Section 2: Frequency Domain Analysis

Contents

2. Frequency Domain Analysis

2.1 Fourier Series

A signal x(t) which has period T can be expressed as:

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn2\pi t/T}$$

$$X_n = \frac{1}{T}\int_{(T)} x(t) e^{-jn2\pi t/T} dt$$

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Notes:

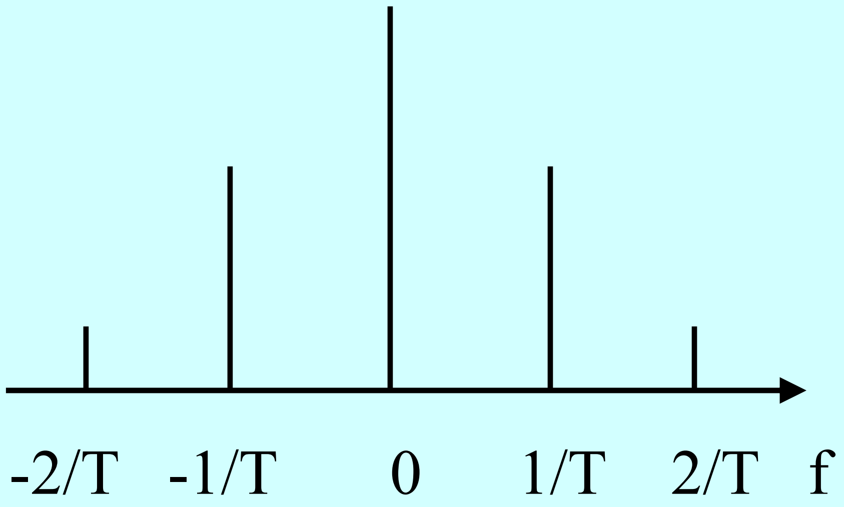
- (T) means integration over any time interval of length T.

- We will use f in Hz as the frequency variable. The symbol ω will only be used to represent 2πf.

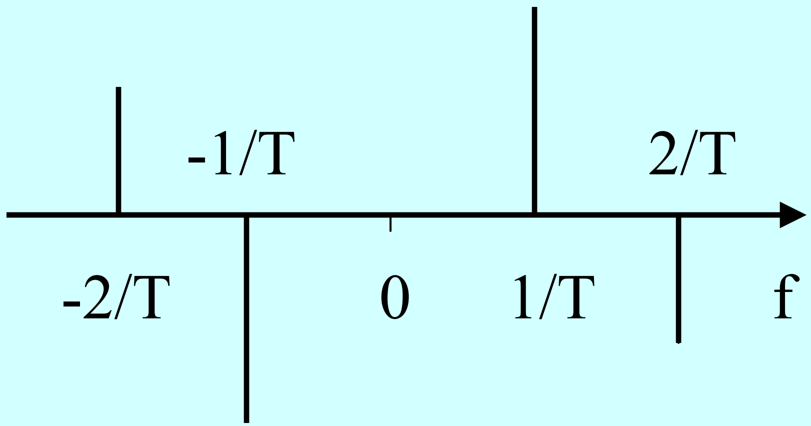
- Frequency f is a property of the complex exponential $e^{j2\pi ft}$ and may be positive or negative. Real signals contain both positive and negative frequencies. eg. $\cos(2\pi ft) = 0.5e^{j2\pi ft} + 0.5e^{-j2\pi ft}$.

- The number X_n is a complex number (dimensions V) and represents a component of frequency n/T Hz, and for a real signal $X_{-n} = X_n^*$. This is called the ***Hermitian*** property.
- The ***fundamental frequency*** is $f_o = 1/T$ corresponding to n = 1. All other frequency components are multiples of this frequency.
- The ***amplitude spectrum*** $|X_n|$ of a real signal is an even function of frequency, whereas the ***phase spectrum*** arg X_n is an odd function.

Amplitude Spectrum



Phase Spectrum



- The average ***power*** is obtained by averaging over one period.

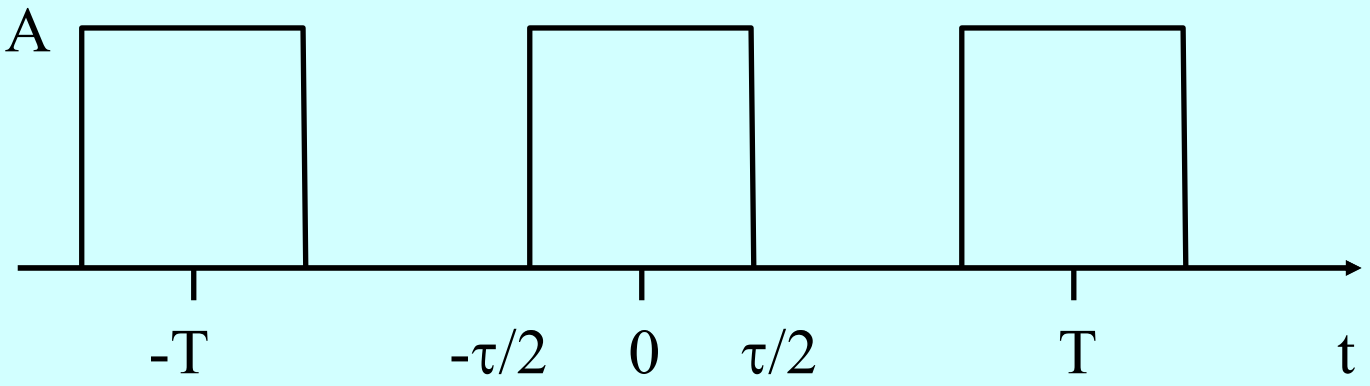
$$P = \frac{1}{T}\int_{(T)} |x(t)|^2 dt = \frac{1}{T}\int_{(T)} x(t)\, x^*(t)\, dt$$

$$= \frac{1}{T}\int_{(T)} x(t)\left[\sum_{n=-\infty}^{\infty} X_n^*\, e^{-jn2\pi t/T}\right] dt$$

$$= \sum_{n=-\infty}^{\infty} X_n\, X_n^* = \sum_{n=-\infty}^{\infty} |X_n|^2$$

This is ***Parseval's Theorem***. Note that P is actually the mean square value.

Example: Rectangular pulse train.



$$x(t) = \text{rep}_T\left\{A\,\text{rect}(t/\tau)\right\}$$

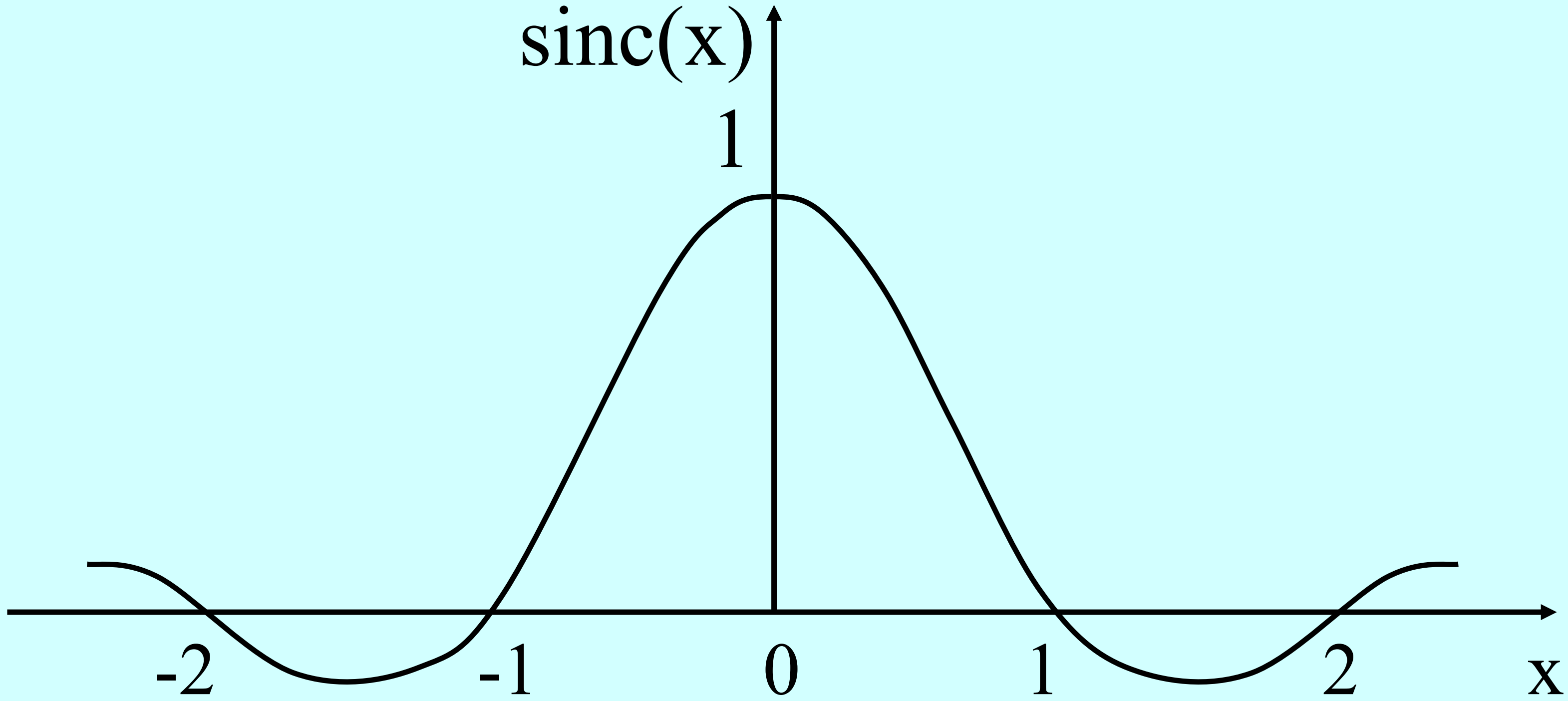
$$X_n = \frac{1}{T}\int_{-\tau/2}^{\tau/2} A\, e^{-jn2\pi t/T} dt$$

$$= \frac{A}{j2\pi n}\left(e^{jn\pi\tau/T} - e^{-jn\pi\tau/T}\right)$$

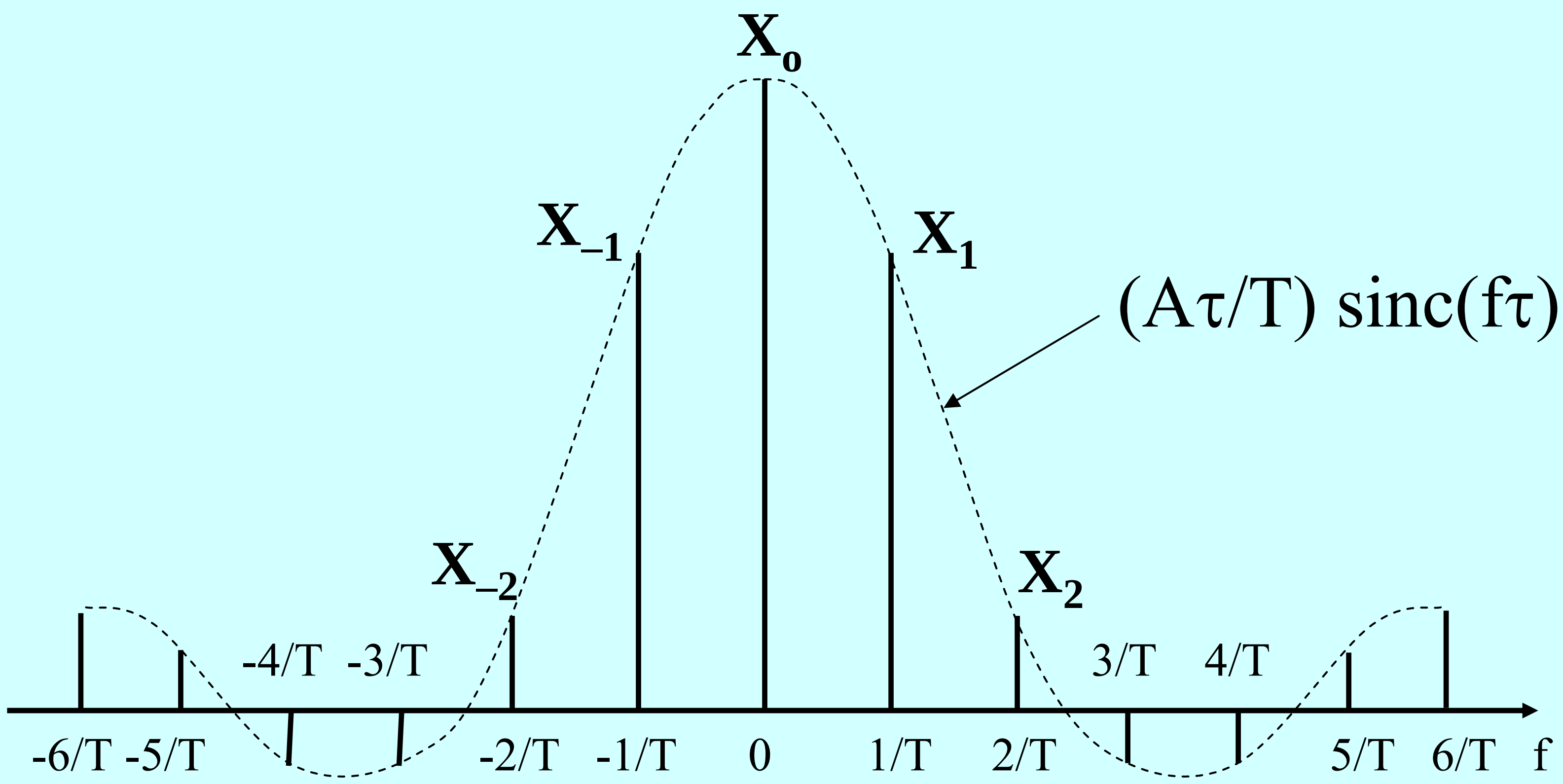
$$= \frac{A\tau}{T}\,\frac{\sin(n\pi\tau/T)}{n\pi\tau/T}$$

$$= \frac{A\tau}{T}\,\mathrm{sinc}(n\tau/T)$$

where sinc(x) = sin(πx)/(πx).



The Fourier series spectrum is:



2.2 Fourier Transforms

The ***Fourier transform*** is an extension of Fourier series to non-periodic signals.

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt$$

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df$$

We will use the notation $x(t) \leftrightarrow X(f)$. If $x(t)$ is in volts, the dimensions of $X(f)$ are volt-sec or V/Hz.

Fourier transforms exist for signals of ***finite energy***.

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} x(t)\, x^*(t)\, dt$$

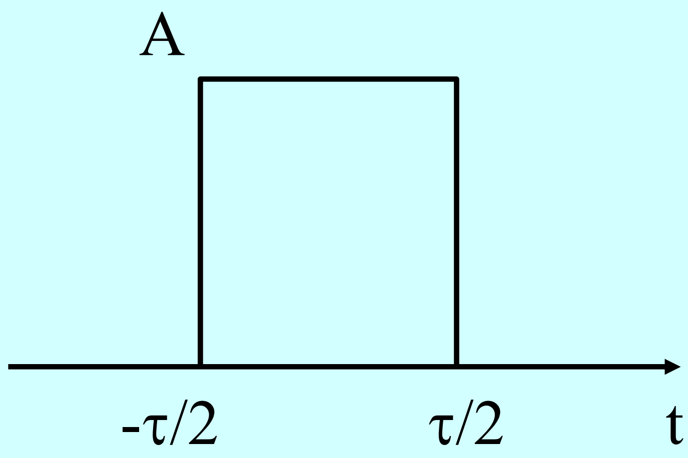
$$= \int_{-\infty}^{\infty} x(t) \left\{ \int_{-\infty}^{\infty} X^*(f)\, e^{-j2\pi f t} df \right\} dt$$

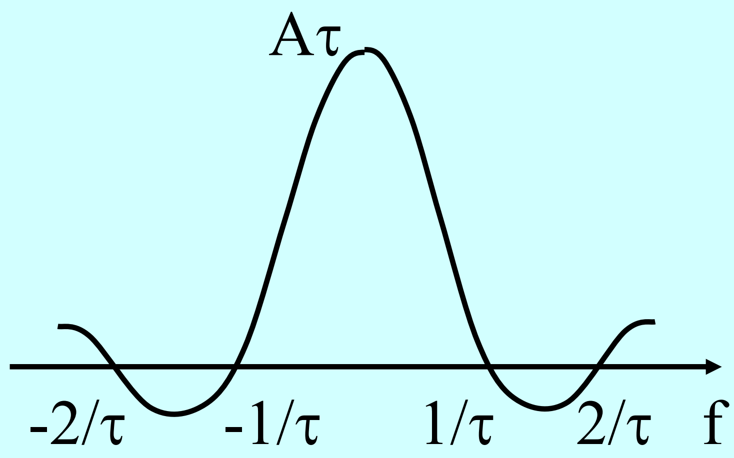
$$= \int_{-\infty}^{\infty} X(f)\, X^*(f)\, df = \int_{-\infty}^{\infty} |X(f)|^2 df$$

This is ***Rayleigh's Energy Theorem***.

Note that $G_{xx}(f) = |X(f)|^2$ is also called the ***energy spectral density*** of the finite energy signal.

Example: Rectangular pulse

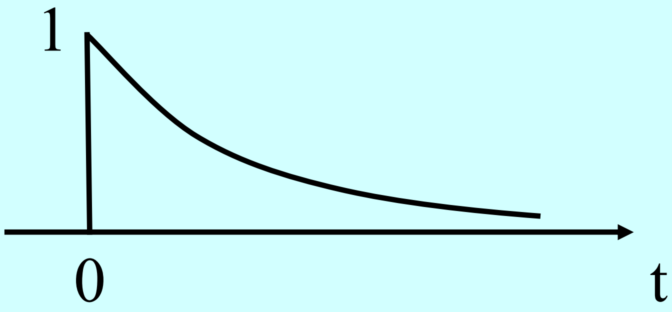


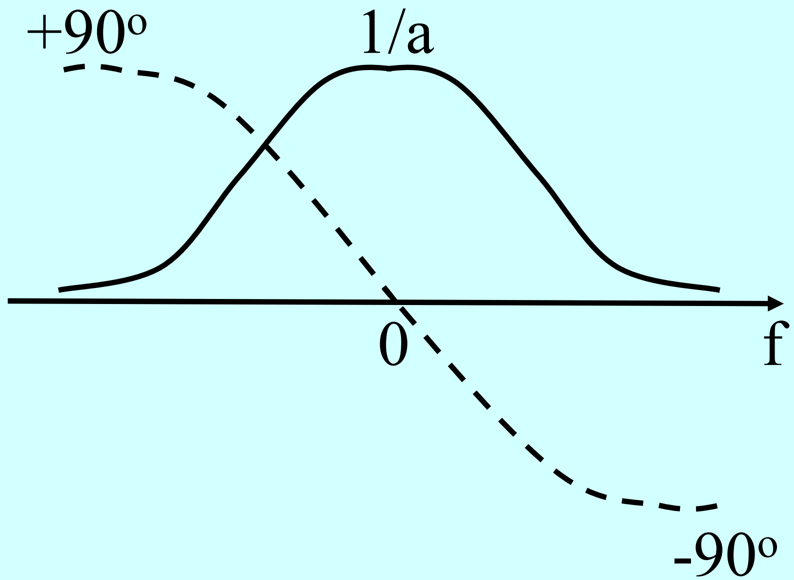


$$x(t) = A\,\mathrm{rect}(t/\tau)$$

$$X(f) = A\tau\,\mathrm{sinc}(f\tau)$$

Example: Exponential pulse





$$x(t) = u(t)\,e^{-at}$$

$$X(f) = \frac{1}{a + j2\pi f}$$

Make sure that you know how to deal with:

- Time scaling
- Frequency scaling
- Time shifts
- Frequency shifts
- Time inversion
- Differentiation
- Integration
- Multiplication by t
- See the Fourier transform sheet provided

For periodic signals we have:

$$x(t) = \sum_{k=-\infty}^{\infty} g(t - kT) = \mathrm{rep}_T\{g(t)\} = \sum_{n=-\infty}^{\infty} X_n \, e^{jn2\pi t/T}$$

$$X(f) = \sum_{n=-\infty}^{\infty} X_n \, \delta(f - n/T)$$

$$X_n = \frac{1}{T}\int_{(T)} x(t)\, e^{-j2\pi nt/T}\, dt = \frac{1}{T} G(n/T)$$

$$G(f) = \int_{-\infty}^{\infty} g(t)\, e^{-j2\pi f t} dt$$

Exercise: Prove that $X_n = (1/T)\ G(n/T)$.

2.3 Convolution

If we have V(f) = X(f)Y(f), what is v(t)?

$$v(t) = \int_{-\infty}^{\infty} X(f)\,Y(f)\,e^{j2\pi ft}\,df$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\lambda)\,e^{-j2\pi f\lambda}\,d\lambda\,Y(f)\,e^{j2\pi ft}\,df$$

$$= \int_{-\infty}^{\infty} x(\lambda)\,y(t-\lambda)\,d\lambda = x(t) \otimes y(t)$$

This is called the ***convolution*** of x(t) and y(t). We can also define convolution in the frequency domain.

$$x(t)\,y(t) \leftrightarrow X(f) \otimes Y(f) = \int_{-\infty}^{\infty} X(\lambda)\,Y(f-\lambda)\,d\lambda$$

$$X(f)\,Y(f) \leftrightarrow x(t) \otimes y(t) = \int_{-\infty}^{\infty} x(\lambda)\,y(f-\lambda)\,d\lambda$$

Exercise: Prove $\int_{-\infty}^{\infty} x(\lambda)\,x^{*}(\lambda-t)\,d\lambda \leftrightarrow |X(f)|^{2} = G_{xx}(f)$

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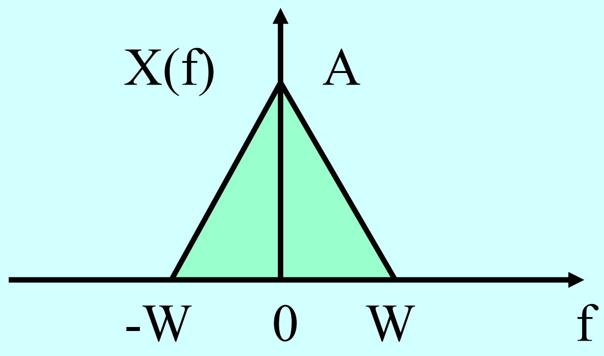
2.4 The Sampling Theorem

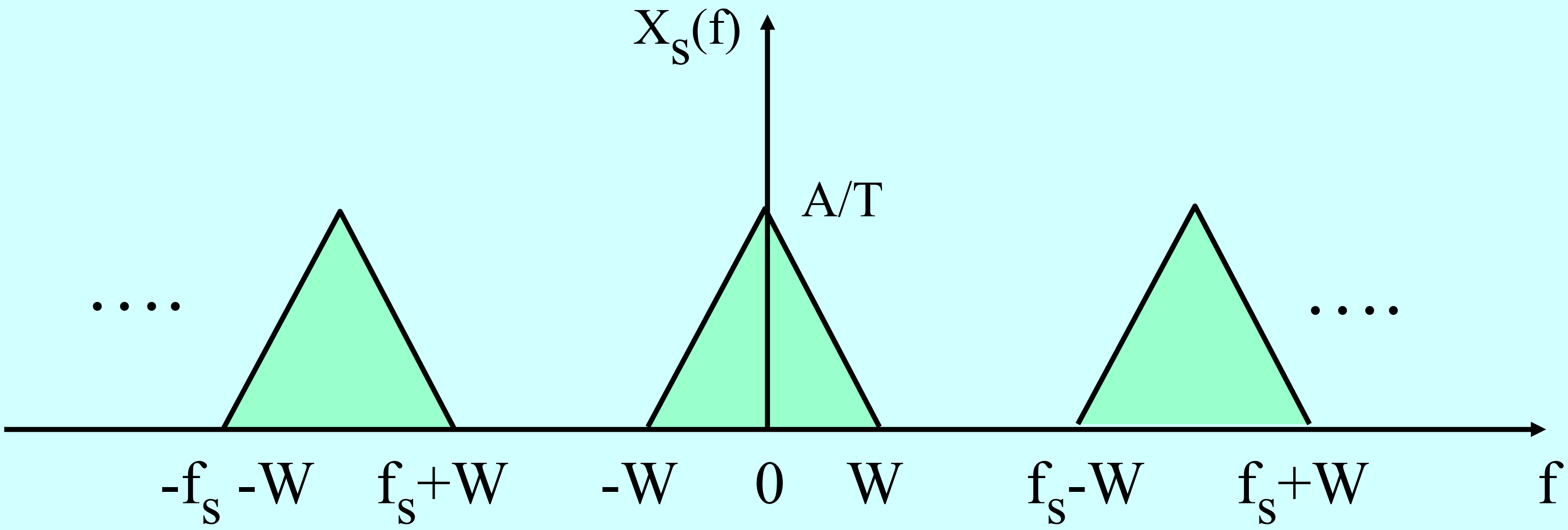
If a signal is sampled at intervals of $T = 1/f_s$ we have:

$$x_s(t) = \sum_{n=-\infty}^{\infty} x(nT)\,\delta(t - nT) = x(t)\,\mathrm{comb}_T(t)$$

$$X_s(f) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(f - k/T) = \frac{1}{T}\,\mathrm{rep}_{1/T}\{X(f)\}$$

We can recover $x(t)$ from $x_s(t)$ by low pass filtering if the bandwidth of $x(t)$ is less than $1/2T$ (ie. less than one half the sampling rate).


X(f)
A
-W
0
W
f


Xs(f)
A/T
....
....
-fs -W
fs+W
-W
0
W
fs-W
fs+W
f

If we sample at lower than the required rate of $f_s = 2W$, we get ***aliasing.*** This is where signals at frequencies greater than the ***Nyquist frequency*** $f_s/2$, reappear at frequencies mirror imaged about $f_s/2$. For instance if the sampling rate is 10 kHz, a 6.5 kHz signal will appear as a 3.5 kHz signal when we try to recover the signal.

To avoid this, frequencies higher than $f_s/2$ must be removed by an anti-aliasing filter before sampling.

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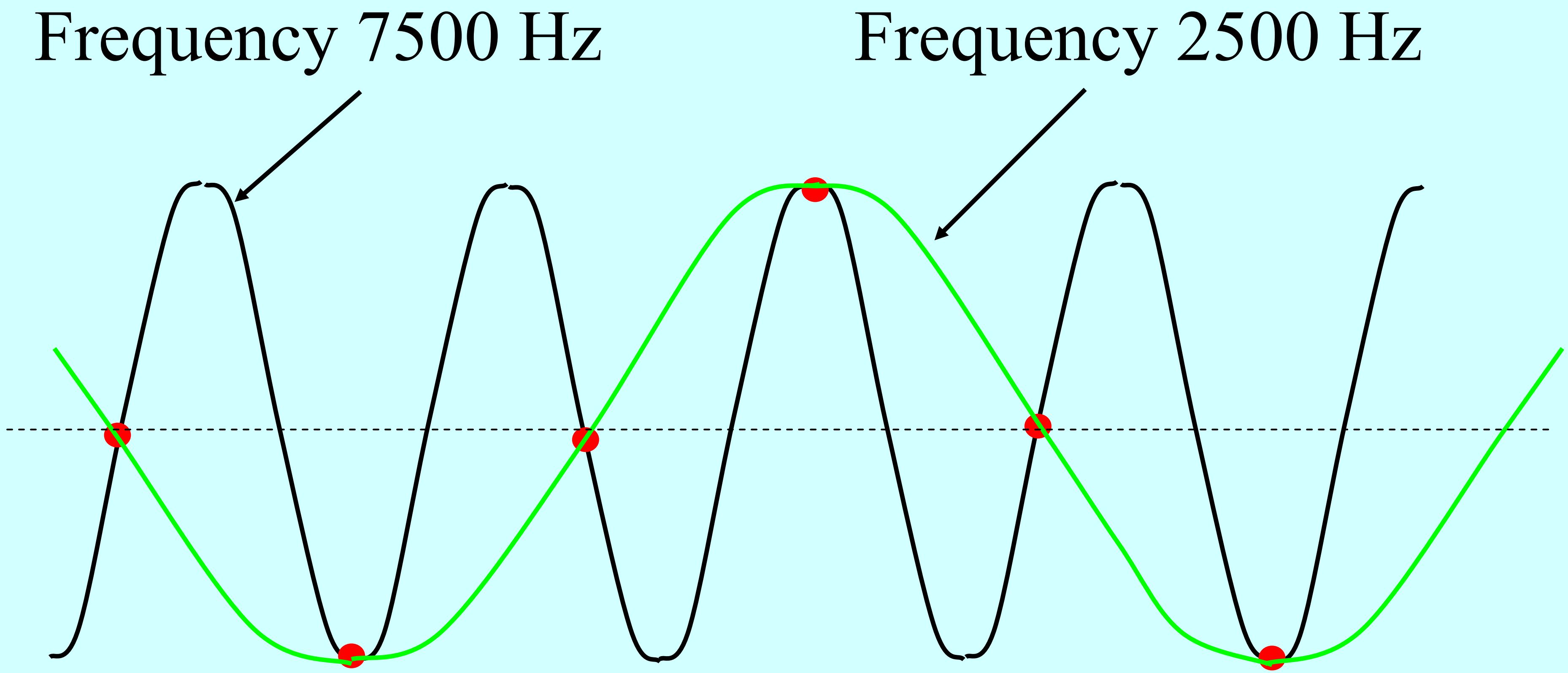


Sampling frequency 10000 Hz

2.5 The Analytic Signal

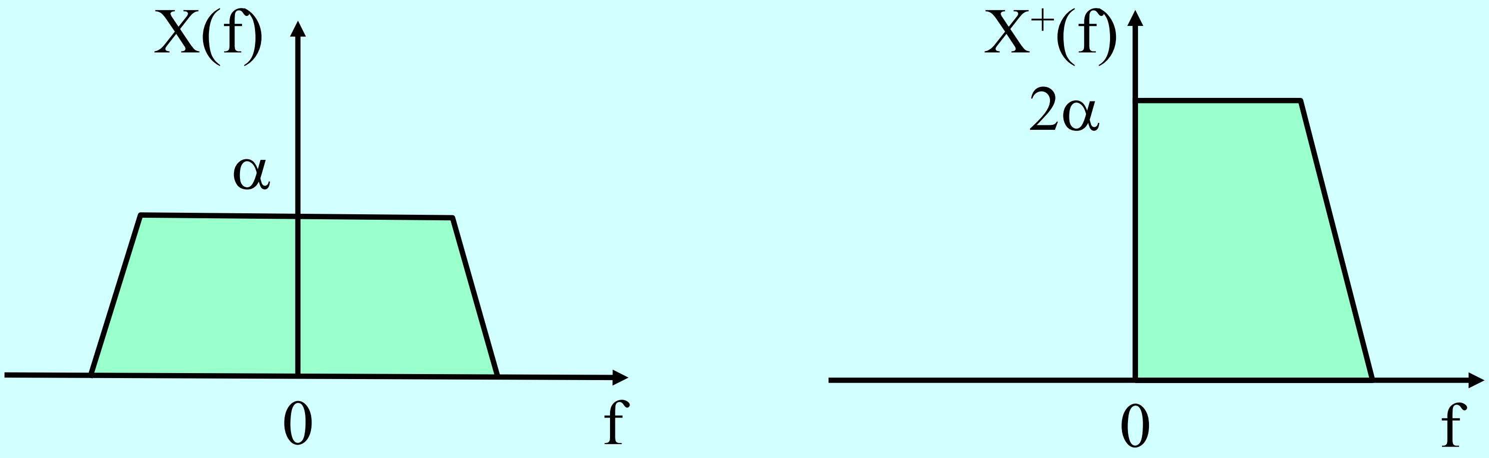
With a real signal x(t) we have $X(-f) = X^*(f)$, so the negative frequency part is redundant. As for sinewaves, it is convenient to deal with signals which contain only positive frequencies.

$$A\cos(2\pi f_o t + \theta) \quad = \quad \mathrm{Re}\{A\, e^{j(2\pi f_o t + \theta)}\}$$

(Real signal) (Analytic signal)

The analytic signal is also called the ***pre-envelope.***

To obtain the analytic signal, we simply discard the negative frequency part and double the positive frequency part. For a real signal x(t), we designate the analytic signal as $x^+(t)$.



$$X^+(f) = 2u(f)X(f)$$

Example:

$$x(t) = \frac{a}{a^2 + t^2}$$

$$X(f) = \pi e^{-2\pi|f|a}$$

$$X^+(f) = 2\pi u(f) e^{-2\pi f a}$$

$$x^+(t) = \frac{1}{a - jt} = \left\{\frac{a}{a^2 + t^2}\right\} + j\left\{\frac{t}{a^2 + t^2}\right\}$$

$$X^{+}(f) = 2\pi u(f) e^{-2\pi fa}$$

$$u(t) e^{-at} \leftrightarrow \frac{1}{a + j2\pi f}$$

$$u(-f) e^{af} \leftrightarrow \frac{1}{a + j2\pi t} \quad \text{using} \quad X(t) \leftrightarrow x(-f)$$

$$u(f) e^{-af} \leftrightarrow \frac{1}{a - j2\pi t} \quad \text{using} \quad x(-t) \leftrightarrow X(-f)$$

$$2\pi u(f) e^{-2\pi af} \leftrightarrow \frac{2\pi}{2\pi a - j2\pi t} = \frac{1}{a - jt} = x^{+}(t)$$

The real part of the analytic signal is the original signal, the imaginary part is called the ***Hilbert transform*** of x(t) and is denoted $\hat{x}(t)$.

$$Ae^{j(2\pi f_o t+\theta)} = A\cos(2\pi f_o t+\theta) + jA\sin(2\pi f_o t+\theta)$$

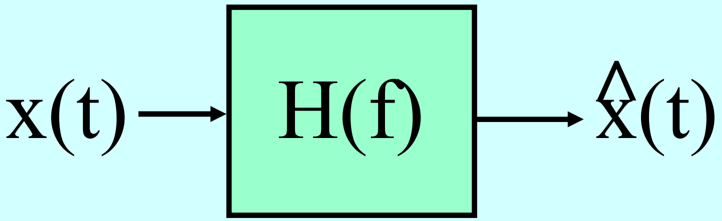
$$x(t) = A\cos(2\pi f_o t+\theta) = \frac{1}{2}Ae^{j(2\pi f_o t+\theta)} + \frac{1}{2}Ae^{-j(2\pi f_o t+\theta)}$$

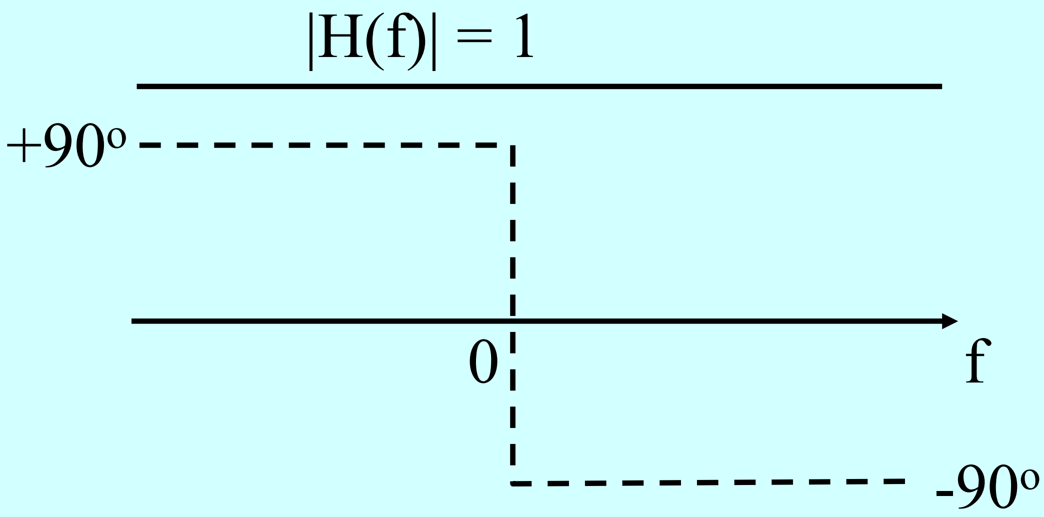
$$\hat{x}(t) = A\sin(2\pi f_o t+\theta) = -\frac{j}{2}Ae^{j(2\pi f_o t+\theta)} + \frac{j}{2}Ae^{-j(2\pi f_o t+\theta)}$$

$$\hat{X}(f) = -j\,\mathrm{sgn}(f)X(f)$$

We recognise this as a filtering operation with a filter:

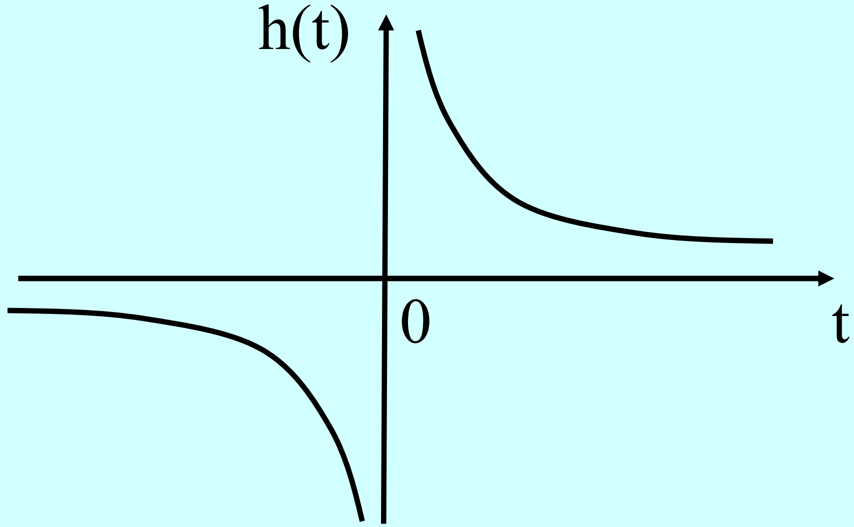
$$H(f) = -j\ \mathrm{sgn}(f).$$





The impulse response of the Hilbert transformer is:

$$h(t) = \frac{1}{\pi t} \quad \Rightarrow \quad \hat{x}(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x(\lambda)}{t-\lambda} d\lambda$$



The Hilbert transformer is ***non-causal***.

Bandlimiting removes the infinity at $t = 0$.

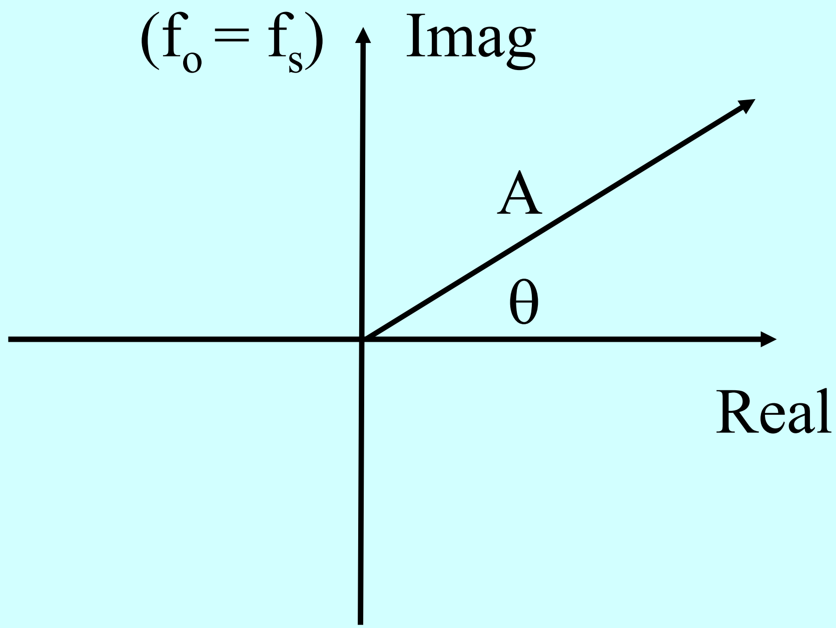
2.6 Applications of the Analytic Signal

(i) Phasors

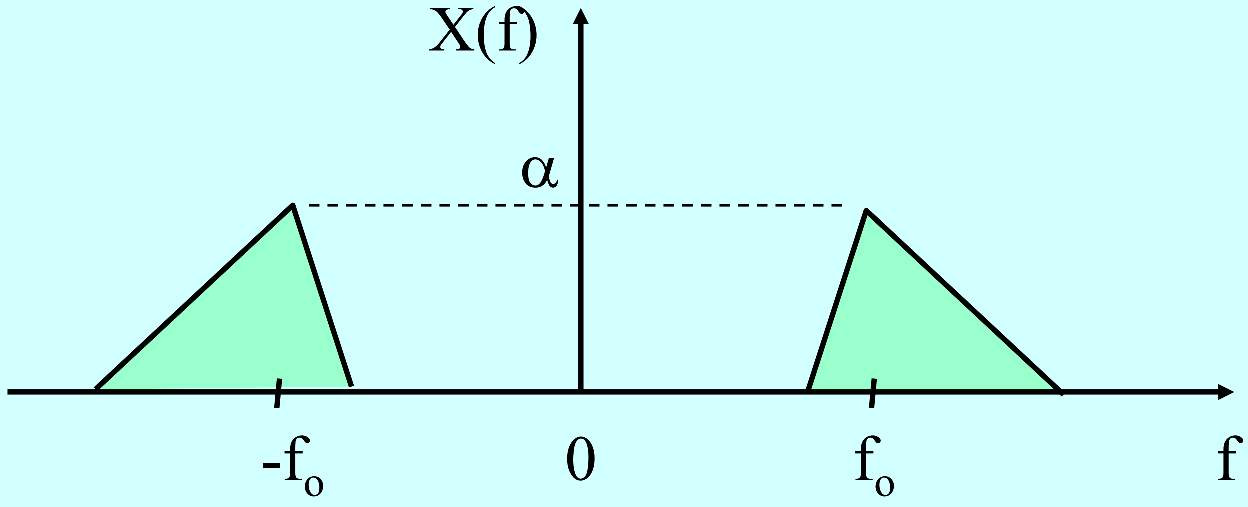
With a sinusoid $x(t) = A\cos(2\pi f_o t + \theta)$ the analytic signal is $x^+(t) = A\, e^{j(2\pi f_o t + \theta)}$. If we factor out the $e^{j2\pi f_o t}$ term, the remaining factor $Ae^{j\theta}$ is the ***phasor*** representing x(t).

An important aspect of the phasor is the reference frequency f_o (specified separately).

The reference frequency f_o is arbitrary, but for a sinewave $A\cos(2\pi f_s t+\theta)$ we usually choose it as the frequency of the sinewave, since then the phasor is a constant.

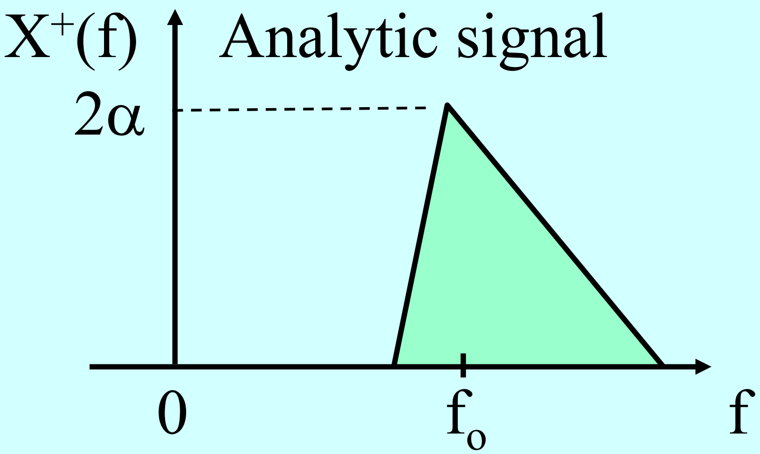


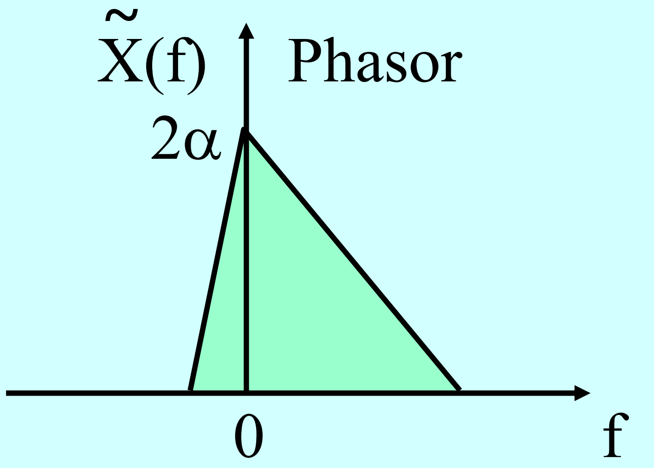
Phasors are of most use when we consider ***narrowband*** signals. These are signals which have their frequency components concentrated near some frequency f_o.



The phasor is denoted $\tilde{x}(t)$ and is simply a frequency down-shifted version of the analytic signal. It contains all the information about x(t) **except** the carrier frequency f_o.

$$\tilde{x}(t) = x_c(t) + jx_s(t) = x^+(t)e^{-j2\pi f_o t}$$





$$\tilde{x}(t) = x^{+}(t)e^{-j2\pi f_o t}$$

$$= x_c(t) + jx_s(t) = r(t)e^{j\theta(t)}$$

$$x(t) = \mathrm{Re}\left\{x^{+}(t)\right\} = \mathrm{Re}\left\{\tilde{x}(t)e^{j2\pi f_o t}\right\}$$

$$= x_c(t)\cos(2\pi f_o t) - x_s(t)\sin(2\pi f_o t)$$

$$= r(t)\cos\{2\pi f_o t + \theta(t)\}$$

- **Envelope** $r(t) = |\tilde{x}(t)|$
- **Relative phase** $\theta(t) = \arg \tilde{x}(t)$
- **Frequency deviation** $= d\theta(t)/dt$; rad/sec

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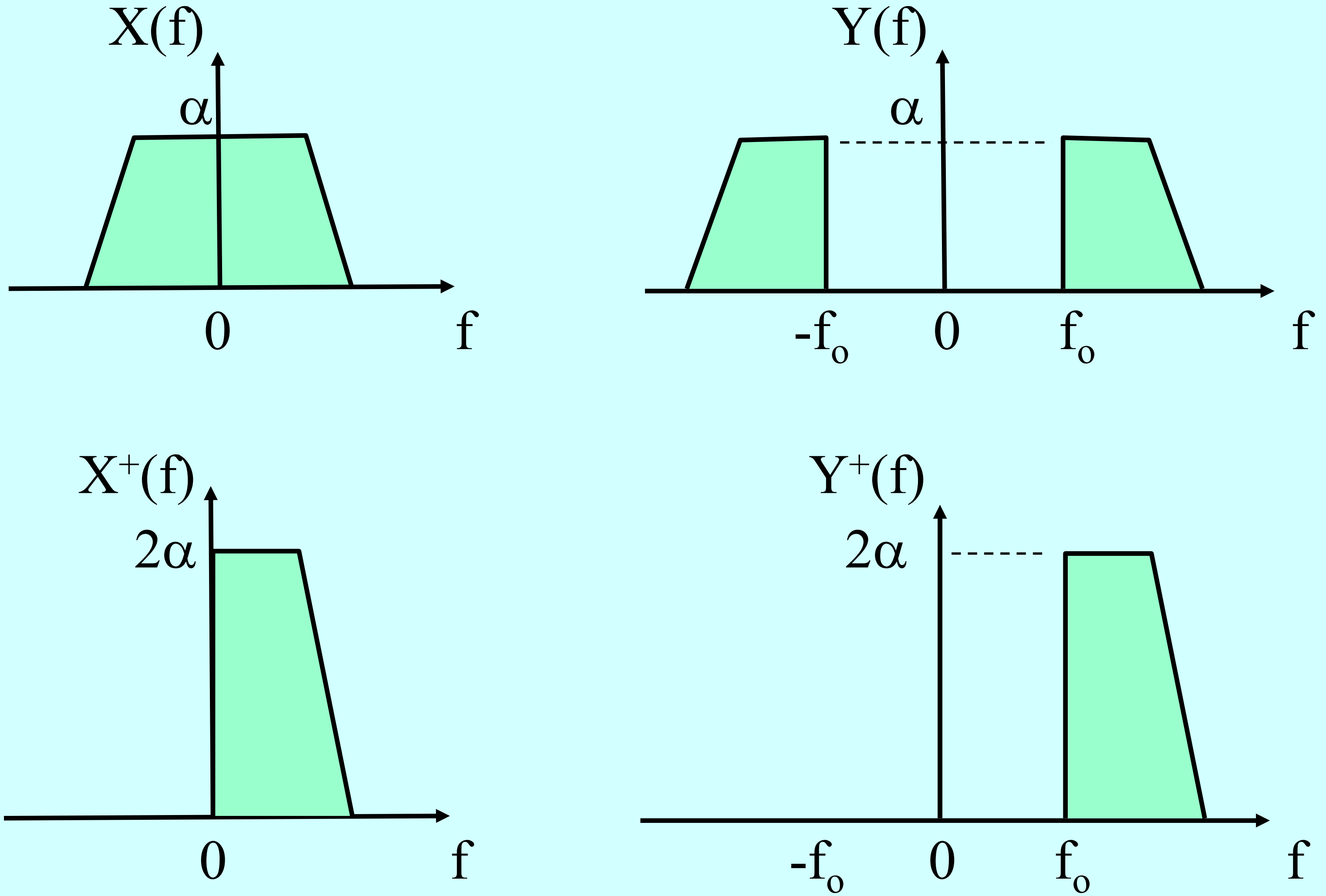
(ii) Single Sideband Signals

A single sideband (SSB) signal is one in which the positive frequency components of a baseband signal x(t) are translated up by f_o and the negative frequency components down by f_o.

To determine an expression describing how the SSB signal y(t) is related to the baseband signal x(t), we first consider the relation between the respective analytic signals.


X(f)
α
0
f
Y(f)
α
-f_o
0
f_o
f
X⁺(f)
2α
0
f
Y⁺(f)
2α
-f_o
0
f_o
f

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$$Y^{+}(f) = X^{+}(f - f_o)$$

$$y^{+}(t) = x^{+}(t)e^{j2\pi f_o t}$$

$$y(t) + j\hat{y}(t) = \{x(t) + j\hat{x}(t)\}e^{j2\pi f_o t}$$

$$y(t) = x(t)\cos(2\pi f_o t) - \hat{x}(t)\sin(2\pi f_o t)$$

Exercise: For $x(t) = 4\,\mathrm{sinc}(2t)$ and $f_o = 10$ Hz, calculate the spectrum of $\hat{x}(t)$ and hence that of $y(t)$ from the expression above. Sketch the spectra obtained in each case.

Exercises: You are expected to attempt the following exercises in Proakis & Salehi. Completion of these exercises is part of the course. Solutions will be available later.

2.23

2.24

2.25

2.28

2.49

2.55

2.58 (f = 0 should be f > 0)