(t) + n_o(t)$$
$$P_r = \tfrac{1}{2} A^2 \left\langle m^2(t) \right\rangle + \tfrac{1}{2} A^2 \left\langle \hat{m}^2(t) \right\rangle$$
$$= A^2 \left\langle m^2(t) \right\rangle$$
$$SNR_{ssbsc} = \frac{A^2 \left\langle m^2 \right\rangle}{N_o W} = \frac{P_r}{N_o W}$$

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Hence, SSBSC has the same SNR performance as DSBSC (and a baseband system), the advantage being that it only requires half the bandwidth of AM or DSBSC (ie. B = W).

The ***pre-detection signal to noise ratio*** is given by:

$$SNR_p = \frac{P_r}{\left\langle n_p^2 \right\rangle} = \frac{P_r}{N_o B} = \frac{A^2 < m^2 >}{N_o B}$$

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5.5 Carrier Phase Estimation

To use a synchronous detector in DSBSC, it is necessary to generate a local carrier signal. However, the signal does not contain any component at the carrier frequency, so we have to resort to non-linear processing.

We can generate a double frequency component by squaring the received signal. To reduce the noise components we do this after RF and IF filtering.

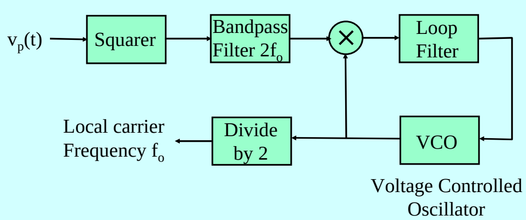
$$v_p(t) = A\,m(t)\cos(2\pi f_o t) + n_p(t)$$

$$v_p^2(t) = A^2 m^2(t)\cos^2(2\pi f_o t) + \text{noise terms}$$

$$= \frac{A^2}{2}m^2(t) + \frac{A^2}{2}m^2(t)\cos(4\pi f_o t) + \text{noise terms}$$

The second term is the one of interest. Because $m^2(t) > 0$, this term contains a component at double the carrier frequency (actually it will be double the IF frequency at this point). We can filter this signal to obtain a double carrier frequency signal and then divide this by two.

A possible carrier recovery system is shown below. The prefilter removes the extraneous components from the squarer, and the ***phase locked loop*** extracts the double frequency carrier.



If the component of interest from the squarer is $v_1(t) = C_1\cos(2\omega_o t)$ and the VCO signal is $v_2(t) = C_2\sin(2\omega_o t+\phi)$, then the output of the multiplier is:

$$v_3(t) = v_1(t)\,v_2(t)$$

$$= C_1C_2\cos(2\omega_o t)\sin(2\omega_o t + \phi)$$

$$= \frac{1}{2}C_1C_2\sin(\phi) + \frac{1}{2}C_1C_2\sin(4\omega_o t + \phi)$$

The $4\omega_o$ term is rejected by the loop filter and if ϕ is small the first term is approximately $\frac{1}{2}C_1C_2\,\phi$ and the phase locked loop will force $\phi = 0$.

The closed loop response of the phase locked loop is designed to have a narrow bandwidth to provide a jitter free carrier reference.

The sign ambiguity when we divide the frequency by two does not matter since for a synchronous detector either $\pm\cos(\omega_o t)$ can be used.

For details on the phase locked loop design, see Proakis.

5.6 Frequency Modulation

The received signal in a ***frequency modulation*** (FM) system has the form:

$$v(t) = A\cos\left(2\pi f_o\,t + 2\pi f_d\int m(\tau)\,d\tau\right) + n(t)$$

$$v_p(t) = A\cos\left(2\pi(f_o\,t + 2\pi f_d\int m(\tau)\,d\tau\right) + n_p(t)$$

In this case the RF and IF amplifiers have a bandwidth of approximately $B = 2(f_d + W)$, which is usually >> 2W.

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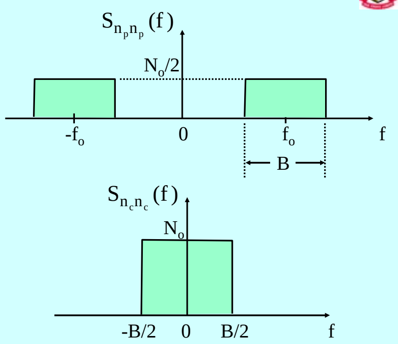

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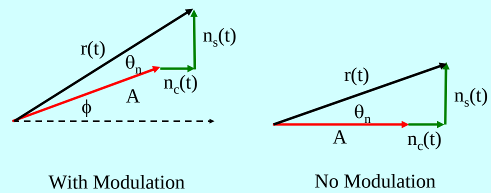
The phasor diagram of the predetection signal is as shown below, with $\phi(t) = 2\pi f_d \int m(\tau)\,d\tau$.



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The frequency detector produces a voltage proportional to the instantaneous frequency deviation from f_o.

$$v_d(t) = \frac{1}{2\pi}\frac{d}{dt}\left(\phi(t) + \theta_n(t)\right)$$

$$= f_d\, m(t) + \frac{1}{2\pi}\frac{d\theta_n(t)}{dt}$$

Proakis analyses the noise in FM with modulation present, which is somewhat complicated.

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We will compute θ_n for the case where:

(1) $n_c(t)$ and $n_s(t)$ are both << A
(2) there is no modulation.

The first assumption requires that the **pre-detection** signal to noise ratio be greater than about **10 dB***
and it can be shown that the modulation has a negligible effect on the output noise. [* Previously I have used 12 dB, but this is a bit conservative].

$$SNR_p = \frac{P_r}{N_o B} = \frac{A^2}{2N_o B} \geq 10 \text{ (10 dB)}$$

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$$\theta_n(t) \approx \frac{n_s(t)}{A}$$

$$n_d(t) = \frac{1}{2\pi}\frac{d\theta_n}{dt} = \frac{1}{2\pi A}\frac{dn_s(t)}{dt}$$

$$S_{n_d,n_d}(f) = S_{n_s,n_s}(f)\left|H(f)\right|^2$$

$$= \frac{N_o f^2}{A^2}$$

since $S_{n_s,n_s}(f) = N_o$ for $-B/2 \leq f \leq B/2$

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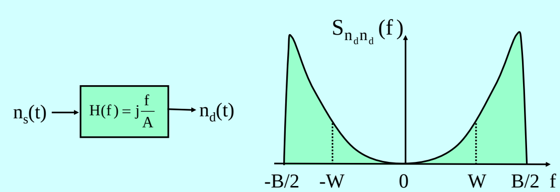
The transfer function H(f) connecting $n_s(t)$ and $n_d(t)$ is H(f) = jf/A.



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$$\left\langle n_o^2(t)\right\rangle = \int_{-W}^{W} \frac{N_o f^2}{A^2} df = \frac{2N_o W^3}{3A^2}$$

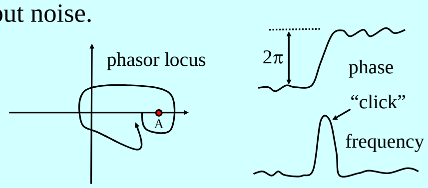
$$\left\langle s_o^2(t)\right\rangle = f_d^2 \left\langle m^2(t)\right\rangle$$

$$SNR_{fm} = \frac{3A^2 f_d^2 \left\langle m^2\right\rangle}{2N_o W^3} = 3\left\langle m^2\right\rangle \left(\frac{f_d}{W}\right)^2 \left(\frac{P_r}{N_o W}\right)$$

Hence SNR_{fm} is much greater than that of a baseband system if $\beta = f_d/W >> 1$ (ie. for wideband FM). However, as we increase β we require more bandwidth, and if **SNR_p** falls below **10 dB** the output SNR falls rapidly, and the system is said to be **below threshold**. The critical value of P_r/N_oW at threshold is:

$$\frac{P_r}{N_o W}(th) = \left(\frac{P_r}{N_o B}\right)\left(\frac{B}{W}\right) = 10\frac{B}{W}$$

The threshold in FM is caused by the noise phasor encircling the origin. The 2π jump in phase is converted into an impulse by the frequency detector. As SNR_p falls below **10 dB**, the number of impulses increases rapidly, and so does the output noise.



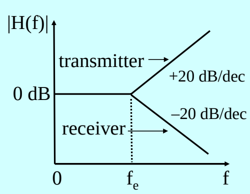
For an FM system with modulation index β, the bandwidth required is $B = 2W(\beta + 1)$, so **at threshold** where $P_r/N_oB = 10$, we have $P_r/(N_oW) = 20(\beta + 1)$. [This is consistent with Proakis].

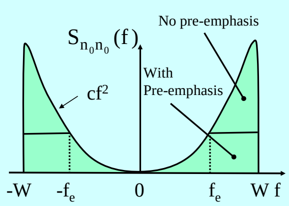
For β <<1, $P_r/(N_oW)$ (th) = 20 (**13.0 dB**)

For β = 2, $P_r/(N_oW)$ (th) = 60 (**17.8 dB**)

For β = 5, $P_r/(N_oW)$ (th) = 120 (**20.8 dB**).

Most FM systems use ***pre-emphasis***. At the transmitter, high frequencies are boosted and this is compensated by a de-emphasis in the receiver. There is no nett effect on modulation, but the noise is substantially reduced.





Without pre-emphasis:

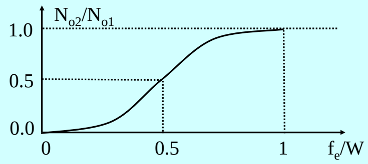
$$N_{o1} = \int_{-W}^{W} cf^2 df = \tfrac{2}{3}cW^3$$

With pre-emphasis:

$$N_{o2} \approx \int_{-f_e}^{f_e} cf^2 df + 2cf_e^2(W - f_e) = 2cf_e^2 W - \tfrac{4}{3}cf_e^3$$

$$\frac{N_{o2}}{N_{o1}} = 3\left(\frac{f_e}{W}\right)^2 - 2\left(\frac{f_e}{W}\right)^3$$

For f_e << W substantial gains can be made. However, f_e must be high enough that the amplitude of the signal is not significantly increased, otherwise the pre-emphasised signal will have to be reduced in amplitude (at the transmitter).



Example: In Australia, broadcast FM uses β =5 and a pre-emphasis time constant of 50 us, which corresponds to f_e = 3.18 kHz. For a baseband bandwidth of 15 kHz, this gives an improvement of 9.4 dB.

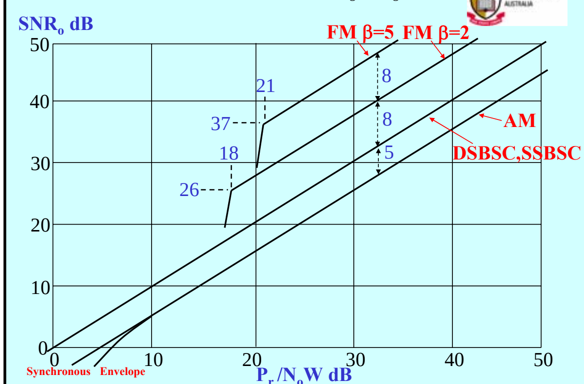
Exercise:

(i) Do the necessary calculations to verify this.
(ii) The above analysis is approximate. Try an exact analysis using a de-emphasis filter $H(f) = 1/(1+jf/f_e)$, it gives essentially the same answer.

5.7 Comparison of Analog Modulation Systems

To compare the various analog modulation systems we will plot the output SNR against P_r/N_oW for 100% sinusoidal modulation. In general the output SNR for a typical speech or music signal will not be as high.

The SNR for FM is that without pre-emphasis. Also FM is usually transmitted in a stereo format, and the SNR for this is lower than for monaural transmission.



Exercises: You are expected to attempt the following exercises in Proakis & Salehi. Completion of these exercises is part of the course. Solutions will be available later.

5.4
5.5
5.9 (1)
5.10
5.11

Section 6: Information Theory

Contents

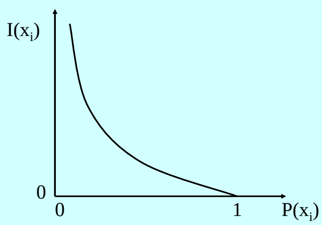
6. Information Theory

6.1 Information Content

If we have an experiment or measurement in which the outcomes are $x_1, x_2, \ldots, x_n$ with probabilities $P(x_1), P(x_2), \ldots, P(x_n)$, then we define the ***uncertainty*** of a particular outcome x_i as $-\log P(x_i)$.

When that outcome occurs, the uncertainty is removed and ***information*** is gained.

$$I(x_i) = -\log(P(x_i))$$



The base of the logarithm determines the units:

Base 10 ($\log_{10}$) gives "dits" (1 dit = 3.322 bits)
Base 2 ($\log_2$) gives "bits"
Base e (ln) gives "nits" (1 nit = 1.443 bits)

We will mostly be interested in "bits", but for calculation purposes it is easiest to use the natural logarithm (hence giving a result in "nits"), and then divide by ln(2) to give the result in "bits".

$$\log_2(x) = \ln(x)/\ln(2)$$

Information is ***additive***. Suppose x_i and y_j are two independent outcomes of an experiment. The joint event (x_i,y_j) has a probability $P(x_i,y_j) = P(x_i)P(y_j)$.

$$\begin{aligned} I(x_i, y_j) &= -\log_2 P(x_i, y_j) \\ &= -\log_2 P(x_i) - \log_2 P(y_j) \\ &= I(x_i) + I(y_j) \end{aligned}$$

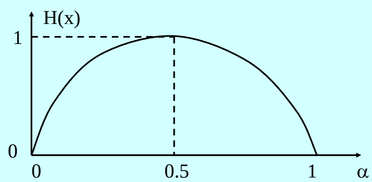
We will usually be interested in events such as the occurrence of 0's and 1's in a digital communication system, or of ASCII characters in a telex system.

For a source x with possible symbols $x_1, x_2, \ldots, x_n$ the ***source entropy*** H(x) is the average information content per symbol.

$$H(x) = -\sum_{i=1}^{n} P(x_i)\log_2 P(x_i) \qquad \text{bits/symbol}$$

Example: For a binary source P(1) = α, P(0) = 1–α. Hmax = 1 bit/symbol when they are equally likely.

$$H(x) = -\alpha \log_2 \alpha - (1-\alpha)\log_2(1-\alpha) \quad \text{bits/symbol}$$



Exercise: For n symbols, prove $H_{max} = \log_2(n)$ when all symbols are equally likely.

6.2 Joint and Conditional Entropy

If source x produces symbols $x_1, x_2, \ldots, x_n$ and source y produces symbols $y_1, y_2, \ldots, y_m$ then the ***joint entropy*** of the two sources is:

$$H(x,y) = -\sum_{i=1}^{n}\sum_{j=1}^{m} P(x_i, y_j)\log_2 P(x_i, y_j) \quad \text{bits/symbol pair}$$

If x_i and y_j are independent, then H(x,y) = H(x) + H(y). You should prove this.

Now if a symbol x_i has already occurred, then the uncertainty of y_j is $-\log_2\{P(y_j|x_i)\}$. If we average this over all symbol pairs, we have the ***conditional entropy***.

$$H(y|x) = -\sum_{i=1}^{n}\sum_{j=1}^{m} P(x_i, y_j)\log_2 P(y_j|x_i)$$

$$= -\sum_{i=1}^{n}\sum_{j=1}^{m} P(x_i, y_j)\left\{\log_2 P(x_i, y_j) - \log_2 P(x_i)\right\}$$

$$= H(x,y) - H(x)$$

To calculate the conditional entropies, it is usually easier to use the relations below.

$$H(x,y) = H(x) + H(y|x) = H(y) + H(x|y)$$

In a communication system, we will be interested in the situation where x is the transmitted information and y is the received information. These will not be the same due to errors which occur during transmission.

6.3 Source Coding Theorem

A source with entropy H bits/symbol can be encoded with arbitrarily small error at any rate R bits/symbol as long as R > H. Conversely, if R < H it is not possible for there to be arbitrarily small error.

This theorem was proved by Shannon in 1948. It sets the limits on source coding, but does not define an algorithm for achieving it.

Source coding is the problem of matching the source to the channel. A mismatch occurs if

- the number of symbols is different
- the symbol probabilities are not optimum

Example: Consider a multisymbol source and a binary channel.

Symbol	A	B	C	D	E	F
Probability	0.5	0.15	0.12	0.10	0.08	0.05

A simple code is shown below:

Symbol	A	B	C	D	E	F
Code	000	001	010	011	100	101

$$H(x) = -\sum_{i=1}^{6} P(x_i)\log_2 P(x_i) = 2.117 \text{ bits/symbol}$$

But we require 3 bits/symbol of channel capacity, so the coding efficiency is:

$$\eta = \frac{\text{source entropy (bits/symbol)}}{\text{channel capacity (bits/symbol)}} = \frac{2.117}{3} = 71\%$$

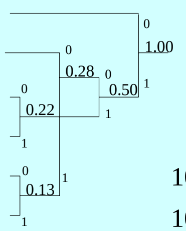
6.4 Huffman Codes

A Huffman Code (1952) uses the following procedure:

- List symbols in descending order of probability
- Group the two symbols of lowest probability
- Continue grouping until symbols are exhausted
- Assign digits 0/1 at each merger point
- Read the code from right to left

This achieves an efficient match to the channel.

Symbol	Prob	Code	n_iP_i	$-\log_2P_i$
A	0.50	0	0.50	1.00
B	0.15	100	0.45	2.74
C	0.12	110	0.36	3.06
D	0.10	111	0.30	3.32
E	0.08	1010	0.32	3.64
F	0.05	1011	0.20	4.43
			2.13	



The average code length is 2.13 bits/symbol so the efficiency is:

$$\eta = \frac{2.117}{2.130} = 99.4\%$$

Properties of a Huffman code:

1. It is ***comma free*** (it needs no spaces between symbols and is uniquely decipherable). No symbol code forms the prefix of another code.

0 | 1 1 0 | 1 1 1 | 0 | 1 1 0 | 1 0 1 0 | ...
A | C | D | A | C | E |

2. Errors tend to propagate

0 | **0** | 1 0 1 1 | 1 0 1 1 | 0 | 1 0 1 0 | …
A | **A** | **F** | **F** | **A** | E |

The single bit error has resulted in several symbol errors.

3. Buffering is necessary, since if the symbols arrive periodically, only a few bits are required if they are A's, whereas many more are required if they are F's. A buffer is needed to store the coded bits so that they can be output at a constant rate.

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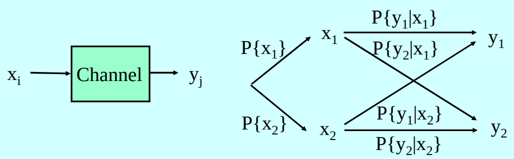
6.5 Mutual Information

A binary communication channel can be represented as shown.



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Before a symbol is sent, the average uncertainty is H(x). After it is transmitted and the received symbol observed, the remaining uncertainty is H(x|y). The nett gain of information is therefore:

$$I(x,y) = H(x) - H(x\,|\,y) \quad \text{bits/symbol}$$

I(x,y) is called the ***mutual information***.

$$\begin{aligned} I(x,y) &= H(y) - H(y\,|\,x) \\ &= H(x) + H(y) - H(x,y) \end{aligned}$$

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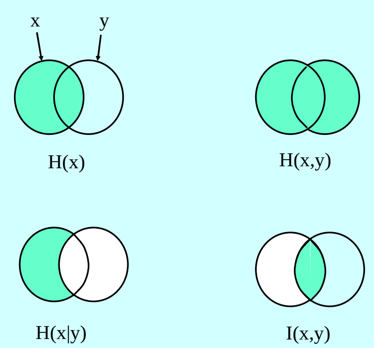

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The ***channel capacity*** is defined as the maximum information rate possible **by choosing the source probabilities**. If "r" is the symbol rate, then:

$$C = r \max\{I(x,y)\} \quad \text{bits/sec}$$

Example: Error free channel - n symbols

There are no errors so H(x|y) = 0. Hence I(x,y) = H(x) and this is maximum when the symbols are equally likely, giving $C = r \log_2(n)$ bits/sec.

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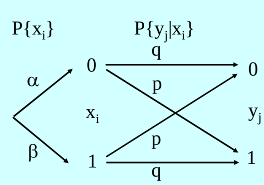
Example: Binary symmetric channel



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$P(x_i,y_j)$	y_j = 0	y_j = 1	$P(x_i)$
x_i = 0	αq	αp	α
x_i = 1	βp	βq	β
$P(y_j)$	?	?	1

$$\begin{aligned} H(y\,|\,x) &= -\alpha q \log_2 q - \alpha p \log_2 p - \beta p \log_2 p - \beta q \log_2 q \\ &= -p \log_2 p - q \log_2 q \end{aligned}$$

$$I(x,y) = H(y) + p\log_2 p + q \log_2 q$$

$$C = r(1 + p\log_2 p + q\log_2 q) \quad \text{bits/sec}$$

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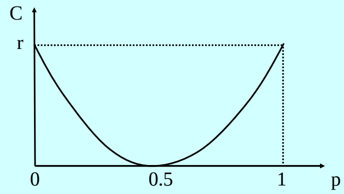
The maximum of I(x,y) corresponds to the maximum of H(y) which we know is 1 when the received symbols are equally likely. By symmetry, this is also when the transmitted symbols are equally likely.



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6.6 Pulse Code Modulation

Pulse code modulation (PCM) is not a digital modulation system as such, but describes the conversion of an analog signal into a form suitable for digital transmission.

The analog signal is:

- Filtered to prevent aliasing
- Sampled by an A/D converter (8 kHz for speech)
- The n bit sample is transmitted serially

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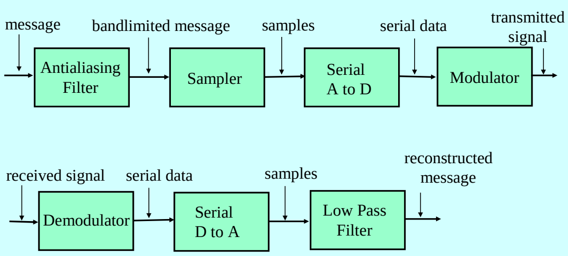

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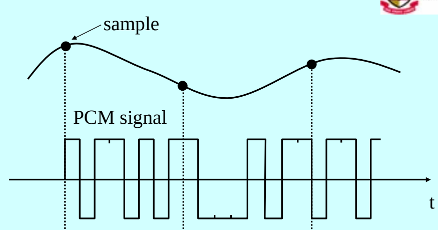

Speech	300-3300 Hz
Sampling rate	8000 Hz
Bits/sample	8
Bits/sec	64 kbits/sec

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Quantisation Noise

If the least significant bit of the A/D converter represents Δ volt, then the digital representation can be in error by up to ±Δ/2. This error $n_q(t)$ is ***quantisation noise***, is uniformly distributed and appears random, and as a consequence it sounds like random noise. Its mean square value is:

$$E\left\{n_q^2(t)\right\} = \frac{1}{\Delta}\int_{-\Delta/2}^{\Delta/2} n_q^2 \, dn_q = \frac{\Delta^2}{12}$$

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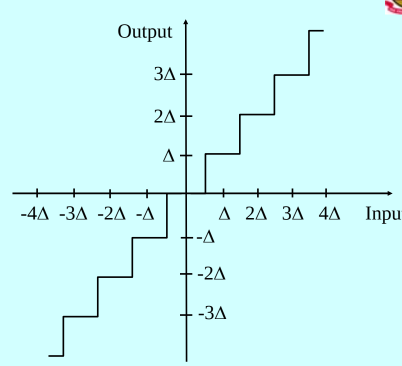

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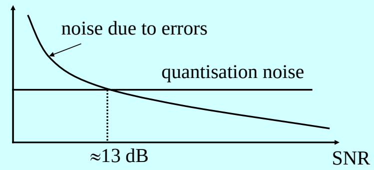
With n bits there are $q = 2^n$ quantisation levels, so $q\Delta = 2$ if $|m(t)| \le 1$. The quantisation SNR is therefore:

$$SNR_q = \frac{\langle m^2(t) \rangle}{\Delta^2/12} = 3q^2 \langle m^2 \rangle = 3 \times 4^n \langle m^2 \rangle$$

For a sinewave, we have $SNR_q = 6n+2$ dB, so with n = 8 this achieves $SNR_q = 50$ dB. For speech with $\langle m^2 \rangle = 0.1$, $SNR_q = 43$ dB.

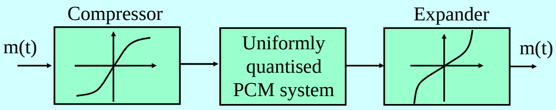
Bit Errors

Bit errors cause random errors which are perceived as noise after D/A conversion. For n = 8, a bit error rate of about 4 x 10^{-6} gives noise equal to the quantisation noise, and this occurs when $SNR_s \approx$ 13 dB.



Companding

In order to improve the dynamic range of PCM it is usual to use non-linear quantisation and this is achieved by ***companding***.

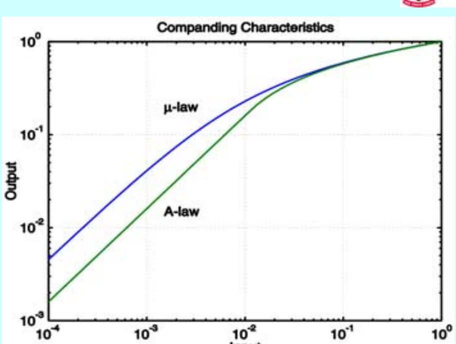


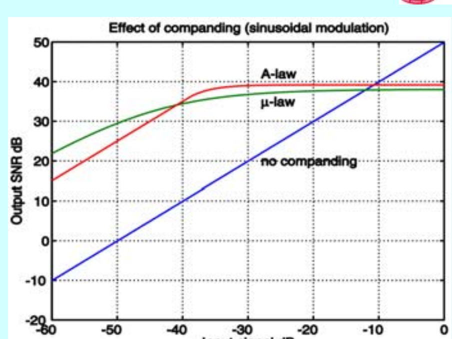
Two forms of companding are in common use, the so-called μ-law and A-law.

$$y_\mu = \text{sgn}(x)\frac{\ln(1+\mu|x|)}{\ln(1+\mu)} \quad ; -1 \le x \le 1, \quad \mu = 255$$

$$y_A = \text{sgn}(x)\frac{A|x|}{1+\ln A} \quad ; |x| \le \tfrac{1}{A}, \qquad A = 87.5$$

$$= \text{sgn}(x)\frac{1+\ln(A|x|)}{1+\ln A} \quad ; \tfrac{1}{A} < |x| \le 1$$





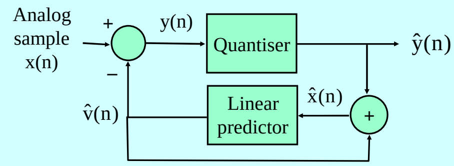
6.7 Differential Pulse Code Modulation

For speech signals it is found that the samples are highly correlated, so a differential PCM system simply transmits the difference between successive samples rather than the samples.

Because the differences are usually smaller than the original signal, DPCM can transmit the same signal with fewer bits per sample, and hence the bit rate can be reduced. This principle is widely used in speech and image compression.

In its simplest form we have $\hat{v}(n) = \hat{x}(n-1)$, but in its general form, it uses a ***linear predictor*** to generate an estimate of the next sample, and then transmits the difference between this and the actual sample.

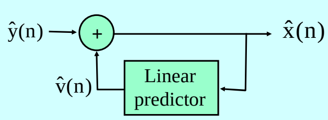
DPCM Transmitter



The linear predictor forms an estimate of the next sample as a linear combination of p previous outputs.

$$\hat{v}(n) = \sum_{i=1}^{p} a(i)\,\hat{x}(n-i)$$

DPCM Receiver



Exercises: You are expected to attempt the following exercises in Proakis & Salehi. Completion of these exercises is part of the course. Solutions will be available later.

6.1
6.7
6.22 (1,2,3)
6.56

Section 7: Digital Modulation Systems

Contents

7. Digital Modulation Systems

These are systems in which the transmitted signal consists of a sequence of ***symbols*** (pulses), each of which represents binary data 0 or 1, or some combination of 0's and 1's.

If the symbols are transmitted at a rate f_s symbols/sec, then the ***symbol period*** T is $1/f_s$. In many cases the symbol pulse is of length T or less, but as we shall see later, this is not absolutely necessary.

Binary Systems

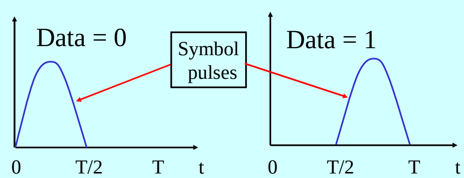
The simplest situation is where we only transmit two symbols, one representing a "0" and the other representing a "1". ***Bit rate*** (bits/sec) = Symbol rate.

M-ary Systems

In this case each symbol represents K bits, so we require $M = 2^K$ symbols. For example, with M = 4, (K = 2) the symbols would represent 00, 01, 10, 11 respectively. ***Bit rate*** (bits/sec) = K × Symbol rate.

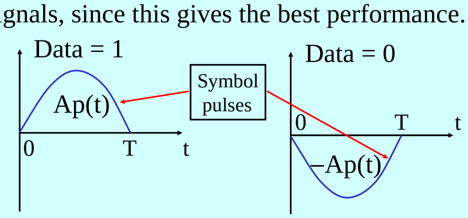
7.1 Pulse Position Modulation

In **baseband** pulse position modulation (PPM), the information is carried by the position of the pulse. This is not very bandwidth efficient, but is simple. Larger values of M require more bandwidth.



7.2 Pulse Amplitude Modulation

In **baseband** pulse amplitude modulation (PAM), the information is carried as the amplitude of the pulse. In the binary case we use ***antipodal*** signals, since this gives the best performance.

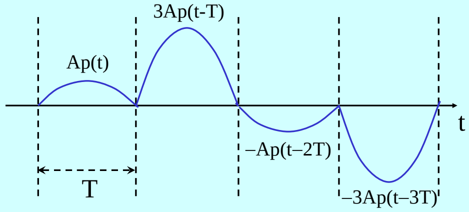


An M-ary **baseband** PAM signal is of the form:

$$s(t) = \sum_{k=-\infty}^{\infty} A\, a_k\, p(t - kT)$$

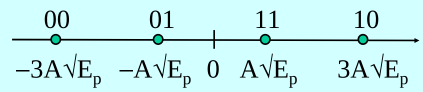
where a_k can take any of M possible values. These values are usually equally spaced and symmetrical about 0, since this gives the best performance in the presence of noise. With M = 4 we would use $a_k = \pm 1$ or ± 3. We note that the same pulse shape p(t) is used, all that is varied is the amplitude.

A typical 4PAM signal might look like this:



Data	11	10	01	00
	$a_0 = 1$	$a_1 = 3$	$a_2 = -1$	$a_3 = -3$

For each symbol $A\,a_k\,p(t-kT)$, if we plot the possible values of the square root of the symbol energy we obtain the ***signal constellation***. For the 4PAM signal on the previous page, these points would be as shown below.

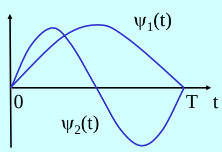


$$\text{where } E_p = \int_{-\infty}^{\infty} p^2(t)\,dt$$

7.3 Two Dimensional Signals

Consider two orthogonal signals $\psi_1(t)$ and $\psi_2(t)$ such as those shown below. Orthogonality means that:

$$\int_{-\infty}^{\infty} \psi_1(t)\,\psi_2(t)\,dt = 0$$



The pulses shown span the interval (0,T) but in general may span $(-\infty,\infty)$.

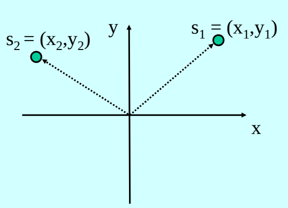
These are called ***basis functions***, and are chosen so that they have ***unit energy***.

$$\int_{-\infty}^{\infty} \psi_1^2(t)\,dt = \int_{-\infty}^{\infty} \psi_2^2(t)\,dt = 1$$

Suppose we transmit signals

$$s_1(t) = x_1\psi_1(t) + y_1\psi_2(t)$$
$$s_2(t) = x_2\psi_1(t) + y_2\psi_2(t)$$

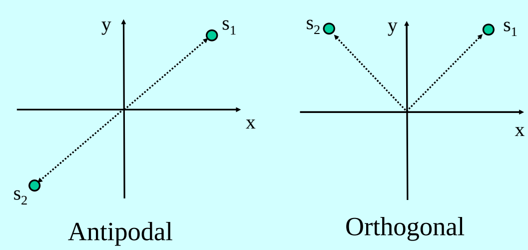
These can be represented geometrically.



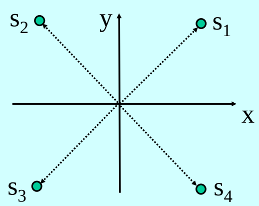
Each signal is called a ***symbol***, and the square of the distance from the origin represents the ***signal energy*** E.

$$E_1 = \int_0^T s_1^2(t)\,dt = x_1^2 + y_1^2 = r_1^2$$

If $x_2 = -x_1$ and $y_2 = -y_1$ then the signals $s_1(t)$ and $s_2(t)$ are ***antipodal***. If the angle between the vectors is 90^o, then they are ***orthogonal***.

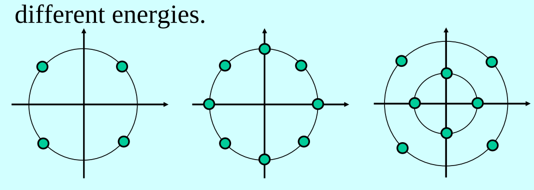


We can generate larger sets of signal waveforms by adding various weights of $\psi_1(t)$ and $\psi_2(t)$. For example the signal set below is called a ***bi-orthogonal*** set.



Bi-orthogonal

The diagram showing the points corresponding to the various symbols is called the ***signal constellation.*** Various possible constellations are shown below. Note that the symbols may have different energies.



M = 4 M = 8 M = 8

With a symbol set of size $M = 2^K$, we can transmit K bits of information in each symbol interval T. If E_s is the average energy per symbol, then the (average) transmitter power is $P_{av} = E_s/T$.

E_s = average energy per symbol.
E_b = average energy per bit = E_s/K.

The energy per bit E_b is a useful way of comparing modulation schemes of different sizes.

7.4 Carrier Modulation

We can form **carrier modulated** signals by choosing the basis functions (with unit energy) to be:

$$\psi_1(t) = C\,p(t)\cos(2\pi f_o t)$$

$$\psi_2(t) = C\,p(t)\sin(2\pi f_o t)$$

$$C = \sqrt{\frac{2}{E_p}}, \quad E_p = \int_{-\infty}^{\infty} p^2(t)\,dt$$

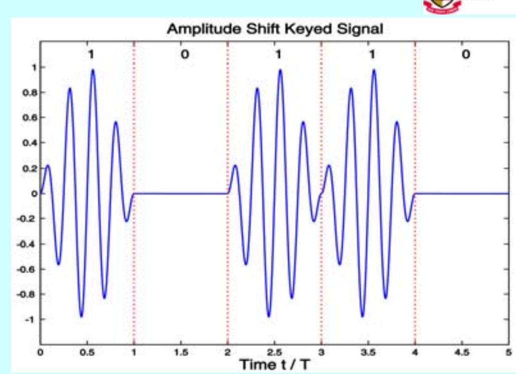
where p(t) is the pulse shape. The required bandwidth is usually **twice** that required for p(t).

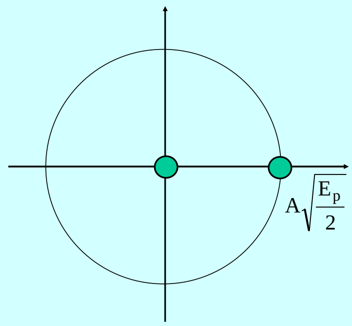
7.5 Amplitude Shift Keying

The simplest form of carrier modulation is ***amplitude shift keying*** (ASK) which is the digital equivalent of AM. (The energy of the "1" symbol is $0.5A^2E_p$, that of a "0" symbol is 0, so the average symbol energy $E_s = 0.25A^2E_p$).

$$s(t) = \sum_{k=-\infty}^{\infty} s_k(t)$$

$$s_k(t) = \begin{cases} A\,p(t-kT)\cos(2\pi f_o t) & \text{for a "1"} \\ 0 & \text{for a "0"} \end{cases}$$



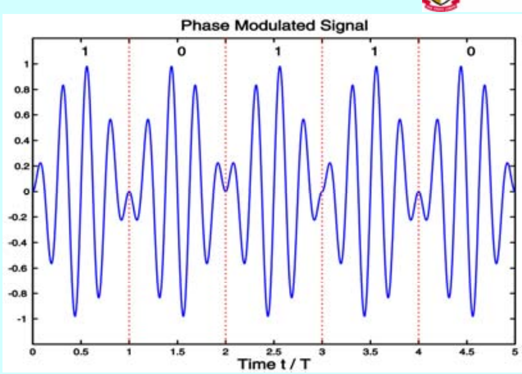


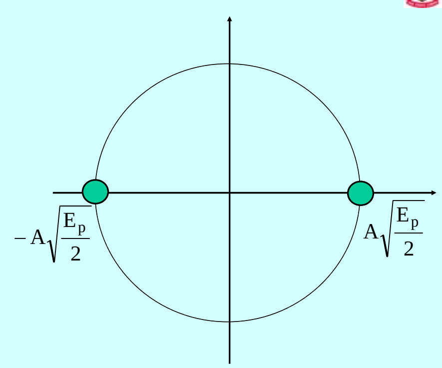
7.6 Phase Shift Keying

The simplest form of this is ***binary phase shift keying*** (BPSK) where the signal s(t) is:

$$s(t) = A \sum_{k=-\infty}^{\infty} a_k \, p(t - kT) \cos(2\pi f_o t)$$

where p(t) is the pulse shape, $a_k = \pm 1$ is the digital data and $E_s = 0.5A^2|a_k|^2E_p = 0.5A^2E_p$ is the energy per symbol. Note that the signal s(t) is a superposition of all the individual signals for each symbol interval.

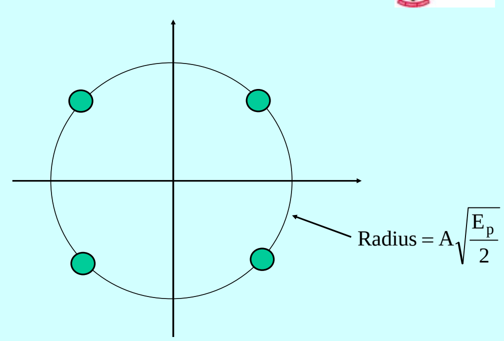




The M = 4 version is called ***quaternary phase shift keying*** (QPSK) which can transmit two bits of information per symbol.

$$s(t) = \sum_{k=-\infty}^{\infty} A \, p(t - kT) \cos(2\pi f_o t + \theta_k)$$

Where θ_k takes values $\pm 45^o$ and $\pm 135^o$. In this case $E_s = 0.5A^2E_p$ and $E_b = E_s/2$. This form of modulation is very popular in satellite systems and HF communication systems.



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7.7 Frequency Shift Keying

In ***binary frequency shift keying*** (BFSK) we have:

$$s(t) = \sum_{k=-\infty}^{\infty} s_k(t)$$

$$s_k(t) = \begin{cases} A\cos(2\pi f_1 t + \theta_{1k}) & \text{for a "1"} \\ A\cos(2\pi f_2 t + \theta_{2k}) & \text{for a "0"} \end{cases}$$

The phases of the two sinewaves are usually adjusted so that the signal s(t) is continuous when switching from a “0” to “1” and vice versa.

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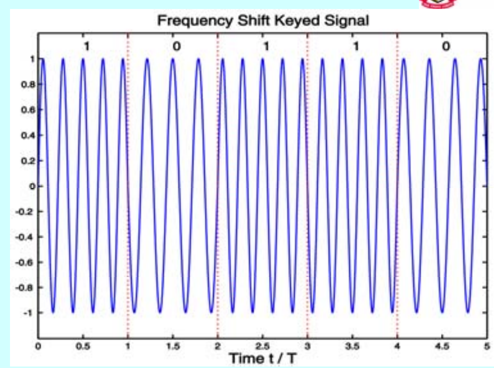

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FSK has the advantage that simple detectors are possible, but because FSK does not use antipodal signals, it requires 3 dB more power than BPSK for the same error rate.

The ***modulation index*** **h = (f_1 – f_2)T** is usually about unity, but special forms with h = 1/2 exist.

Demodulation for h = 1 is achieved by filtering the received signal with filters tuned to the frequencies f_1 and f_2 and selecting whichever output is greater, or alternatively a correlation detector can be used.

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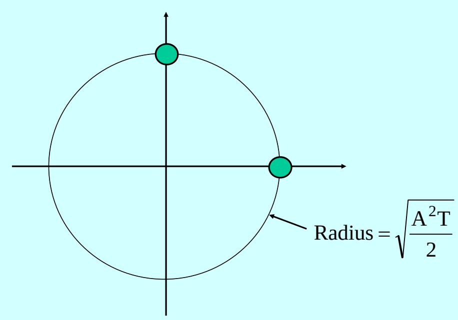

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We can also express the FSK signal as:

$$s(t) = A\cos[2\pi f_o t + \theta(t)]$$

$$\frac{d\theta(t)}{dt} = \frac{\pi h a_k}{T} \quad ; kT \le t < (k+1)T$$

$$f_o = \frac{f_1 + f_2}{2}$$

where $a_k = \pm 1$ is the digital data and the instantaneous frequency of s(t) is either f_1 or f_2.

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FSK is digital frequency modulation, since the information is carried in the frequency of the signal.

In some FSK systems the frequency change is smoothed by using a pulse shape p(t) with unit area

$$\frac{d\theta(t)}{dt} = \sum_{k=-\infty}^{\infty} \pi h a_k p(t - kT)$$

With binary data the phase changes by πh in each symbol interval.

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7.8 Quadrature Amplitude Modulation

The two simplest versions of this are equivalent to BPSK and QPSK. For larger symbol sets we have:

$$s(t) = \sum_{k=-\infty}^{\infty} A\,p(t-kT)\,[a_k \cos(2\pi f_o t) - b_k \sin(2\pi f_o t)]$$

where a_k and b_k are multilevel signals. The symbol energy is $E_k = 0.5A^2E_p(a_k^2+b_k^2)$ so E_s is the average over all possible values of a_k and b_k. For instance, with 16QAM, we would have a_k and b_k chosen from the values ±1 and ±3.

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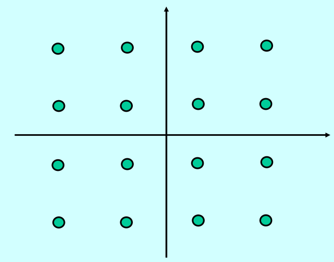
The constellation would appear as:



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7.9 The Matched Filter

We will derive the optimum receiver for a baseband binary PAM signal and extend the result to other situations. We will assume we have a symbol which is received accompanied by additive white Gaussian noise (AWGN).

$$v(t) = \pm s(t) + n(t)$$

where s(t) is the received signal of energy E_s and n(t) is white Gaussian noise of power spectral density $S_{nn}(f) = \alpha = N_o/2$.

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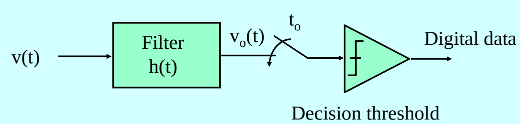
The receiver will be assumed to consist of a filter (to reduce the noise) and whose output is sampled at some time t_o. On the basis of this sample we we decide whether the symbol was +1 or –1.



The decision threshold will be zero volts.

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The signal to noise ratio at the sampling instant is:

$$\Gamma = \frac{s_o^2(t_o)}{\langle n_o^2(t) \rangle}$$

$$s_o(t_o) = \int_{-\infty}^{\infty} h(\lambda)s(t_o - \lambda)\,d\lambda$$

$$\langle n_o^2(t) \rangle = \alpha \int_{-\infty}^{\infty} |H(f)|^2 df = \alpha \int_{-\infty}^{\infty} h^2(\lambda)\,d\lambda$$

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To maximise Γ we will maximise $s_o(t_o)$ while keeping $\langle n_o^2(t) \rangle$ constant. For a perturbation δh(t):

$$\delta s_o(t_o) = \int_{-\infty}^{\infty} \delta h(\lambda)s(t_o - \lambda)\,d\lambda = 0 \quad \text{(for a maximum)}$$

$$\delta\langle n_o^2(t) \rangle = 2\alpha \int_{-\infty}^{\infty} \delta h(\lambda)h(\lambda)\,d\lambda = 0 \quad \text{(constant output noise)}$$

Since δh(t) is arbitrary except that the second relation must be zero, we must have $h(t) = c\,s(t_o - t)$.

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$$h(t) = c\,s(t_o - t)$$

$$s_o(t_o) = c\int_{-\infty}^{\infty} s^2(t_o - t)\,dt = c\int_{-\infty}^{\infty} s^2(t)\,dt = c\,E_s$$

$$\langle n_o^2(t)\rangle = \alpha c^2 \int_{-\infty}^{\infty} s^2(t_o - t)\,dt = \alpha c^2 E_s$$

$$\Gamma_{max} = \frac{E_s}{\alpha} = \frac{E_s}{\frac{1}{2}N_o}$$

In the frequency domain we have

$$H(f) = c\,S^*(f)\,e^{-j2\pi f t_o}$$

where S(f) is the Fourier transform of s(t). This is called a ***matched filter***. It is equivalent to a ***correlator***, where the received signal is multiplied by a replica of the known signal s(t) and integrated.

$$v_o(t_o) = \int_{-\infty}^{\infty} v(\lambda)h(t_o - \lambda)\,d\lambda = \int_{-\infty}^{\infty} v(\lambda)s(\lambda)\,d\lambda$$

7.10 Receiver for Carrier Systems

For carrier systems, the received signal is synchronously demodulated using $\cos(2\pi f_o t)$ and $\sin(2\pi f_o t)$ to obtain baseband signals which can then be applied to baseband receivers of the type discussed in the previous section.

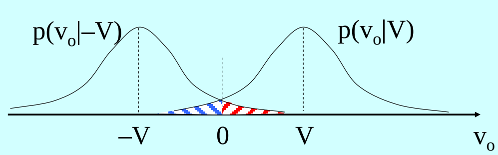
In some cases, such as FSK for example, simple non-optimal receivers can be used. Alternatively, the matched filter can be realised by correlation applied directly to the RF signal.

7.11 Probability of Error

We will derive the result for a ***baseband binary PAM signal*** and extend the result to other situations. We will assume we have a symbol which is received accompanied by additive white Gaussian noise (AWGN).

We will assume the sample value is $v_o = s_o(t_o) + n_o(t_o) = \pm V + n_o$. The noise sample n_o is of variance $\sigma^2 = \langle n_o^2(t)\rangle$.

The conditional probability density functions of v_o are called a ***likelihood functions***. They are simply the pdf of the noise n_o, shifted by the signal component ±V.



An error occurs if $n_o > V$ when $s_o = -V$, or if $n_o < -V$ when $s_o = V$. The probabilities of these are equal, so the probability of a bit error is:

$$P_b = \int_V^{\infty} p(n_o)\,dn_o = \frac{1}{\sigma\sqrt{2\pi}}\int_V^{\infty} e^{-n_o^2/2\sigma^2}\,dn_o$$

$$= \frac{1}{\sqrt{2\pi}}\int_{V/\sigma}^{\infty} e^{-t^2/2}\,dt = Q\left(\frac{V}{\sigma}\right)$$

$$P_b = Q\left(\sqrt{\Gamma}\right)$$ (always true)

If the filter is a matched filter, then:

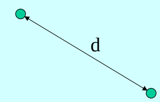
$$P_b = Q\left(\sqrt{\frac{E_s}{\alpha}}\right) = Q\left(\sqrt{\frac{E_s}{\frac{1}{2}N_o}}\right)$$

Note that this is ***only true for a matched filter*** receiver.

7.12 Probability of Error (General Constellation)

The square of the distance of a constellation point from the origin is the ***symbol energy***. If two symbols in a constellation are spaced by a distance "d" on the diagram, then with a ***matched filter*** receiver and a noise spectral density $S_{nn}(f) = N_o/2$, the probability of selecting the wrong symbol is:

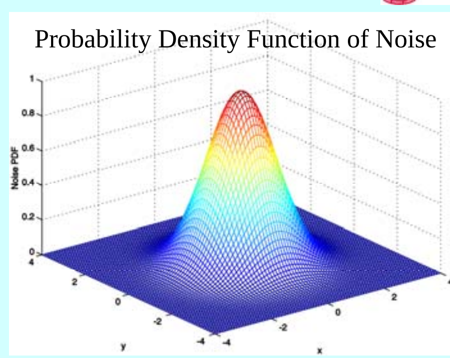
$$P_{sym} \approx Q\left\{\sqrt{\frac{d^2}{2N_o}}\right\}$$

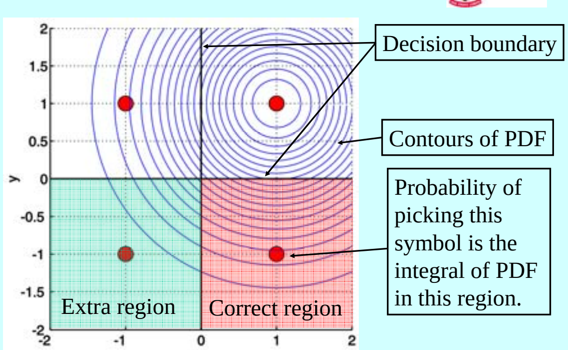


This formula is approximate as it calculates the probability of a received symbol being in the half plane containing the other symbol, and not the actual ***decision region*** for the other symbol.

The ***decision regions*** are defined by ***decision boundaries*** which are equidistant from each pair of symbols.

Hence the result will be greater than the true value, but in most cases the difference is negligible.





When we compare digital modulation systems, we do it on the basis of ***energy per bit* (E_b)**, and not ***energy per symbol* (E_s)**. For a system which transmits K bits per symbol (which requires M = 2^K symbols) we have:

$$E_b = \frac{E_s}{K} = \frac{P_s(av)T}{K}$$

Where $P_s(av)$ is the average power of each symbol, and T is the symbol interval.

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To calculate the energy per symbol E_s or the energy per bit E_b for an arbitrary constellation of $M = 2^K$ symbols represented by phasors Z_k:

$$E_s = av\left\{|Z_k|^2\right\} \quad \text{; average energy per symbol}$$

$$E_b = \frac{E_s}{K} \quad \text{; average energy per bit}$$

$$P_s = E_s / T \quad \text{; average power}$$

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Example: For BPSK, $E_b = E_s$, and $d = 2\sqrt{E_s}$:

$$P_b = P_{sym} = Q\left\{\sqrt{\frac{4E_b}{2N_o}}\right\} = Q\left\{\sqrt{\frac{E_b}{\frac{1}{2}N_o}}\right\}$$

Example: For QPSK, $E_s = 2E_b$ and $d = \sqrt{(2E_s)}$ and hence:

$$P_{sym} = Q\left\{\sqrt{\frac{2E_s}{2N_o}}\right\} = Q\left\{\sqrt{\frac{E_s}{N_o}}\right\} = Q\left\{\sqrt{\frac{E_b}{\frac{1}{2}N_o}}\right\}$$

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If there are a number of symbols near a particular symbol, then the ***symbol error probability*** is the sum of the probabilities of picking each of the wrong ones. We can ignore symbols far away.

For instance, with QPSK the probability of picking an adjacent symbol is actually $2*Q\{\sqrt{(E_s/N_o)}\}$ because there are two adjacent symbols. However, this will usually cause only one of the two bits to be in error, so the ***bit error probability*** is $Q\{\sqrt{(E_s/N_o)}\} = Q\{\sqrt{(2E_b/N_o)}\}$.

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Exercise: In M-ary PSK, ($M=2^K$), show that the probability of choosing an adjacent symbol (the most likely error event) is given by:

$$P_{sym} = Q\left(\sin\left(\frac{\pi}{M}\right)\sqrt{\frac{E_s}{\frac{1}{2}N_o}}\right) = Q\left(\sin\left(\frac{\pi}{M}\right)\sqrt{\frac{KE_b}{\frac{1}{2}N_o}}\right)$$

Hence determine how much transmitter power is required for 8-PSK compared with 4-PSK for the same probability of error for the same a) symbol rate, b) bit rate. **Ans:** 5.3dB, 3.6dB more.

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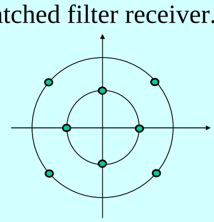
Exercise: For the 8QAM constellation shown calculate the radii such that the nearest symbol is distance d in all cases. Hence find the probability of a symbol error if E_b/N_o = 13 dB, assuming a matched filter receiver.



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7.13 Non-Matched Filters

When the receiver filter is matched to p(t), we have the useful result that in a binary PAM or BPSK system, the signal to noise ratio Γ is:

$$\Gamma_{matched} = \frac{E_b}{\frac{1}{2}N_o}$$

If the receiver filter is not a matched filter, the analysis becomes more complicated in that the signal and noise components at the receiver output must be calculated individually.

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7.14 Carrier and Clock Recovery

In digital communication systems neither the ***carrier*** nor the symbol ***clock*** is transmitted, and these must be recovered from the received signal.

The carrier is required for demodulation of the RF signal (except for FSK) and the clock is required to time the digital data signal. The recovery circuits usually involve some sort of non-linear operation on the received signal, followed by a phase locked loop to extract the required signal.

Exercises: You are expected to attempt the following exercises in Proakis & Salehi. Completion of these exercises is part of the course. Solutions will be available later.

7.10
7.15
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7.34
7.43

Section 8: Digital Transmission in Bandlimited Channels

Contents

8. Digital Transmission in Bandlimited Channels

So far we have not considered pulse shape or bandwidth. In practice all channels are bandlimited, either because they are physically limited in bandwidth or because of a need to share spectrum space with other users.

We will consider transmission of baseband signals and extend the results to carrier modulated signals.

For an M-ary **baseband** PAM signal s(t) of the form:

$$s(t) = \sum_{k=-\infty}^{\infty} A\, a_k\, p(t - kT)$$

$$S_{ss}(f) = \frac{A^2}{T} E\{a_k^2\} |P(f)|^2$$

where a_k are the symbol amplitudes (k = 1, 2, .. M), $S_{ss}(f)$ is its power spectral density, p(t) is the pulse shape and T is the symbol interval. This result follows from the derivation in Chapter 4.

8.1 Pulse Shape

The optimum pulse to use from bandwidth considerations is a ***sinc pulse***:

$$p(t) = \mathrm{sinc}\left(\frac{t}{T}\right)$$

This has a Fourier transform P(f) = T rect(fT), so the power spectrum of s(t) is:

$$S_{ss}(f) = A^2 T\, \mathrm{rect}(fT) \quad V^2/Hz$$

This has a bandwidth equal to **1/2T Hz**.

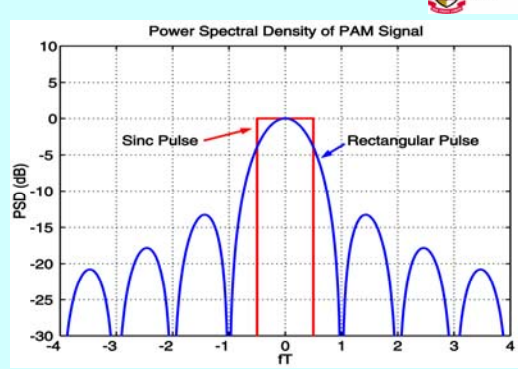
If the pulse has a ***rectangular pulse*** shape:

$$p(t) = \mathrm{rect}\left(\frac{t}{T}\right)$$

This has a Fourier transform P(f) = T sinc(fT), so the power spectral density of s(t) is:

$$S_{ss}(f) = A^2 T\, \mathrm{sinc}^2(fT) \quad V^2/Hz$$

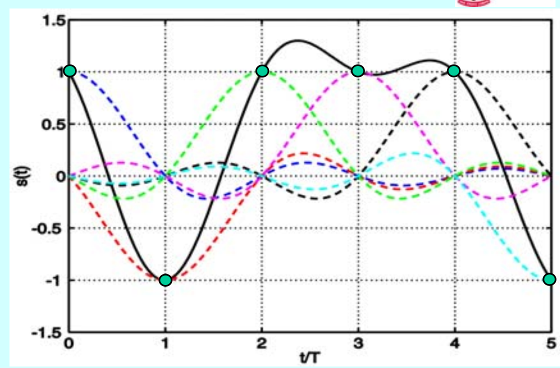
This has a bandwidth in excess of **1/T Hz**, with significant power at frequencies outside this band. Hence rectangular pulses are not used in practice.



Any pulse which has the property that p(kT) = 0 (for all k except k = 0) has the property of having zero ***intersymbol interference*** (ISI). This is because when the signal

$$s(t) = \sum_{k=-\infty}^{\infty} A\, a_k\, p(t - kT)$$

is sampled at t = kT, the only contribution is from the symbol a_k of interest. The sinc pulse is the simplest pulse with this property, but there are others.



With sinc pulses, although the pulses overlap, there is no ***inter-symbol interference*** (ISI). The zero crossings in the sinc function give the signal at the symbol intervals t = kT, $s(kT) = A\, a_k$, with no interference from adjacent data values.

However, as we see shortly, sinc pulses are not suitable for transmitting data in practice. The desirable feature is the ***periodic zero crossings***, and other pulses besides sinc pulses have this property.

8.2 Signal Design and ISI

In practice we do not use sinc pulses because:

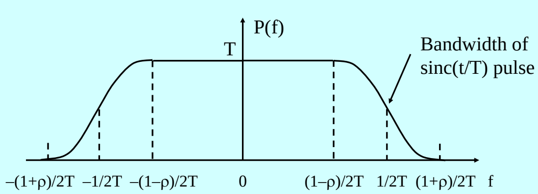
• they are hard to generate.

• it is very difficult to maintain zero ISI because the pulses die away slowly and the zero crossings are significantly modified by a non-ideal channel response.

• there are large peak voltages between samples.

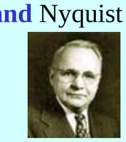
• very accurate timing is necessary.

The solution is to use pulses which die away more quickly, but which still have the zero ISI property. This usually requires more bandwidth. Pulses which have this property are ***Nyquist*** pulses, and these have a Fourier transform as shown below.



The **bandwidth** required for a **baseband** Nyquist pulse is:

$$B = \frac{(1+\rho)}{2T}\ \text{Hz}$$



The pulse p(t) can be found by forming the inverse fourier transform of P(f). The expression for p(t) is shown below, and appears on the data sheet

$$p(t) = \frac{\pi}{4}\,\text{sinc}\left(\frac{t}{T}\right)\left[\text{sinc}\left(\frac{\rho t}{T} - \frac{1}{2}\right) + \text{sinc}\left(\frac{\rho t}{T} + \frac{1}{2}\right)\right]$$

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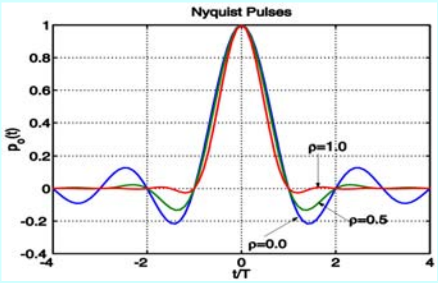
These pulses are characterised by an ***excess bandwidth factor*** ρ, where $0 \le \rho \le 1$.



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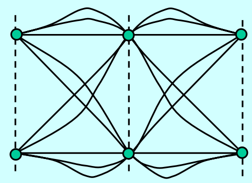
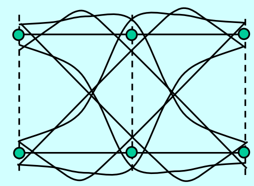
The ISI properties of a pulse can be illustrated by an eye diagram, which is simply a superposition of all possible signals over a 2T interval.



No intersymbol interference



With intersymbol interference

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The height of the eye tells us the margin against noise, whereas the width of the eye tells us how accurate the timing has to be.

The pulse shape of importance is actually the pulse shape $p_o(t)$ at the ***output*** of the receiver filter, so the actual transmitted pulse shape p(t) may be somewhat different. If we know the receiver filter transfer function H(f), then we have $P_o(f) = P(f)\,H(f)$, so we can find P(f) and hence p(t).

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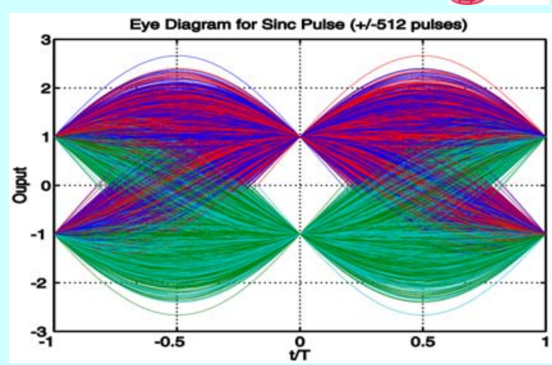

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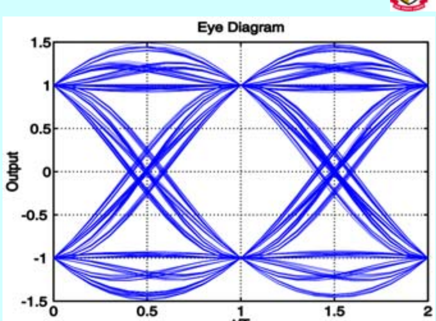

Eye Diagram of Nyquist Pulse (ρ = 0.5)

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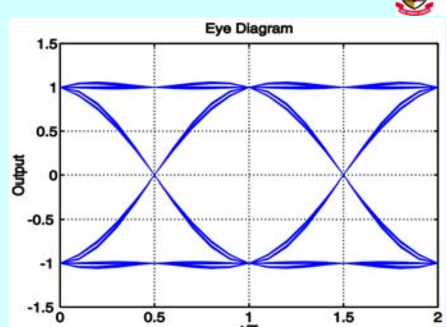

Eye Diagram of Nyquist Pulse (ρ = 1.0)

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8.3 Bandwidth in Carrier Systems

For carrier systems the bandwidth is usually double that of a baseband system. If T is the symbol period:

$$B = \frac{(1+\rho)}{T}\ \text{Hz} \quad \text{for M-PSK and M-QAM}$$

$$B = \frac{(2+\rho)}{T}\ \text{Hz} \quad \text{for binary FSK with } h = 1$$

For binary FSK, the bandwidth is greater by an amount equal to the separation of the two carriers, which for h = 1 will be 1/T.

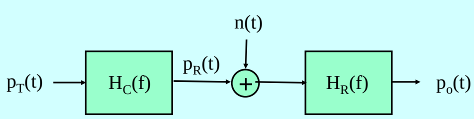
A carrier system is used when we wish to modulate the digital signal onto an RF carrier. However it may also be used for baseband channels when the frequency response does not go down to DC.

For example, for a telephone line the bandwidth is 300 Hz to 3300 Hz, so data modems usually use M-QAM with a sub-carrier frequency of 1800 Hz.

The doubling of the bandwidth is not a problem, because we can use both the cosine and sine subcarrier frequencies to carry data.

8.4 Overall System Design

The overall binary baseband PAM system can be considered to be as shown. The transmitted pulse is $p_T(t)$ with Fourier transform $P_T(f)$, $H_C(f)$ is the channel response and $H_R(f)$ is the receiver filter. The received pulse is $p_R(t)$ with transform $P_R(f)$ and the output pulse is $p_o(t)$ with transform $P_o(f)$.



We specify the output pulse $p_o(t)$, and require $H_R(f)$ to be matched to $p_R(t)$.

$$P_R(f) = P_T(f)H_C(f)$$

$$H_R(f) = P_R^*(f)e^{-j2\pi f t_o} \quad \text{(matched filter)}$$

$$P_o(f) = P_R(f)H_R(f) = |P_R(f)|^2 e^{-j2\pi f t_o}$$

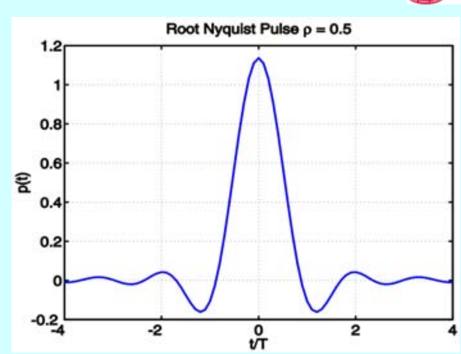
$$P_R(f) = \sqrt{|P_o(f)|}\, e^{j\theta(f)} \quad [\theta(f) \text{ is arbitrary}]$$

$$H_R(f) = \sqrt{|P_o(f)|}\ e^{-j[\theta(f)+2\pi f t_o]}$$

$$P_T(f) = \frac{1}{H_C(f)}\sqrt{|P_o(f)|}\ e^{j\theta(f)}$$

If the channel response is known, we can design a suitable transmitter pulse (if not, we need to use an ***adaptive equaliser*** at the receiver). The phase $\theta(f)$ is arbitrary, but may be chosen to make the transmitted pulse causal (ie. start from t = 0). Note that $p_T(t)$ and $p_R(t)$ may not have the zero ISI properties of $p_o(t)$.

Example: If $H_C(f) = 1$ and $p_o(t)$ is a $\rho = 0.5$ Nyquist pulse, the transmitter pulse is a **Root-Nyquist** pulse as shown.



8.5 Orthogonal Frequency Division Multiplex

When the channel is non-ideal, ISI is a problem. However, if we can increase the symbol time T, the effect of the channel response can be significantly reduced. One such way of doing this is to send a large number of bits in one symbol, and one way of doing this is OFDM.

If we have a channel of bandwidth W, we create M sub-channels of bandwidth W/M and in each sub-channel use a carrier frequency f_i, i = 1,2, … , M.

By selecting the symbol rate T equal to M/W, the carrier signals are orthogonal. Each sub-channel can be modulated using any of the modulation methods discussed earlier, and BPSK and QPSK are common choices.

The main problem with OFDM is that large peak voltages compared to the RMS value can occur, which is undesirable in any power limited system since this may cause ***intermodulation distortion*** if the amplifiers saturate.

OFDM systems are very popular for HF communication channels, because the channels often vary at a rate that makes adaptive equalisation difficult.

Also, by spreading the symbol over a long time, the effects of fading can be reduced, since (hopefully) fades will only occur during a relatively small fraction of the symbol time.

Exercises: You are expected to attempt the following exercises in Proakis & Salehi. Completion of these exercises is part of the course. Solutions will be available later.

8.2 (Use Matlab to plot P_e for SNR = 0:12 dB)
8.3 (Calculate P_e if $P_e = 10^{-3}$ with no error)
8.10
8.15 (use ordinary PAM & Nyquist pulse)
8.17

Section 9: Channel Capacity and Coding

Contents

9. Channel Capacity and Coding

9.1 Discrete Channels

We have already derived the result for a discrete channel in Chapter 6. For a channel with a symbol rate r symbols/sec,

$$C = r \max\{I(x,y)\} \quad \text{bits/sec}$$
$$\text{where} \quad I(x,y) = H(x) - H(x|y) = H(y) - H(y|x)$$

and the maximisation is by varying the source probabilities.

9.2 Continuous Channels

Consider a continuous channel of bandwidth B in which the received signal is x(t) = s(t) + n(t). We consider samples of the signal sampled at a rate 2B.

$$x(i\delta t) = s(i\delta t) + n(i\delta t), \qquad \delta t = \frac{1}{2B}$$
$$x_i = s_i + n_i, \qquad i = 1,2,\ldots,N$$
$$\mathbf{x} = [x_1, x_2, \ldots, x_N]$$

For N samples, we can calculate the sums of squares:

$$|\mathbf{s}|^2 = \sum_{i=1}^{N} s_i^2, \quad |\mathbf{n}|^2 = \sum_{i=1}^{N} n_i^2, \quad |\mathbf{x}|^2 = \sum_{i=1}^{N} x_i^2$$

Since both the signal and noise are random processes, $|\mathbf{s}|^2$, $|\mathbf{n}|^2$ and $|\mathbf{x}|^2$ will be random variables, with means equal to NP_s, NP_n and NP_x respectively, and variances which become (relatively) small for large N. [P_s, P_n and P_x are the average powers.]

For a Gaussian process, the standard deviation of $|\mathbf{x}|^2$ is $P_x\sqrt{(2N)}$, so we find that for large N, $|\mathbf{x}|^2 \approx NP_x$ and all possible received sequences $\mathbf{x} = [x_1,x_2,\ldots,x_N]$ effectively lie near the surface of a hypersphere of radius $(NP_x)^{(1/2)}$.

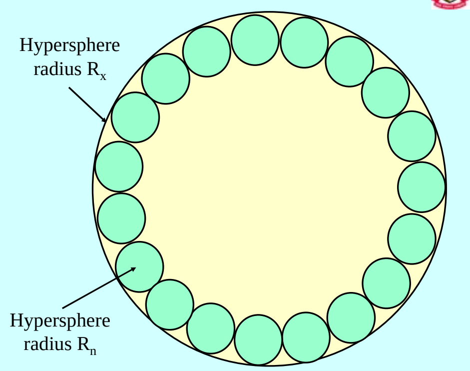
For a particular signal sequence $\mathbf{s} = [s_1,s_2,\ldots,s_N]$, the received sequence $\mathbf{x} = [x_1,x_2,\ldots,x_N]$ will lie near the surface of a hypersphere of radius $(NP_n)^{(1/2)}$ with centre $\mathbf{s} = [s_1,s_2,\ldots,s_N]$.

A hypersphere of dimension N is the region

$$x_1^2 + x_2^2 + \cdots + x_N^2 \le R_x^2$$

It has a volume proportional to R_x^N.

Ideally we require that the hyperspheres for each signal sequence not overlap, so that the signal sequence can be uniquely identified by observation of the received sequence $[x_1,x_2,\ldots,x_N]$.

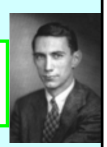


The channel capacity is related to the number M of different signal sequences $[s_1, s_2, \ldots, s_N]$ which can be distinguished by examining the received sequence $[x_1, x_2, \ldots, x_N]$. The number of such sequences is roughly equal to the ratio of the volumes of the $|\mathbf{x}|^2$ and $|\mathbf{n}|^2$ hyperspheres (and becomes more accurate as $N \to \infty$).

$$M \approx \frac{\{NP_x\}^{N/2}}{\{NP_n\}^{N/2}} = \left\{1 + \frac{P_s}{P_n}\right\}^{N/2}$$

since $P_x = P_s + P_n$ if s(t) and n(t) are uncorrelated.

Hence in a time $T = N\delta t = N/2B$ seconds we can send $\log_2(M)$ bits of information. The channel capacity is the information rate which is:



$$C = \frac{\log_2(M)}{T} = B\log_2\left(1 + \frac{P_s}{P_n}\right) \text{ bits/sec}$$

This result was first proved by Shannon in 1948 and is called ***Shannon's Theorem***. Shannon also proved that it was possible to transmit information at a rate R < C with ***arbitrarily small error***.

Shannon's theorem is important in that it sets an upper limit to the channel capacity of any communication system.

In practice we can get fairly close to Shannon's limit by the use of coding, but as we approach the limit the complexity and time delay required increase rapidly.

One of the counter-intuitive results of Shannon's theorem is that best performance is obtained using an infinite bandwidth.

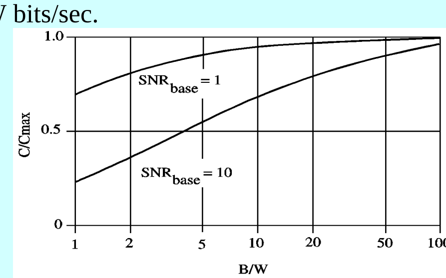
However we have seen that wide band systems such as FM perform better than narrow band systems such as AM or DSBSC.

Suppose we have a baseband signal of bandwidth W, noise of spectral density $N_o/2$ and a channel of bandwidth B. Then $P_n = N_oB$, and we have:

$$C = B\log_2\left(1 + \frac{P_s}{P_n}\right) = B\log_2\left(1 + \frac{P_s}{N_oW}\frac{W}{B}\right)$$

$$\frac{C}{W} = \frac{B}{W}\log_2\left(1 + SNR_{base}\frac{W}{B}\right)$$

where $SNR_{base} = P_s/N_oW$. The maximum value of C is reached as $B \to \infty$ and $C_{max} = 1.44\ SNR_{base}$ W bits/sec.



9.3 Error Detection and Correction

In order to approach Shannon's channel capacity it is necessary to use coding. If we have a channel with a certain error rate, then we can reduce this by the use of coding.

Error detection is more simple than ***error correction***, since error detection only indicates that there is an error in a block of data without saying where it is, whereas error correction requires that the error location be known.

If we have detected an error in a block of data, we can request it be transmitted. This is called ***automatic repeat request*** (ARQ).

If retransmission is impossible (eg. one way transmission) or impractical (eg. real time speech), then error control must be by ***forward error correction*** (FEC).

With FEC the object is to have a code from which the receiver can determine if an error has occurred, and to be able to correct it.

9.4 Repetition and Parity Check Codes

Suppose errors occur randomly and with a bit error probability of P_b. A simple error control strategy is to repeat each bit a number of times.

Data	1	0	1	1	0	1
Transmit	111	000	111	111	000	111

For each bit, P_b is the probability of error and $Q_b = 1 - P_b$ is the probability of being correct.

The probability of i errors in a block of n is given by the binomial distribution.

$$P\{i \text{ errors}\} = \binom{n}{i} P_b^i Q_b^{n-i}$$

For a triple repetition code (n = 3), single or double errors can be detected, but a triple error would be undetected. But for $P_b = 10^{-3}$, $P(1) = 3\times10^{-3}$, $P(2) = 3\times10^{-6}$ and $P(3) = 1.0\times10^{-9}$.

For error correction, use a majority decision decoder.

000, 001, 010, 100	all decode to 0
111, 110, 101, 011	all decode to 1

With this decoder, single errors are corrected, but double or triple errors result in a decoding error. Hence the probability of error with correction is

$$P_{cbe} = 3P_b^2 Q_b + P_b^3 \approx 3P_b^2 \quad \text{for our example}$$

Repetition codes are not very efficient. More efficient codes operate on blocks of digits rather than each digit separately.

A simple ***parity check code*** takes n-1 message digits and adds a check digit so that the overall parity is always even or always odd.

This code can detect single errors, but is not able to correct them.

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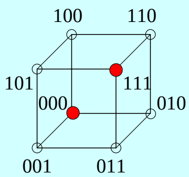
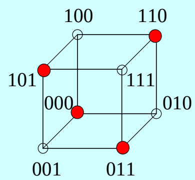
9.5 Hamming Distance

An n bit codeword can be visualised as a point in n dimensional space. For repetition and parity check codes with n = 3, we have:



Repetition Code



Parity Check Code

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Note that the repetition code vectors are separated further than those of the parity check code. This separation is expressed in terms of ***Hamming distance***, which is simply the number of positions where the digits are different.

eg. X = 1 0 1
Y = 1 1 0 d(X,Y) = 2

Hamming distance is the square of the Euclidean distance.

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The minimum distance d_{min} between code words determines the ***power*** of the code.

Detect s errors	$\Rightarrow d \geq s + 1$
Correct t errors	$\Rightarrow d \geq 2t + 1$
Correct t, detect s > t errors	$\Rightarrow d \geq s + t + 1$

Hence a triple repetition code can detect 2 errors **or** correct 1 error (in a block of 3). With $d_{min} = 7$, could correct 3 errors, **or** correct 2 & detect 4.

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To achieve error correction, we need to add ***check digits***. These are an overhead and do not carry any message information.

An (n,k) ***block code*** consists of a block of n digits, of which k are message digits and q = n – k are check digits. The ***code rate*** R is the factor by which the message rate is reduced.

$$R = k / n$$

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9.6 Probability of Error

For a matched filter receiver we have:

$$E_b = \text{Energy per message bit} = \frac{\text{Received signal power}}{\text{Message bit rate}}$$

$$E_c = \text{Energy per channel bit} = \frac{\text{Received signal power}}{\text{Channel bit rate}} = R\,E_b$$

$$P_b = Q\left\{\sqrt{E_c/\alpha}\right\} = Q\left\{\sqrt{R\,E_b/\alpha}\right\}$$

$$P_{cwe} \approx \binom{n}{t+1} P_b^{t+1} \text{ (with error correction)}$$

[P_{cwe} is the probability of a word (block) error]

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If there are t+1 errors (the most likely error scenario), the decoder will pick an adjacent code word which is distance 2t+1 away.

Hence, 2t+1 bits will be in error, giving a bit error probability after correction of:

$$P_{cbe} = \frac{2t+1}{n} P_{cwe} \approx \frac{2t+1}{n} \binom{n}{t+1} P_b^{t+1}$$

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Example: A (15,11) block code has $d_{min} = 3$, so $t = 1$ and $R = 11/15$.

With $E_b/\alpha = 13$ dB, the uncoded bit error rate is $P_{ube} = Q(\sqrt{20}) = 3.9\times10^{-6}$.

With coding, $P_b = Q\{\sqrt{(20 \times 11/15)}\} = 6.4\times10^{-5}$ before correction and after correction $P_{cbe} \approx 21P_b^2 = 8.6\times10^{-8}$. For the same message bit rate we would require a higher channel bit rate.

The error correction performance improves as E_b/α increases.

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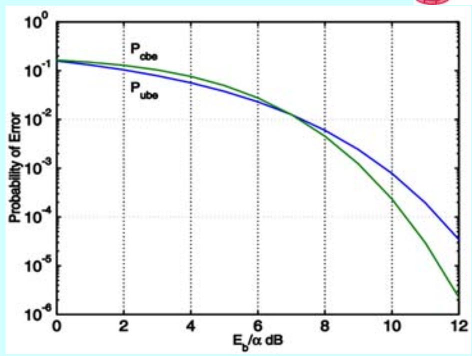

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9.7 Linear Block Codes

Block codes may be ***linear*** or ***non-linear***. A linear block code is one in which the bitwise sum (modulo 2) of any two code words is also a code word.

A code is ***systematic*** if it is formed by adding check digits to the message digits.

$$x = [m_1, m_2, \ldots, m_k, c_1, \ldots, c_q]$$

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Weight and Distance

The ***weight*** of a code word is the number of 1's in it. If x and y are two code words, then the Hamming distance between them is

$$d(x,y) = w(x+y)$$

Hence if we wish to correct one error, then $d_{min} = 3$ and all code words must have a weight of 3 or more (except the all zeros code word).

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Matrix representation

For a systematic code:

$$\tilde{x} = [x_1, x_2, \ldots, x_n]$$
$$= [m_1, \ldots, m_k, c_1, \ldots, c_q] = [\tilde{m}\ \tilde{c}]$$
$$\tilde{x} = [\tilde{m}\ \tilde{c}] = [\tilde{m}][I_k\ \ P] = \tilde{m}G$$
$$\tilde{c} = \tilde{m}P$$

I_k is (k×k) unit matrix, P is a (k×q) ***parity generation matrix*** (all elements 0 or 1). The matrix G (k×n) = $[I_k\ \ P]$ is the ***generator matrix***.

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9.8 Hamming Codes

Hamming codes are (n,k) block codes with $q \geq 3$ check digits and $n = 2^q - 1$. The minimum distance of all Hamming codes is $\mathbf{d_{min} = 3}$.

Example: q = 3, n = 7 so k = 4.

Note that all arithmetic is modulo 2, and adding is equivalent to doing a parity check.

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$$P = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$c_1 = m_1 + m_2 + m_3$$
$$c_2 = m_2 + m_3 + m_4$$
$$c_3 = m_1 + m_2 + m_4$$

(We will discuss how to find a suitable P later).

Syndrome Decoding

Decoding is conveniently carried out using a ***parity check matrix*** H (n×q) . With I_q (q×q) :

$$H = \begin{bmatrix} P \\ I_q \end{bmatrix} \quad \text{[Note that Proakis uses } H^t]$$

$$\tilde{x} H = [\tilde{m}][I_k \ \ P]\begin{bmatrix} P \\ I_q \end{bmatrix} = [\tilde{m}][P + P] = 0$$

$$\tilde{y} = \tilde{x} + \tilde{e}$$
$$\tilde{s} = \tilde{y} H = \tilde{x} H + \tilde{e} H = \tilde{e} H$$

The transmitted code word is $\tilde{x}$, the received vector $\tilde{y}$ has errors, and $\tilde{s}$ is a (1×q) vector called the ***syndrome***, which only depends on the error pattern $\tilde{e}$.

There are 2^n possible error patterns and only 2^q syndromes. This simply means that we cannot correct all possible errors. We choose to correct only the most likely, so we have ***maximum likelihood decoding***.

The most likely patterns are no errors (1) or single errors (n), a total of n+1, which matches the number of syndromes 2^q. Each of the single errors gives a row of the H matrix, so the rows of P must be different and not a row of the unit matrix.

Example:

$$H = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{matrix} \}\ P\ (k\times q) \\ \}\ I_q\ (q\times q) \end{matrix}$$

$$\tilde{m} = [1 \ \ 1 \ \ 0 \ \ 1]$$
$$\tilde{x} = [1 \ \ 1 \ \ 0 \ \ 1 \ \ 0 \ \ 0 \ \ 1]$$

$$\tilde{y} = [1 \ \ \underline{0} \ \ 0 \ \ 1 \ \ 0 \ \ 0 \ \ 1]$$
$$\tilde{s} = [1 \ \ 1 \ \ 1] = \text{second row of } H$$
$$\tilde{x} = [1 \ \ 1 \ \ 0 \ \ 1 \ \ 0 \ \ 0 \ \ 1] = \text{correct}$$

$$\tilde{y} = [1 \ \ 1 \ \ 0 \ \ 1 \ \ \underline{1} \ \ \underline{1} \ \ 1]$$
$$\tilde{s} = [1 \ \ 1 \ \ 0] = \text{third row of } H$$
$$\tilde{x} = [1 \ \ 1 \ \ 1 \ \ 1 \ \ 1 \ \ 1 \ \ 1] = \text{wrong}$$

If a code is to correct t errors, then the number of error patterns and check digits are:

$$N_e = \sum_{i=0}^{t} \binom{n}{i} \leq 2^q$$

$$\boxed{q \geq \log_2 \left\{ \sum_{i=0}^{t} \binom{n}{i} \right\}}$$

Example: To correct 2 errors with n = 15 requires q ≥ 6.92, but a code may not exist with q = 7.

9.9 Cyclic Codes

Cyclic codes are a subclass of block codes in which the cyclic structure leads to simpler coders and decoders.

A cyclic code has the property that if $\tilde{x}$ is code word, then all cyclic shifted versions of $\tilde{x}$ are also a code words.

eg. $[x_1\ x_2\ x_3\ x_4]$ & $[x_2\ x_3\ x_4\ x_1]$ are code words.

In general, if $[x_1\ x_2 \ldots x_{n-1}\ x_n]$ is a code word, then $[x_2\ x_3 \ldots x_n\ x_1]$ is also a code word.

Code words are constructed using a generator polynomial G(p) of degree q, and all code word polynomials are multiples of G(p).

For this to be true, we require that G(p) must be a factor of $p^n + 1$, although not all such factors lead to good codes.

$$X_1(p) = x_1 p^{n-1} + x_2 p^{n-2} + \ldots + x_{n-1} p + x_n$$

$$X_2(p) = x_2 p^{n-1} + x_3 p^{n-2} + \ldots + x_n p + x_1$$

$$= p X_1(p) + x_1 (p^n + 1)$$

$$G(p) = p^q + g_{q-1} p^{q-1} + \ldots + g_1 p + 1$$

$$X_1(p) = Q_1(p) G(p)$$

$$X_2(p) = p Q_1(p) G(p) + x_1 (p^n + 1)$$

$$= Q_2(p) G(p)$$

Example: A (7,4) code has $G(p) = p^3 + p + 1$, since $p^7 + 1 = (p^3 + p + 1)(p^4 + p^2 + p + 1)$.

To design a systematic cyclic code, we have:

$$X(p) = p^q M(p) + C(p)$$

$$M(p) = m_1 p^{k-1} + \ldots + m_k$$

$$C(p) = c_1 p^{q-1} + \ldots + c_q$$

$$C(p) = \text{rem}\left\{ \frac{p^q M(p)}{G(p)} \right\}$$

Example: $G(p) = p^3 + p + 1$, $M(p) = p^3 + 1$.

$p^3 M(p) = p^6 + p^3 = 1\ 0\ 0\ 1\ 0\ 0\ 0$

```
p3 p2 p1 p0     p6 p5 p4 p3 p2 p1 p0
1  0  1  1   )  1  0  0  1  0  0  0
                1  0  1  1
                   1  0  0  0
                   1  0  1  1
                      1  1  0   =  C(p)
```

$$\tilde{x} = [1\quad 0\quad 0\quad 1\quad 1\quad 1\quad 0]$$

Exercises: You are expected to attempt the following exercises in Proakis & Salehi. Completion of these exercises is part of the course. Solutions will be available later.

9.2
9.27

Trigonometric Identities

$$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$
$$\cos(A-B) = \cos(A)\cos(B) + \sin(A)\sin(B)$$
$$\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)$$
$$\sin(A-B) = \sin(A)\cos(B) - \cos(A)\sin(B)$$

$$2\cos(A)\cos(B) = \cos(A+B) + \cos(A-B)$$
$$2\cos(A)\sin(B) = \sin(A+B) - \sin(A-B)$$
$$2\sin(A)\cos(B) = \sin(A+B) + \sin(A-B)$$
$$2\sin(A)\sin(B) = \cos(A-B) - \cos(A+B)$$

$$\cos(A) + \cos(B) = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$
$$\cos(A) - \cos(B) = 2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{B-A}{2}\right)$$
$$\sin(A) + \sin(B) = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$
$$\sin(A) - \sin(B) = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

$$\cos 2\theta = 2\cos^2\theta - 1 \quad = \quad 1 - 2\sin^2\theta \quad = \quad \cos^2\theta - \sin^2\theta$$
$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$
$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$$

$$\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$$
$$\sin\ 4\theta = 4\sin\theta\cos\theta - 8\sin^3\theta\cos\theta$$

$$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$$
$$\sin 5\theta = 5\sin\theta - 20\sin^3\theta + 16\sin^5\theta$$

$$x\cos\omega t - y\sin\omega t = \sqrt{x^2 + y^2}\ \cos\{\omega t + \arg(x,y)\}$$

$$\arg(x,y) = \begin{cases} \arctan(y/x) & ; x > 0 \\ \arctan(y/x) + \pi\,\mathrm{sgn}(y) & ; x < 0 \\ \frac{\pi}{2}\mathrm{sgn}(y) & ; x = 0 \end{cases}$$

Complex Numbers

$j = \sqrt{-1}$

$z = x + jy = re^{j\theta}$ (cartesian and polar forms)

$z^* = x - jy = re^{-j\theta}$ (complex conjugate)

$x = \mathrm{Re}\{z\} = \frac{1}{2}\{z + z^*\}$ (real part)

$y = \mathrm{Im}\{z\} = \frac{1}{2j}\{z - z^*\}$ (imaginary part)

$|z| = r = \sqrt{x^2 + y^2}$ (magnitude)

$\arg(z) = \theta = \arg(x,y)$ (angle or argument)

$$\arg(x,y) = \begin{cases} \arctan(y/x) & ; x > 0 \\ \arctan(y/x) + \pi\,\mathrm{sgn}(y) & ; x < 0 \\ \frac{\pi}{2}\mathrm{sgn}(y) & ; x = 0 \end{cases}$$

$z z^* = r^2 = x^2 + y^2$

$z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$ (addition)

$z_1 z_2 = (x_1 + jy_1)(x_2 + jy_2)$ (multiplication)

$\quad = (x_1 x_2 - y_1 y_2) + j(x_1 y_2 + x_2 y_1)$

$\quad = r_1 r_2\, e^{j(\theta_1 + \theta_2)}$

$\frac{z_1}{z_2} = \frac{x_1 + jy_1}{x_2 + jy_2} = \frac{(x_1 + jy_1)(x_2 - jy_2)}{x^2 + y^2}$ (division)

$\quad = \frac{x_1 x_2 + y_1 y_2}{x^2 + y^2} + j\frac{x_2 y_1 - x_1 y_2}{x^2 + y^2}$

$\quad = \frac{r_1}{r_2} e^{j(\theta_1 - \theta_2)}$

$\ln(z) = \ln(r) + j\theta$

Phasors

$x + jy \Leftrightarrow x\cos(\omega t) - y\sin(\omega t)$ (cartesian phasor)

$r e^{j\theta} \Leftrightarrow r\cos(\omega t + \theta)$ (polar phasor)

$v(t) = \mathrm{Re}\left\{ \text{phasor} \times e^{j\omega t} \right\}$ (peak phasor)

For RMS phasors, multiply the time function by $\sqrt{2}$

Fourier Transforms

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt$$

$$x(t) = \int_{-\infty}^{\infty} X(f)\, e^{+j2\pi f t}\, df$$

Theorems

$x(t)$	$X(f)$
$x(t/T)$	$\lvert T \rvert X(fT)$
$x(t-T)$	$X(f)\, e^{-j2\pi f T}$
$x(t)\, e^{j2\pi F t}$	$X(f-F)$
$x(-t)$	$X(-f)$
$\frac{dx(t)}{dt}$	$j2\pi f\, X(f)$
$\int_{-\infty}^{t} x(\lambda)\, d\lambda$	$\frac{X(f)}{j2\pi f} + \frac{1}{2} X(0)\, \delta(f)$
$t\, x(t)$	$-\frac{1}{j2\pi} \frac{dX(f)}{df}$
$X(t)$	$x(-f)$
$rep_T \{x(t)\}$	$\lvert F \rvert comb_F(f)\, X(f)$
$\lvert T \rvert comb_T(t)\, x(t)$	$rep_F \{X(f)\}$
$x(t)\, y(t)$	$X(f) \otimes Y(f)$
$x(t) \otimes y(t)$	$X(f)\, Y(f)$
$x^*(t)$	$X^*(-f)$

Note that F and T are real constants, with FT = 1.

Transforms

$u(t)\, e^{-at}$	$\frac{1}{a + j2\pi f}$; $a > 0$
$e^{-a\lvert t \rvert}$	$\frac{2a}{a^2 + (2\pi f)^2}$; $a > 0$
$\frac{1}{a^2 + t^2}$	$\frac{\pi}{a} e^{-\lvert 2\pi f a \rvert}$
$\delta(t)$	1
1	$\delta(f)$
$u(t)$	$\frac{1}{j2\pi f} + \frac{1}{2}\delta(f)$
$sgn(t)$	$\frac{1}{j\pi f}$
$\frac{1}{\pi t}$	$-j\, sgn(f)$
$rect(t/T)$	$\lvert T \rvert sinc(fT)$
$sinc(t/T)$	$\lvert T \rvert rect(fT)$
$\Delta(t/T)$	$\lvert T \rvert sinc^2(fT)$
$comb_T(t)$	$\lvert F \rvert comb_F(f)$
$e^{-t^2/2T^2}$	$\lvert T \rvert \sqrt{2\pi}\, e^{-\frac{1}{2}(2\pi f T)^2}$
$sgn(t)\, rect(t/T)$	$\frac{1 - \cos(\pi f T)}{j\pi f}$

Note that a is a real positive constant.

Definitions

$$\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$

$$\text{rep}_P\{f(x)\} = \sum_{n=-\infty}^{\infty} f(x - nP)$$

$$\text{comb}_P(x) = \sum_{n=-\infty}^{\infty} \delta(x - nP)$$

$$u(x) = \begin{cases} 0 & ; x < 0 \\ 1 & ; x > 0 \end{cases}$$

$$\delta(x) = \text{unit impulse (area} = 1)$$

$$\text{sgn}(x) = \begin{cases} -1 & ; x < 0 \\ +1 & ; x > 0 \end{cases}$$

$$\text{rect}(x) = \begin{cases} 1 & ; |x| < 0.5 \\ 0 & ; |x| > 0.5 \end{cases}$$

$$\Delta(x) = \begin{cases} 1 - |x| & ; |x| < 1 \\ 0 & ; |x| > 1 \end{cases}$$

$$f(x) \otimes g(x) = \int_{-\infty}^{\infty} f(\lambda)\, g(x - \lambda)\, d\lambda$$

Relations

$$x(0) = \int_{-\infty}^{\infty} X(f)\, df = \text{area of } X(f)$$

$$X(0) = \int_{-\infty}^{\infty} x(t)\, dt = \text{area of } x(t)$$

$$X(-f) = X^*(f) \quad \text{if } x(t) \text{ is real}$$

$$X(f) = \text{real \& even if } x(t) \text{ real \& even}$$

$$X(f) = \text{imaginary \& odd if } x(t) \text{ real \& odd}$$

$$\int_{-\infty}^{\infty} x(t)\, y^*(t)\, dt = \int_{-\infty}^{\infty} X(f)\, Y^*(f)\, df$$

$$\int_{-\infty}^{\infty} |x(t)|^2\, dt = \int_{-\infty}^{\infty} |X(f)|^2\, df$$

Unless otherwise stated, these relations are true for x(t) real or complex.

Table of the Q Function

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-t^2/2}\, dt$$

	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	5.000E-01	4.960E-01	4.920E-01	4.880E-01	4.840E-01	4.801E-01	4.761E-01	4.721E-01	4.681E-01	4.641E-01
0.1	4.602E-01	4.562E-01	4.522E-01	4.483E-01	4.443E-01	4.404E-01	4.364E-01	4.325E-01	4.286E-01	4.247E-01
0.2	4.207E-01	4.168E-01	4.129E-01	4.090E-01	4.052E-01	4.013E-01	3.974E-01	3.936E-01	3.897E-01	3.859E-01
0.3	3.821E-01	3.783E-01	3.745E-01	3.707E-01	3.669E-01	3.632E-01	3.594E-01	3.557E-01	3.520E-01	3.483E-01
0.4	3.446E-01	3.409E-01	3.372E-01	3.336E-01	3.300E-01	3.264E-01	3.228E-01	3.192E-01	3.156E-01	3.121E-01
0.5	3.085E-01	3.050E-01	3.015E-01	2.981E-01	2.946E-01	2.912E-01	2.877E-01	2.843E-01	2.810E-01	2.776E-01
0.6	2.743E-01	2.709E-01	2.676E-01	2.643E-01	2.611E-01	2.578E-01	2.546E-01	2.514E-01	2.483E-01	2.451E-01
0.7	2.420E-01	2.389E-01	2.358E-01	2.327E-01	2.296E-01	2.266E-01	2.236E-01	2.206E-01	2.177E-01	2.148E-01
0.8	2.119E-01	2.090E-01	2.061E-01	2.033E-01	2.005E-01	1.977E-01	1.949E-01	1.922E-01	1.894E-01	1.867E-01
0.9	1.841E-01	1.814E-01	1.788E-01	1.762E-01	1.736E-01	1.711E-01	1.685E-01	1.660E-01	1.635E-01	1.611E-01
1.0	1.587E-01	1.562E-01	1.539E-01	1.515E-01	1.492E-01	1.469E-01	1.446E-01	1.423E-01	1.401E-01	1.379E-01
1.1	1.357E-01	1.335E-01	1.314E-01	1.292E-01	1.271E-01	1.251E-01	1.230E-01	1.210E-01	1.190E-01	1.170E-01
1.2	1.151E-01	1.131E-01	1.112E-01	1.093E-01	1.075E-01	1.056E-01	1.038E-01	1.020E-01	1.003E-01	9.853E-02
1.3	9.680E-02	9.510E-02	9.342E-02	9.176E-02	9.012E-02	8.851E-02	8.692E-02	8.534E-02	8.379E-02	8.226E-02
1.4	8.076E-02	7.927E-02	7.780E-02	7.636E-02	7.493E-02	7.353E-02	7.215E-02	7.078E-02	6.944E-02	6.811E-02
1.5	6.681E-02	6.552E-02	6.426E-02	6.301E-02	6.178E-02	6.057E-02	5.938E-02	5.821E-02	5.705E-02	5.592E-02
1.6	5.480E-02	5.370E-02	5.262E-02	5.155E-02	5.050E-02	4.947E-02	4.846E-02	4.746E-02	4.648E-02	4.551E-02
1.7	4.457E-02	4.363E-02	4.272E-02	4.182E-02	4.093E-02	4.006E-02	3.920E-02	3.836E-02	3.754E-02	3.673E-02
1.8	3.593E-02	3.515E-02	3.438E-02	3.362E-02	3.288E-02	3.216E-02	3.144E-02	3.074E-02	3.005E-02	2.938E-02
1.9	2.872E-02	2.807E-02	2.743E-02	2.680E-02	2.619E-02	2.559E-02	2.500E-02	2.442E-02	2.385E-02	2.330E-02
2.0	2.275E-02	2.222E-02	2.169E-02	2.118E-02	2.068E-02	2.018E-02	1.970E-02	1.923E-02	1.876E-02	1.831E-02
2.1	1.786E-02	1.743E-02	1.700E-02	1.659E-02	1.618E-02	1.578E-02	1.539E-02	1.500E-02	1.463E-02	1.426E-02
2.2	1.390E-02	1.355E-02	1.321E-02	1.287E-02	1.255E-02	1.222E-02	1.191E-02	1.160E-02	1.130E-02	1.101E-02
2.3	1.072E-02	1.044E-02	1.017E-02	9.903E-03	9.642E-03	9.387E-03	9.137E-03	8.894E-03	8.656E-03	8.424E-03
2.4	8.198E-03	7.976E-03	7.760E-03	7.549E-03	7.344E-03	7.143E-03	6.947E-03	6.756E-03	6.569E-03	6.387E-03
2.5	6.210E-03	6.037E-03	5.868E-03	5.703E-03	5.543E-03	5.386E-03	5.234E-03	5.085E-03	4.940E-03	4.799E-03
2.6	4.661E-03	4.527E-03	4.397E-03	4.269E-03	4.145E-03	4.025E-03	3.907E-03	3.793E-03	3.681E-03	3.573E-03
2.7	3.467E-03	3.364E-03	3.264E-03	3.167E-03	3.072E-03	2.980E-03	2.890E-03	2.803E-03	2.718E-03	2.635E-03
2.8	2.555E-03	2.477E-03	2.401E-03	2.327E-03	2.256E-03	2.186E-03	2.118E-03	2.052E-03	1.988E-03	1.926E-03
2.9	1.866E-03	1.807E-03	1.750E-03	1.695E-03	1.641E-03	1.589E-03	1.538E-03	1.489E-03	1.441E-03	1.395E-03
3.0	1.350E-03	1.306E-03	1.264E-03	1.223E-03	1.183E-03	1.144E-03	1.107E-03	1.070E-03	1.035E-03	1.001E-03
3.1	9.676E-04	9.354E-04	9.043E-04	8.740E-04	8.447E-04	8.164E-04	7.888E-04	7.622E-04	7.364E-04	7.114E-04
3.2	6.871E-04	6.637E-04	6.410E-04	6.190E-04	5.976E-04	5.770E-04	5.571E-04	5.377E-04	5.190E-04	5.009E-04
3.3	4.834E-04	4.665E-04	4.501E-04	4.342E-04	4.189E-04	4.041E-04	3.897E-04	3.758E-04	3.624E-04	3.495E-04
3.4	3.369E-04	3.248E-04	3.131E-04	3.018E-04	2.909E-04	2.803E-04	2.701E-04	2.602E-04	2.507E-04	2.415E-04
3.5	2.326E-04	2.241E-04	2.158E-04	2.078E-04	2.001E-04	1.926E-04	1.854E-04	1.785E-04	1.718E-04	1.653E-04
3.6	1.591E-04	1.531E-04	1.473E-04	1.417E-04	1.363E-04	1.311E-04	1.261E-04	1.213E-04	1.166E-04	1.121E-04
3.7	1.078E-04	1.036E-04	9.961E-05	9.574E-05	9.201E-05	8.842E-05	8.496E-05	8.162E-05	7.841E-05	7.532E-05
3.8	7.235E-05	6.948E-05	6.673E-05	6.407E-05	6.152E-05	5.906E-05	5.669E-05	5.442E-05	5.223E-05	5.012E-05
3.9	4.810E-05	4.615E-05	4.427E-05	4.247E-05	4.074E-05	3.908E-05	3.747E-05	3.594E-05	3.446E-05	3.304E-05
4.0	3.167E-05	3.036E-05	2.910E-05	2.789E-05	2.673E-05	2.561E-05	2.454E-05	2.351E-05	2.252E-05	2.157E-05
4.1	2.066E-05	1.978E-05	1.894E-05	1.814E-05	1.737E-05	1.662E-05	1.591E-05	1.523E-05	1.458E-05	1.395E-05
4.2	1.335E-05	1.277E-05	1.222E-05	1.168E-05	1.118E-05	1.069E-05	1.022E-05	9.774E-06	9.345E-06	8.934E-06
4.3	8.540E-06	8.163E-06	7.801E-06	7.455E-06	7.124E-06	6.807E-06	6.503E-06	6.212E-06	5.934E-06	5.668E-06
4.4	5.413E-06	5.169E-06	4.935E-06	4.712E-06	4.498E-06	4.294E-06	4.098E-06	3.911E-06	3.732E-06	3.561E-06
4.5	3.398E-06	3.241E-06	3.092E-06	2.949E-06	2.813E-06	2.682E-06	2.558E-06	2.439E-06	2.325E-06	2.216E-06
4.6	2.112E-06	2.013E-06	1.919E-06	1.828E-06	1.742E-06	1.660E-06	1.581E-06	1.506E-06	1.434E-06	1.366E-06
4.7	1.301E-06	1.239E-06	1.179E-06	1.123E-06	1.069E-06	1.017E-06	9.680E-07	9.211E-07	8.765E-07	8.339E-07
4.8	7.933E-07	7.547E-07	7.178E-07	6.827E-07	6.492E-07	6.173E-07	5.869E-07	5.580E-07	5.304E-07	5.042E-07
4.9	4.792E-07	4.554E-07	4.327E-07	4.111E-07	3.906E-07	3.711E-07	3.525E-07	3.348E-07	3.179E-07	3.019E-07

Table of the Q Function

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-t^2/2}\, dt$$

	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
5.0	2.867E-07	2.722E-07	2.584E-07	2.452E-07	2.328E-07	2.209E-07	2.096E-07	1.989E-07	1.887E-07	1.790E-07
5.1	1.698E-07	1.611E-07	1.528E-07	1.449E-07	1.374E-07	1.302E-07	1.235E-07	1.170E-07	1.109E-07	1.051E-07
5.2	9.964E-08	9.442E-08	8.946E-08	8.476E-08	8.029E-08	7.605E-08	7.203E-08	6.821E-08	6.459E-08	6.116E-08
5.3	5.790E-08	5.481E-08	5.188E-08	4.911E-08	4.647E-08	4.398E-08	4.161E-08	3.937E-08	3.724E-08	3.523E-08
5.4	3.332E-08	3.151E-08	2.980E-08	2.818E-08	2.664E-08	2.518E-08	2.381E-08	2.250E-08	2.127E-08	2.010E-08
5.5	1.899E-08	1.794E-08	1.695E-08	1.601E-08	1.512E-08	1.428E-08	1.349E-08	1.274E-08	1.203E-08	1.135E-08
5.6	1.072E-08	1.012E-08	9.548E-09	9.010E-09	8.503E-09	8.022E-09	7.569E-09	7.140E-09	6.735E-09	6.352E-09
5.7	5.990E-09	5.649E-09	5.326E-09	5.022E-09	4.734E-09	4.462E-09	4.206E-09	3.964E-09	3.735E-09	3.519E-09
5.8	3.316E-09	3.124E-09	2.942E-09	2.771E-09	2.610E-09	2.458E-09	2.314E-09	2.179E-09	2.051E-09	1.931E-09
5.9	1.818E-09	1.711E-09	1.610E-09	1.515E-09	1.425E-09	1.341E-09	1.261E-09	1.186E-09	1.116E-09	1.049E-09
6.0	9.866E-10	9.276E-10	8.721E-10	8.198E-10	7.706E-10	7.242E-10	6.806E-10	6.396E-10	6.009E-10	5.646E-10
6.1	5.303E-10	4.982E-10	4.679E-10	4.394E-10	4.126E-10	3.874E-10	3.637E-10	3.414E-10	3.205E-10	3.008E-10
6.2	2.823E-10	2.649E-10	2.486E-10	2.332E-10	2.188E-10	2.052E-10	1.925E-10	1.805E-10	1.693E-10	1.587E-10
6.3	1.488E-10	1.395E-10	1.308E-10	1.226E-10	1.149E-10	1.077E-10	1.009E-10	9.451E-11	8.854E-11	8.294E-11
6.4	7.769E-11	7.276E-11	6.814E-11	6.380E-11	5.974E-11	5.593E-11	5.235E-11	4.900E-11	4.586E-11	4.292E-11
6.5	4.016E-11	3.758E-11	3.515E-11	3.288E-11	3.076E-11	2.877E-11	2.690E-11	2.516E-11	2.352E-11	2.199E-11
6.6	2.056E-11	1.922E-11	1.796E-11	1.678E-11	1.568E-11	1.465E-11	1.369E-11	1.279E-11	1.195E-11	1.116E-11
6.7	1.042E-11	9.731E-12	9.086E-12	8.483E-12	7.919E-12	7.392E-12	6.900E-12	6.439E-12	6.009E-12	5.607E-12
6.8	5.231E-12	4.880E-12	4.552E-12	4.246E-12	3.960E-12	3.692E-12	3.443E-12	3.210E-12	2.993E-12	2.790E-12
6.9	2.600E-12	2.423E-12	2.258E-12	2.104E-12	1.960E-12	1.826E-12	1.701E-12	1.585E-12	1.476E-12	1.374E-12
7.0	1.280E-12	1.192E-12	1.109E-12	1.033E-12	9.612E-13	8.946E-13	8.325E-13	7.747E-13	7.208E-13	6.706E-13
7.1	6.238E-13	5.802E-13	5.396E-13	5.018E-13	4.667E-13	4.339E-13	4.034E-13	3.750E-13	3.486E-13	3.240E-13
7.2	3.011E-13	2.798E-13	2.599E-13	2.415E-13	2.243E-13	2.084E-13	1.935E-13	1.797E-13	1.669E-13	1.550E-13
7.3	1.439E-13	1.336E-13	1.240E-13	1.151E-13	1.068E-13	9.910E-14	9.196E-14	8.531E-14	7.914E-14	7.341E-14
7.4	6.809E-14	6.315E-14	5.856E-14	5.430E-14	5.034E-14	4.667E-14	4.326E-14	4.010E-14	3.716E-14	3.444E-14
7.5	3.191E-14	2.956E-14	2.739E-14	2.537E-14	2.350E-14	2.176E-14	2.015E-14	1.866E-14	1.728E-14	1.600E-14
7.6	1.481E-14	1.370E-14	1.268E-14	1.174E-14	1.086E-14	1.005E-14	9.297E-15	8.600E-15	7.954E-15	7.357E-15
7.7	6.803E-15	6.291E-15	5.816E-15	5.377E-15	4.971E-15	4.595E-15	4.246E-15	3.924E-15	3.626E-15	3.350E-15
7.8	3.095E-15	2.859E-15	2.641E-15	2.439E-15	2.253E-15	2.080E-15	1.921E-15	1.773E-15	1.637E-15	1.511E-15
7.9	1.395E-15	1.287E-15	1.188E-15	1.096E-15	1.011E-15	9.326E-16	8.602E-16	7.934E-16	7.317E-16	6.747E-16
8.0	6.221E-16	5.735E-16	5.287E-16	4.874E-16	4.492E-16	4.140E-16	3.815E-16	3.515E-16	3.238E-16	2.983E-16
8.1	2.748E-16	2.531E-16	2.331E-16	2.146E-16	1.976E-16	1.820E-16	1.675E-16	1.542E-16	1.419E-16	1.306E-16
8.2	1.202E-16	1.106E-16	1.018E-16	9.361E-17	8.611E-17	7.920E-17	7.284E-17	6.698E-17	6.159E-17	5.662E-17
8.3	5.206E-17	4.785E-17	4.398E-17	4.042E-17	3.715E-17	3.413E-17	3.136E-17	2.881E-17	2.646E-17	2.431E-17
8.4	2.232E-17	2.050E-17	1.882E-17	1.728E-17	1.587E-17	1.457E-17	1.337E-17	1.227E-17	1.126E-17	1.033E-17
8.5	9.480E-18	8.697E-18	7.978E-18	7.317E-18	6.711E-18	6.154E-18	5.643E-18	5.174E-18	4.744E-18	4.348E-18
8.6	3.986E-18	3.653E-18	3.348E-18	3.068E-18	2.811E-18	2.575E-18	2.359E-18	2.161E-18	1.979E-18	1.812E-18
8.7	1.659E-18	1.519E-18	1.391E-18	1.273E-18	1.166E-18	1.067E-18	9.763E-19	8.933E-19	8.174E-19	7.478E-19
8.8	6.841E-19	6.257E-19	5.723E-19	5.234E-19	4.786E-19	4.376E-19	4.001E-19	3.657E-19	3.343E-19	3.055E-19
8.9	2.792E-19	2.552E-19	2.331E-19	2.130E-19	1.946E-19	1.777E-19	1.623E-19	1.483E-19	1.354E-19	1.236E-19
9.0	1.129E-19	1.030E-19	9.404E-20	8.584E-20	7.834E-20	7.148E-20	6.523E-20	5.951E-20	5.429E-20	4.952E-20
9.1	4.517E-20	4.119E-20	3.756E-20	3.425E-20	3.123E-20	2.847E-20	2.595E-20	2.365E-20	2.155E-20	1.964E-20
9.2	1.790E-20	1.631E-20	1.486E-20	1.353E-20	1.232E-20	1.122E-20	1.022E-20	9.307E-21	8.474E-21	7.714E-21
9.3	7.022E-21	6.392E-21	5.817E-21	5.294E-21	4.817E-21	4.382E-21	3.987E-21	3.627E-21	3.299E-21	3.000E-21
9.4	2.728E-21	2.481E-21	2.255E-21	2.050E-21	1.864E-21	1.694E-21	1.540E-21	1.399E-21	1.271E-21	1.155E-21
9.5	1.049E-21	9.533E-22	8.659E-22	7.864E-22	7.142E-22	6.485E-22	5.888E-22	5.345E-22	4.852E-22	4.404E-22
9.6	3.997E-22	3.627E-22	3.292E-22	2.986E-22	2.709E-22	2.458E-22	2.229E-22	2.022E-22	1.834E-22	1.663E-22
9.7	1.507E-22	1.367E-22	1.239E-22	1.123E-22	1.018E-22	9.223E-23	8.358E-23	7.573E-23	6.861E-23	6.215E-23
9.8	5.629E-23	5.098E-23	4.617E-23	4.181E-23	3.786E-23	3.427E-23	3.102E-23	2.808E-23	2.542E-23	2.300E-23
9.9	2.081E-23	1.883E-23	1.704E-23	1.541E-23	1.394E-23	1.261E-23	1.140E-23	1.031E-23	9.323E-24	8.429E-24

ELECENG 4035 - COMMUNICATIONS IV
Tutorial 1

1 Exercise 2.10 from Proakis and Salehi

This question is intended to help you practise finding Fourier transforms for the sort of signals we will encounter in this course. Make sure you have the handout showing Fourier transforms and theorems in front of you.

It is also intended to help you become familiar with some of the special functions we use, in particular rect, sinc and Δ.

Determine the Fourier transform of each of the following signals (α is positive).

1. $x(t) = \frac{1}{1+t^2}$
2. $x(t) = \mathrm{rect}(t-3) + \mathrm{rect}(t+3)$
3. $x(t) = \Delta(2t+3) + \Delta(3t-2)$
4. $x(t) = \mathrm{sinc}^3(t)$
5. $x(t) = t\mathrm{sinc}(t)$
6. $x(t) = t\cos(2\pi f_o t)$
7. $x(t) = \exp(-\alpha t)\cos(\beta t)$
8. $x(t) = t\exp(-\alpha t)\cos(\beta t)$

2 Exercise 2.56 from Proakis and Salehi

This question is intended to help you understand why we might use analytic signals.

The bandpass signal $x(t) = \mathrm{sinc}(t)\cos(2\pi f_o t)$ is passed through a bandpass filter with impulse response $h(t) = \mathrm{sinc}^2(t)\sin(2\pi f_o t)$. Using the lowpass equivalents of both the input and the impulse response, find the lowpass equivalent of the output and from it the output $y(t)$.

ELECENG 4035 - COMMUNICATIONS IV
Tutorial 2

1 Exercise 3.2 from Proakis and Salehi

In a double sideband (DSB) system the carrier is $c(t) = A\cos(2\pi f_c t)$ and the message signal is given by $m(t) = \text{sinc}(t) + \text{sinc}^2(t)$. Find the frequency domain representation and the bandwidth of the modulated signal.

[Note that this question, as written in the textbook, is slightly ambiguous; that is, it could be DSB suppressed carrier (DSBSC), or double sideband transmitted carrier (which we call AM). I suggest answering this question for AM, in which case the answer is a function of the modulation index, a. The answer for DSBSC then follows from a simplification of the AM case.]

2 Exercise 3.7 from Proakis and Salehi

An AM signal has the form

$$u(t) = [20 + 2\cos(3000\pi t) + 10\cos(6000\pi t)]\cos(2\pi f_c t),$$

where $f_c = 10^5$ Hz.

1. Sketch the (voltage) spectrum of $u(t)$.
2. Determine the power in each of the frequency components.
3. Determine the modulation index.
4. Determine the power in the sidebands, the total power, and the ratio of the sidebands' power to the total power.

3 Exercise 3.30 from Proakis and Salehi

An FM signal is given as

$$u(t) = 100\cos\left[2\pi f_c t + 100\int_{-\infty}^{t} m(\tau)d\tau\right],$$

where $m(t)$ is shown below.

1. Sketch the instantaneous frequency as a function of time.
2. Determine the peak-frequency deviation.

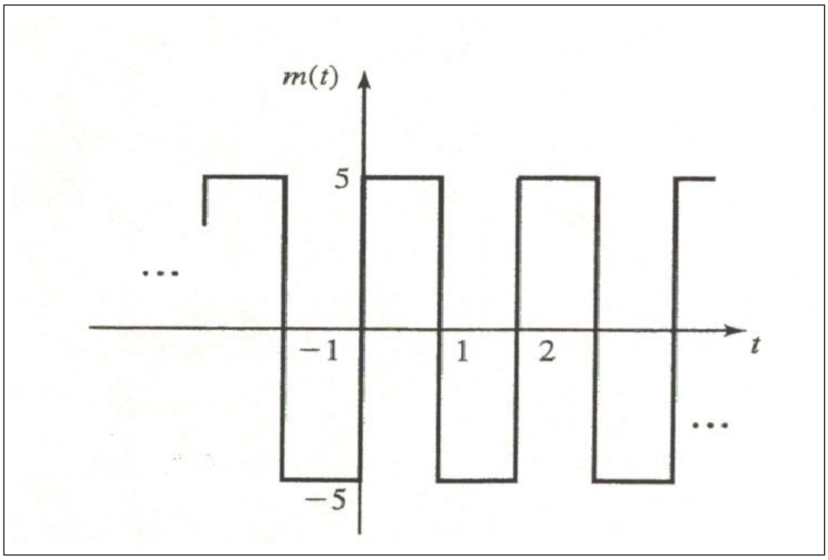


Figure 1:

4 Exercise 4.69 from Proakis and Salehi

A zero-mean white Gaussian noise signal, $n_w(t)$, with power-spectral density $\frac{N_0}{2}$, is passed through an ideal filter whose passband is from 3–11 kHz. the output process is denoted by $n(t)$.

1. If $f_0 = 7$ kHz, find $S_{n_c}(f)$, $S_{n_s}(f)$, and $R_{n_c n_s}(\tau)$, where $n_c(t)$ and $n_s(t)$ are the in-phase and quadrature components of $n(t)$.

2. Repeat part 1 with $f_0 = 6$ kHz.

ELECENG 4035 - COMMUNICATIONS IV
Tutorial 3

1 Question 2(a) from the 2006 Exam

In a broadcast communication system the transmit power is 9 kW, the channel attenuation is 80 dB, the noise power spectral density is $S_{nn}(f) = N_o/2$ with $N_o = 1.5 \times 10^{-10}$ W/Hz and the normalised baseband message signal $m(t)$ has a bandwidth of 15 kHz, $|m(t)| \leq 1$ and a mean square value $\langle m^2(t) \rangle = 0.1$.

(i) If the modulation used is amplitude modulation (AM) with a modulation index a = 0.90, calculate the following for a receiver with bandwidth equal to that of the signal:

- the bandwidth of the signal;
- the predetection signal to noise ratio (SNR_p) in decibels;
- the output signal to noise ratio (SNR_o) in decibels.

(ii) If the modulation used is frequency modulation (FM) with peak frequency deviation 75 kHz, calculate the following for a receiver with a bandwidth given by Carson's rule:

- the (approximate) bandwidth of the signal;
- the predetection signal to noise ratio (SNR_p) in decibels;
- the output signal to noise ratio (SNR_o) in decibels.

(iii) What is the maximum channel attenuation (in decibels) allowed if the FM system in (ii) is to be above threshold?

2 Adapted from Exercise 6.26 from Proakis and Salehi

Design a *ternary* Huffman code for a source with output alphabet probabilities given by $\{0.05, 0.1, 0.15, 0.17, 0.13, 0.4\}$. What is the entropy of the source? What is the average codelength of your Huffman code, and the coding efficiency?

Hint 1: *Ternary* means the Huffman code has three symbols, instead of two.

Hint 2: You can add a dummy source output, with zero probability.

3 Exercise 7.1 from Proakis and Salehi

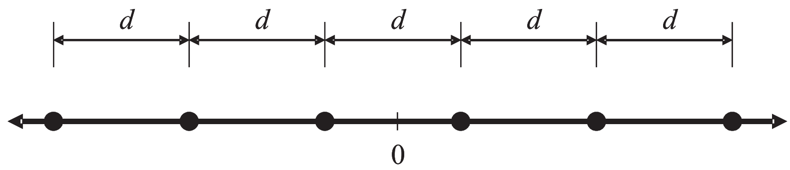
Determine the average energy of a set of M PAM signals of the form

$$s_m(t) = s_m \psi(t), \quad m = 1, 2, .., M \quad 0 \le t \le T,$$

where

$$s_m = \sqrt{\xi_g} A_m, \quad m = 1, 2, .., M.$$

The signals are equally probable with amplitudes that are symmetric about zero and are uniformly spaced with distance d between adjacent amplitudes, as shown below.



Hint:

$$\sum_{m=1}^{M} m = \frac{M(M+1)}{2},$$

and

$$\sum_{m=1}^{M} m^2 = \frac{M(M+1)(2M+1)}{6}.$$

ELECENG 4035 - COMMUNICATIONS IV
Tutorial 4

1 Question 7.19 From Proakis and Selahi

Three messages m_1, m_2 and m_3, are to be transmitted over an AWGN channel with noise power-spectral density $N_0/2$. The messages are

$$s_1(t) = \begin{cases} 1 & 0 \le t \le T \\ 0 & \text{otherwise} \end{cases}$$

$$s_2(t) = -s_3(t) = \begin{cases} 1 & 0 \le t \le T/2 \\ -1 & T/2 \le t \le T \\ 0 & \text{otherwise.} \end{cases}$$

1. What is the dimensionality of the signal space?
2. Find an appropriate basis for the signal space (Hint: You can find the basis without using the Gram-Schmidt procedure outlined in the textbook).
3. Draw the signal constellation for this problem.
4. Derive and sketch the optimal decision regions, R_1, R_2, and R_3.
5. Which of the three messages is more vulnerable to error and why? In other words which of $P(\text{Error}|s_i \text{ transmitted}),\ i = 1, 2, 3$ is larger?

2 Question 8.18 From Proakis and Selahi

A voice-band telephone channel passes the frequencies in the band from 300 to 3300 Hz. It is desired to design a modem that transmits at a symbol rate of 2400 symbols/sec, with the objective of achieving 9600 bits/sec. Select an appropriate QAM signal constellation, carrier frequency, and the roll-off factor of a pulse with a raised cosine spectrum that utilizes the entire frequency band. Sketch the spectrum of the transmitted signal pulse and indicate the important frequencies.

3 Adapted From Question 9.22 in Proakis and Selahi

A $(5, 2)$ code is defined by

$$C = \{00000, 10100, 01111, 11011\}.$$

1. Verify that this code is linear.

2. What is the minimum distance of this code?
3. Which code word(s) is (are) minimum weight?