Section 4: Random Processes and Linear Systems

Contents

4. Random Processes & Linear Systems

Most of the signals in communication signals are random signals, since if they were deterministic (ie. known) there would be no point in transmitting them over a communication channel.

It is assumed that you have had an introduction to random processes previously, but the important relations will be revisited in this chapter.

A discrete random variable x_i will have probabilities $P\{x_i\}$. A continuous random variable x will have a probability density function p(x).

Discrete random variable

$$\sum_{i=1}^{n} P\{x_i\} = 1$$

$$E\{g(x)\} = \sum_{i=1}^{n} g\{x_i\} P\{x_i\}$$

Continuous random variable

$$\int_{-\infty}^{\infty} p(x)\,dx = 1$$

$$E\{g(x)\} = \int_{-\infty}^{\infty} g(x)\,p(x)\,dx$$

4.1 Correlation Functions

The ***autocorrelation function*** of a signal x(t), and the ***crosscorrelation function*** of x(t) and y(t) are:

$$R_{xx}(t_1, t_2) = E\left\{x(t_1)\, x^*(t_2)\right\}$$
$$R_{xy}(t_1, t_2) = E\left\{x(t_1)\, y^*(t_2)\right\}$$

E{.} is the ***expectation operator***, which means taking the ensemble average.

If x(t) and y(t) are ***stationary***, then these are functions only of $\tau = t_1 - t_2$.

$$R_{xx}(\tau) = E\left\{x(t)\,x^*(t-\tau)\right\}$$
$$R_{xy}(\tau) = E\left\{x(t)\,y^*(t-\tau)\right\}$$

If the signals are real, we can ignore the complex conjugate on the second term. If x(t) is in volts, the correlation function has dimensions of volt^2.

If the signals are ***ergodic***, we can also find the correlation functions by a ***time average***.

$$R_{xx}(\tau) = \left\langle x(t)\, x^*(t-\tau) \right\rangle$$

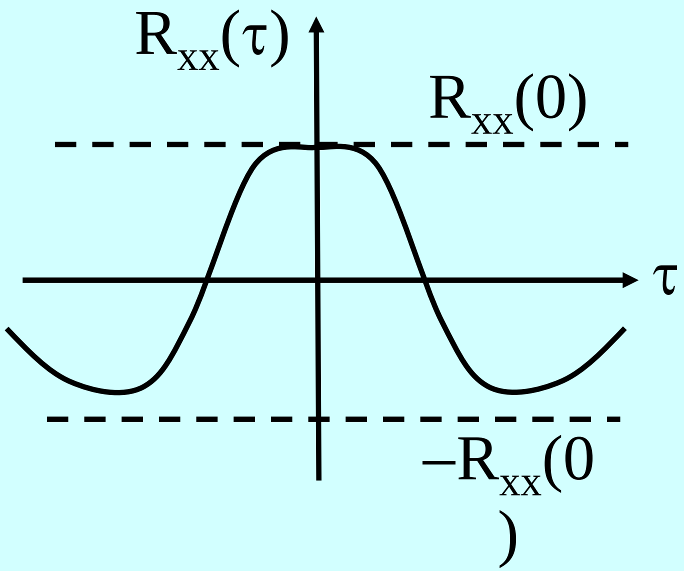
$$R_{xy}(\tau) = \left\langle x(t)\, y^*(t-\tau) \right\rangle$$

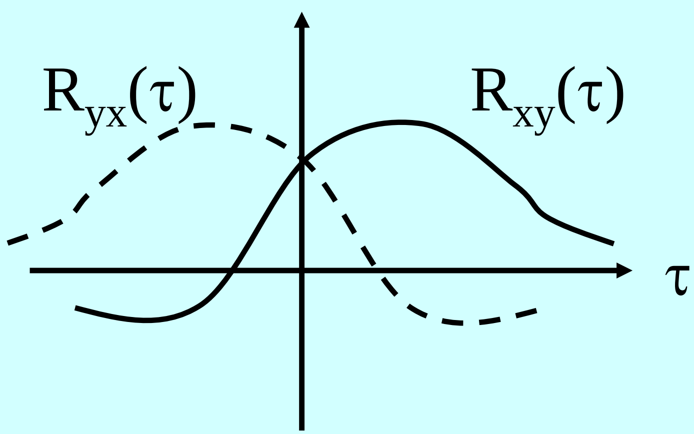
$$\left\langle f(t) \right\rangle = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} f(t)\, dt$$

Exercise: Find the time averages of $\cos(\omega t)$, $\cos^2(\omega t)$, $\cos(\omega t)\sin(\omega t)$, $\cos(\omega_1 t)\cos(\omega_2 t)$.

Properties of correlation functions:

- $R_{xx}(0) = E\{|x(t)|^2\} = \langle|x(t)|^2\rangle = P$
- $|R_{xx}(\tau)| \leq R_{xx}(0)$
- $R_{xx}(-\tau) = R^*_{xx}(\tau)$ ie. an even function if real
- $R_{yx}(\tau) = R^*_{xy}(-\tau)$





4.2 Power Spectral Density

With signals of ***finite energy*** (pulses) we have the energy spectral density $G_{xx}(f) = |X(f)|^2$. For signals of ***finite power*** (random signals, speech, noise) we have a ***power spectral density*** $S_{xx}(f)$.

Suppose x(t) is a finite power process. A time truncated version of this is:

$$x_T(t) = x(t)\,\text{rect}(t/T)$$

The energy density of this signal is $|X_T(f)|^2$, so we can define the power spectral density as the limit as $T \to \infty$ of $|X_T(f)|^2/T$. With random signals, a consistent limit is not reached unless we form the ensemble average over all possible realisations.

$$S_{xx}(f) = \lim_{T \to \infty} \frac{E\left\{ \left| X_T(f) \right|^2 \right\}}{T}$$

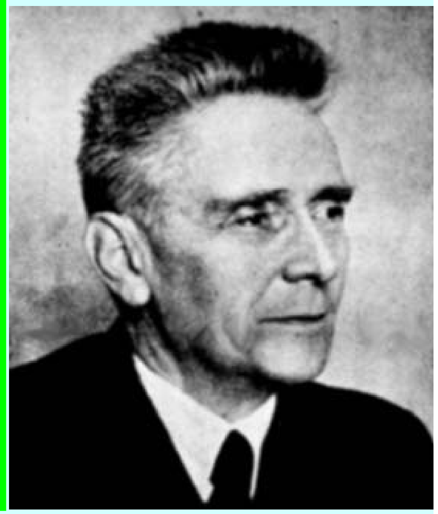
However, this is usually not a satisfactory way to compute the power spectral density.

Wiener-Khinchin Theorem

This theorem states that the power spectral density of an ergodic signal is the Fourier transform of its autocorrelation function.

$$S_{xx}(f) = \int_{-\infty}^{\infty} R_{xx}(\tau) e^{-j2\pi f \tau} d\tau$$

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} S_{xx}(f) e^{j2\pi f \tau} df$$



The power in the signal x(t) at frequencies in the range $f_1 \leq f \leq f_2$ is given by:

$$P_{12} = \int_{f_1}^{f_2} S_{xx}(f)\,df$$

Note that the power spectral density is always ***real*** and ***non-negative*** for all signals, real or complex. For real signals, it is also an ***even function*** of frequency.

We can also define the ***cross power spectral density*** as the Fourier transform of the cross correlation function.

$$S_{xy}(f) = \int_{-\infty}^{\infty} R_{xy}(\tau) e^{-j2\pi f \tau} d\tau$$

$$R_{xy}(\tau) = \int_{-\infty}^{\infty} S_{xy}(f) e^{j2\pi f \tau} df$$

$$S_{yx}(f) = S_{xy}^{*}(f)$$

It can be shown that:

$$\left|S_{xy}(f)\right|^2 \le S_{xx}(f)\,S_{yy}(f)$$

This implies that if x(t) and y(t) have no common frequency components (ie. $S_{xx}(f)S_{yy}(f) = 0$), then x(t) and y(t) are uncorrelated.

Exercise: By considering $z(t) = a\,x(t) + b\,y(t)$, (a,b complex), and using $S_{zz}(f) \ge 0$, prove the above relation.

4.3 Cyclostationary Processes

Many of the processes in communication systems are not strictly stationary, but are ***cyclostationary***. This means the underlying process has a periodic structure, and as a result statistics such as the ***mean*** and ***correlation function*** are periodic.

Hence when we form $E\{x(t)\, x^*(t-\tau)\}$ we find the result is a function of both t and τ but is periodic in t. If we average over t, then we get a result which only depends on τ.

Example: The simplest cyclostationary process is a sinewave.

$$x(t) = A\cos(\omega_o t + \theta)$$

$$E\{x(t)\,x(t-\tau)\} = A^2\cos(\omega_o t + \theta)\cos(\omega_o t - \omega_o \tau + \theta)$$

$$= \frac{1}{2}A^2\cos(\omega_o \tau) + \frac{1}{2}A^2\cos(2\omega_o t - \omega_o \tau + 2\theta)$$

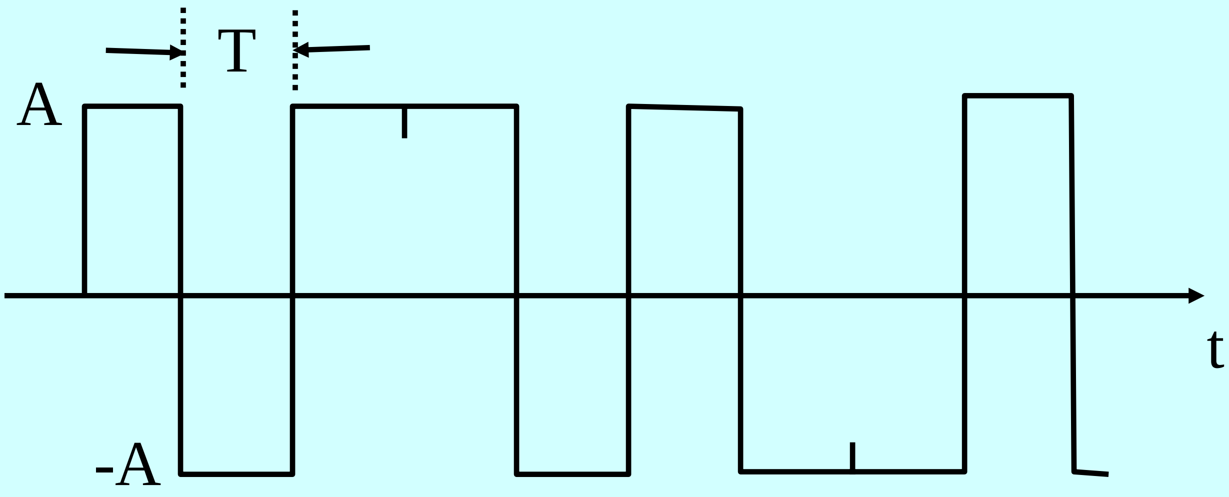
We see that the second term is periodic in t and has an average value of zero.

$$\overline{R}_{xx}(\tau) = \frac{1}{2}A^2\cos(\omega_o \tau)$$

$$S_{xx}(f) = \frac{1}{4}A^2\delta(f - f_o) + \frac{1}{4}A^2\delta(f + f_o)$$

Example: Random Binary Waveform

This consists of rectangular pulses of duration T and amplitude ±A with equal probability and uncorrelated with each other.



We will consider the general case where the pulse shape is p(t), so we can write:

$$x(t) = \sum_{k=-\infty}^{\infty} A\, a_k\, p(t - kT)$$

where p(t) is the pulse shape and $a_k = \pm 1$ is the digital data. For the previous slide we have p(t) = rect(t/T). With the a_k equally likely and uncorrelated, we have $E\{a_k a_r\} = 1$ if k = r and zero otherwise.

The autocorrelation function of x(t) is:

$$R_{XX}(t, t-\tau) = E\left\{x(t)\,x(t-\tau)\right\}$$

$$= \sum_{k=-\infty}^{\infty} \sum_{r=-\infty}^{\infty} A^2 E\left\{a_k a_r\right\} p(t-kT)\,p(t-\tau-rT)$$

$$= A^2 \sum_{k=-\infty}^{\infty} p(t-kT)\,p(t-\tau-kT)$$

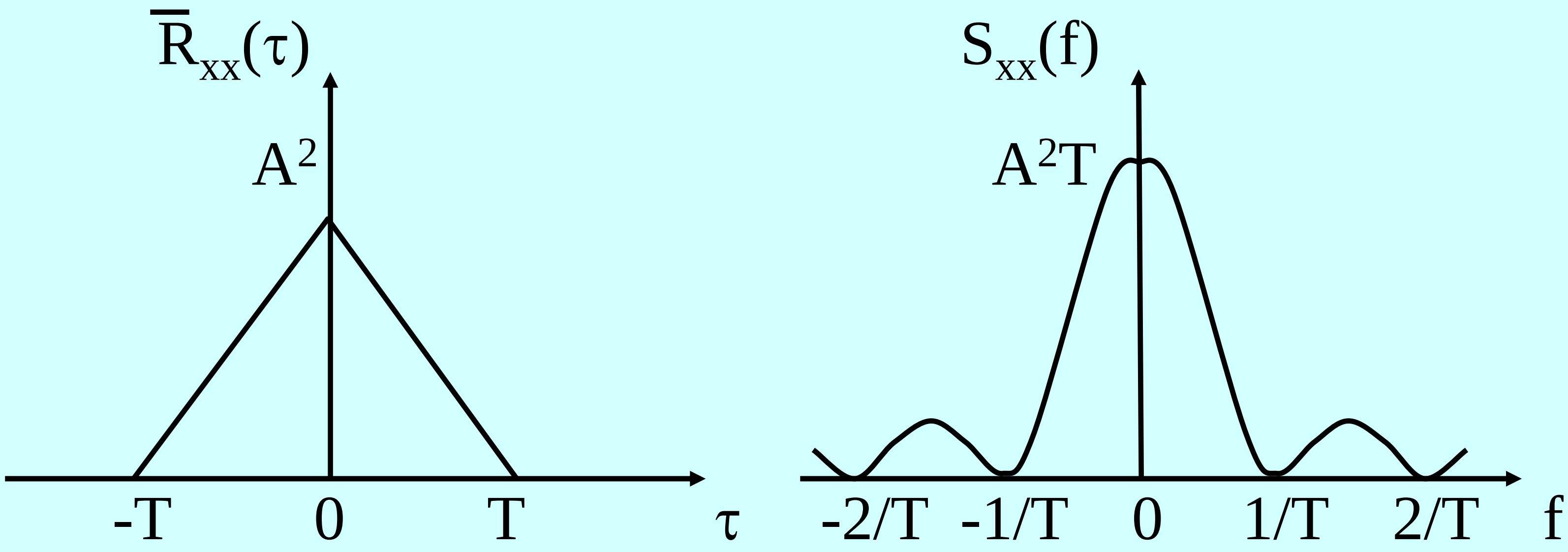
We note that this is periodic in t, so this is a ***cyclo-stationary process***. Hence we must first average the correlation function over one period.

$$\overline{R}_{xx}(\tau) = \frac{A^2}{T} \sum_{k=-\infty}^{\infty} \int_{-T/2}^{T/2} p(t - kT)\, p(t - \tau - kT)\, dt$$

$$= \frac{A^2}{T} \int_{-\infty}^{\infty} p(t)\, p(t - \tau)\, dt = \frac{A^2}{T} p(t) \otimes p(-t)$$

$$S_{xx}(f) = \frac{A^2}{T} \left| P(f) \right|^2$$

For a rectangular pulse we have $\overline{R}_{xx}(\tau) = A^2 \Delta(\tau/T)$ and hence $S_{xx}(f) = A^2T\,\mathrm{sinc}^2(fT)$. The bandwidth required is therefore approximately W = 1/T Hz.



4.4 Dimensions of Power

The dimensions of the autocorrelation function of x(t) is V^2 if x(t) is a voltage and A^2 if it is a current. It is common practice in communication theory to define the ***"power"*** of a signal x(t) as the average value of $x^2(t)$, which is of course actually the ***mean square value***.

The ***power spectral density*** of x(t) then has the dimensions V^2/Hz or A^2/Hz.

If the power spectral density is expressed in terms of W/Hz, then if x(t) is the voltage or current in a resistance R, we must multiply by R to get the power spectral density in V^2/Hz, or divide by R to get it in A^2/Hz.

Alternatively, the power spectral density in V^2/Hz or A^2/Hz is sometimes called the power spectral density of the signal in 1 ohm.

When we calculate power ratios, the resistance R cancels out, so it is not usually of interest.

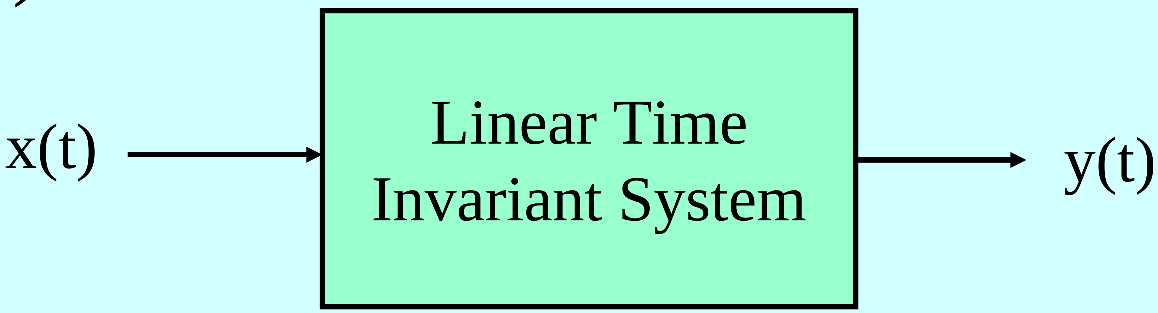
4.5 Linear Time Invariant Systems

A linear time invariant (LTIV) system can be described either by its ***impulse response*** h(t) or its ***frequency response*** H(f) and these are a Fourier transform pair.

$$H(f) = \int_{-\infty}^{\infty} h(t) e^{-j2\pi f t} dt$$

$$h(t) = \int_{-\infty}^{\infty} H(f) e^{j2\pi f t} df$$

Consider a LTIV system with an input x(t) and an output y(t).



$$Y(f) = H(f)X(f)$$

$$y(t) = h(t) \otimes x(t) = \int_{-\infty}^{\infty} h(\lambda)\,x(t-\lambda)\,d\lambda$$

Exercise: If $y(t) = \int_{-\infty}^{t} \left[x(t') - x(t'-T) \right] dt'$ find h(t) & H(f).

The time domain convolution is valid for all signals, but the frequency domain relation is only true if the Fourier transforms exist, which they may not (eg. random signals usually do not have Fourier transforms). However for random signals we have:

$$S_{xy}(f) = S_{xx}(f)H^*(f)$$

$$S_{yx}(f) = S_{xx}(f)H(f)$$

$$S_{yy}(f) = S_{xx}(f)\left|H(f)\right|^2$$

To calculate power spectral density relations.

1. $Y(f) = X(f)\,H(f)$ for finite energy signals.

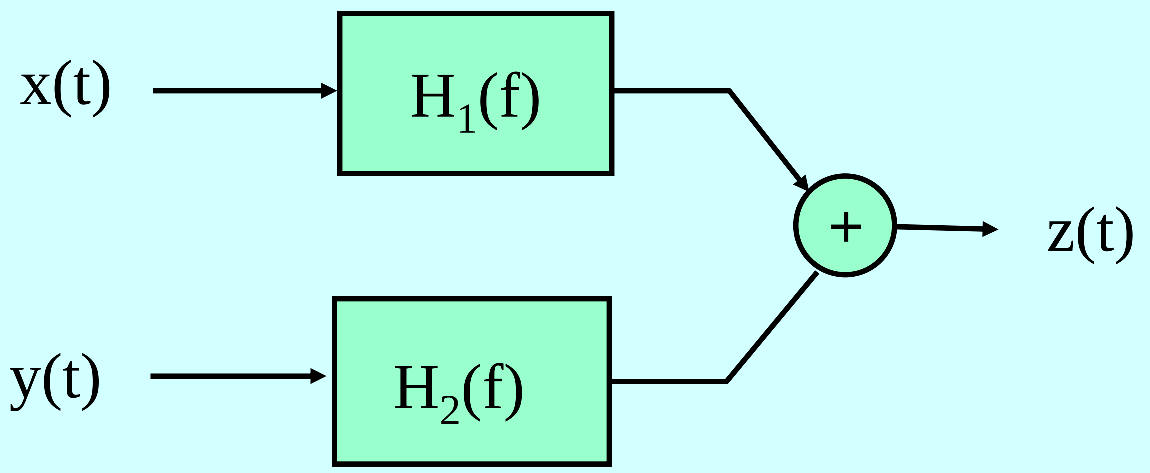
2. $|Y(f)|^2 = X(f)H(f)X^*(f)H^*(f) = |X(f)|^2|H(f)|^2$
   $X(f)Y^*(f) = X(f)X^*(f)H^*(f) = |X(f)|^2H^*(f)$
   $Y(f)X^*(f) = X(f)H(f)X^*(f) = |X(f)|^2H(f)$

3. Replace $|X(f)|^2$ by $S_{xx}(f)$, $|Y(f)|^2$ by $S_{yy}(f)$
   $X(f)Y^*(f)$ by $S_{xy}(f)$, $Y(f)X^*(f)$ by $S_{yx}(f)$

Power spectral densities satisfy the same relations as ***energy spectral densities***.

The results for more complicated situations can be derived in a similar way.

Exercise: Calculate $S_{zz}(f)$.



Answer:

$$S_{zz}(f) = S_{xx}(f)\left|H_1(f)\right|^2 + 2\,\mathrm{Re}\left\{S_{xy}(f)H_1(f)H_2^*(f)\right\} + S_{yy}(f)\left|H_2(f)\right|^2$$

You should verify this result. Note that this also shows that if signals are ***uncorrelated***, we can add their power spectral densities. Unless stated otherwise, ***power spectral densities are always two-sided*** (ie. includes negative frequencies).

4.6 Gaussian Noise

Many of the random signals we will consider, and in particular noise, will have a Gaussian probability density function.

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\eta)^2/2\sigma^2}$$

where $\eta = E\{x\}$ is the mean value and $\sigma^2 = E\{(x-\eta)^2\}$ is the variance.

In digital systems we will be interested in the probability that x exceeds some value x_o, and this is given by:

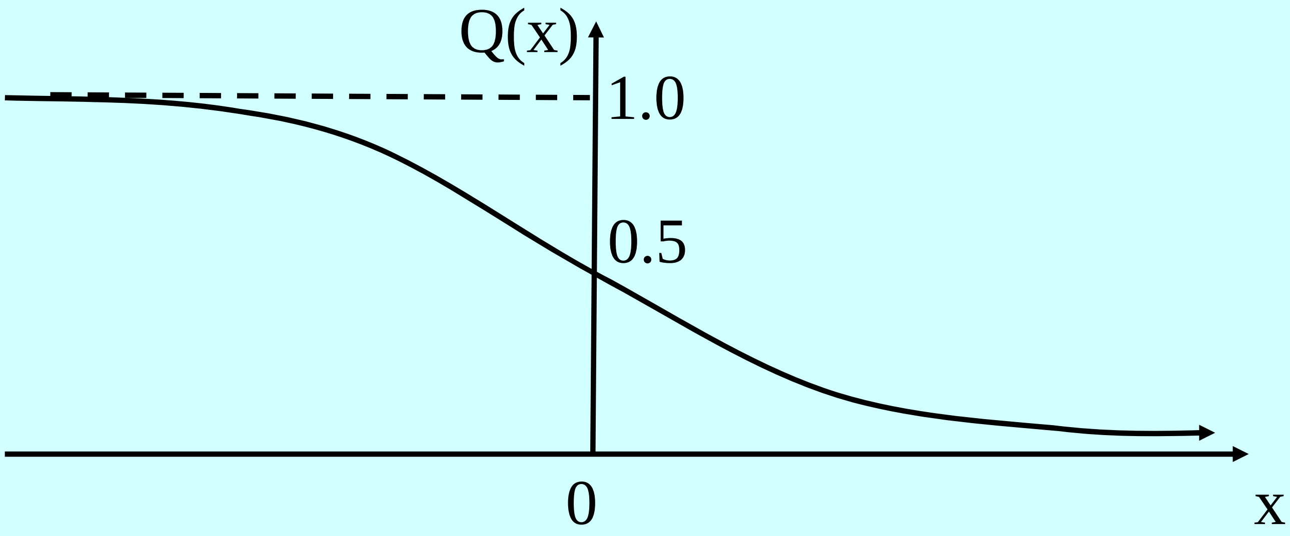
$$P\{x > x_o\} = \frac{1}{\sigma\sqrt{2\pi}} \int_{x_o}^{\infty} e^{-(x-\eta)^2/2\sigma^2} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{(x_o-\eta)/\sigma}^{\infty} e^{-t^2/2} dt \qquad ; \ t = \frac{x-\eta}{\sigma}$$

$$= Q\left\{\frac{x_o-\eta}{\sigma}\right\}$$

Q(x) is the Gaussian error function:

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} e^{-t^2/2}\, dt$$



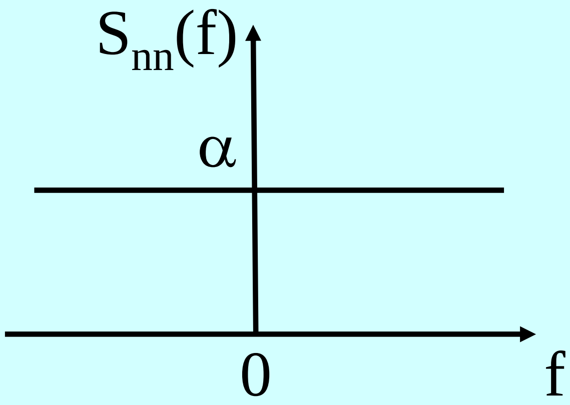
A table of the Q function is provided, but some special values are listed below.

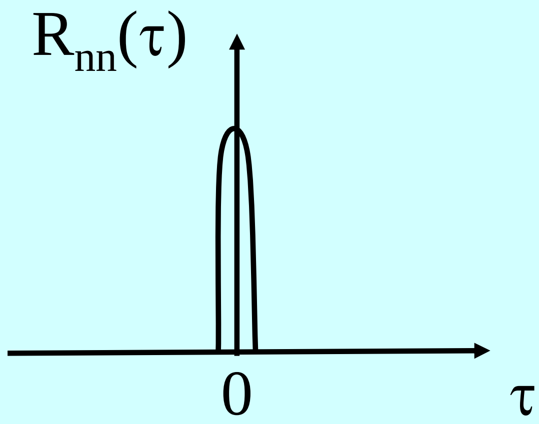
$Q(0) = 0.500$
$Q(1.645) = 0.050$
$Q(1.960) = 0.025$

x	1.282	2.326	3.090	3.719	4.265	4.753	5.199
Q(x)	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}

4.7 White Noise

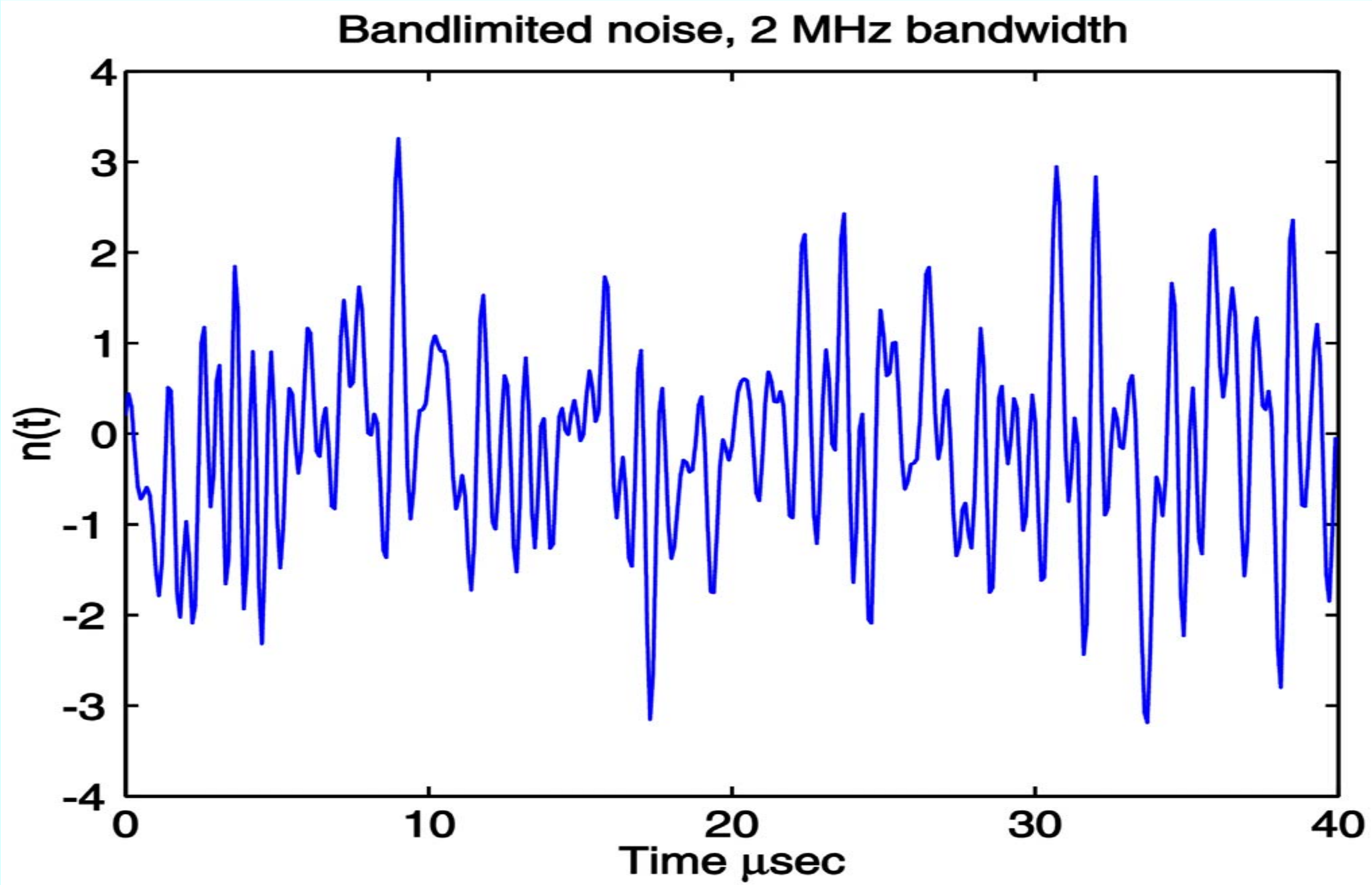
A signal which has $S_{nn}(f) = \alpha = N_o/2$ (a constant) for all frequencies is called ***white noise***. Its autocorrelation function is $R_{nn}(\tau) = \alpha\ \delta(\tau)$.





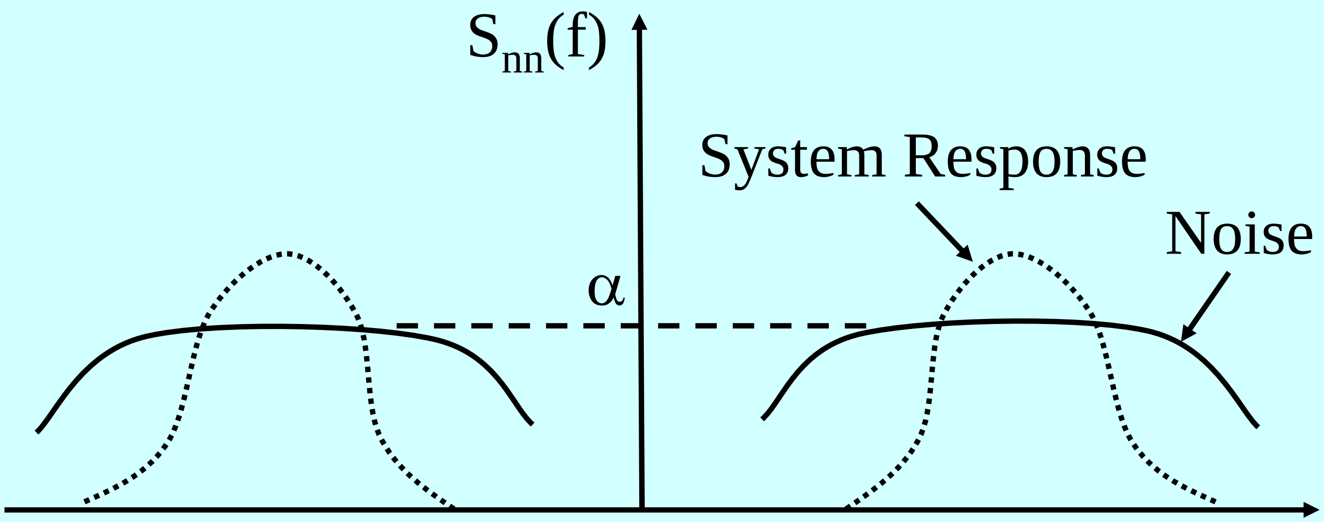
It is usual notation to use $S_{nn}(f) = N_o/2$, where N_o is the ***single-sided power spectral density*** of the noise. In these notes I will often use α instead of $N_o/2$.

True white noise is an idealisation since it has infinite bandwidth and hence infinite power. In practice we encounter ***bandlimited white noise***, but if its bandwidth is greater than that of the system to which it is applied, we can assume it to be white noise without error.


Bandlimited noise, 2 MHz bandwidth
n(t)
4
3
2
1
0
-1
-2
-3
-4
0
10
20
30
40
Time μsec

Bandlimited white noise

If the noise spectral density is constant over the range of frequencies to which the system responds, then we get the same result if we assumed it was constant at all frequencies.



4.8 Noise Bandwidth

If white noise is applied to a filter H(f), the mean square output noise is:

$$\left\langle n_o^2(t) \right\rangle = \int_{-\infty}^{\infty} \alpha \, |H(f)|^2 \, df$$

$$= 2\alpha \int_{0}^{\infty} |H(f)|^2 \, df$$

$$= 2\alpha \, |H_o|^2 \, B_n$$

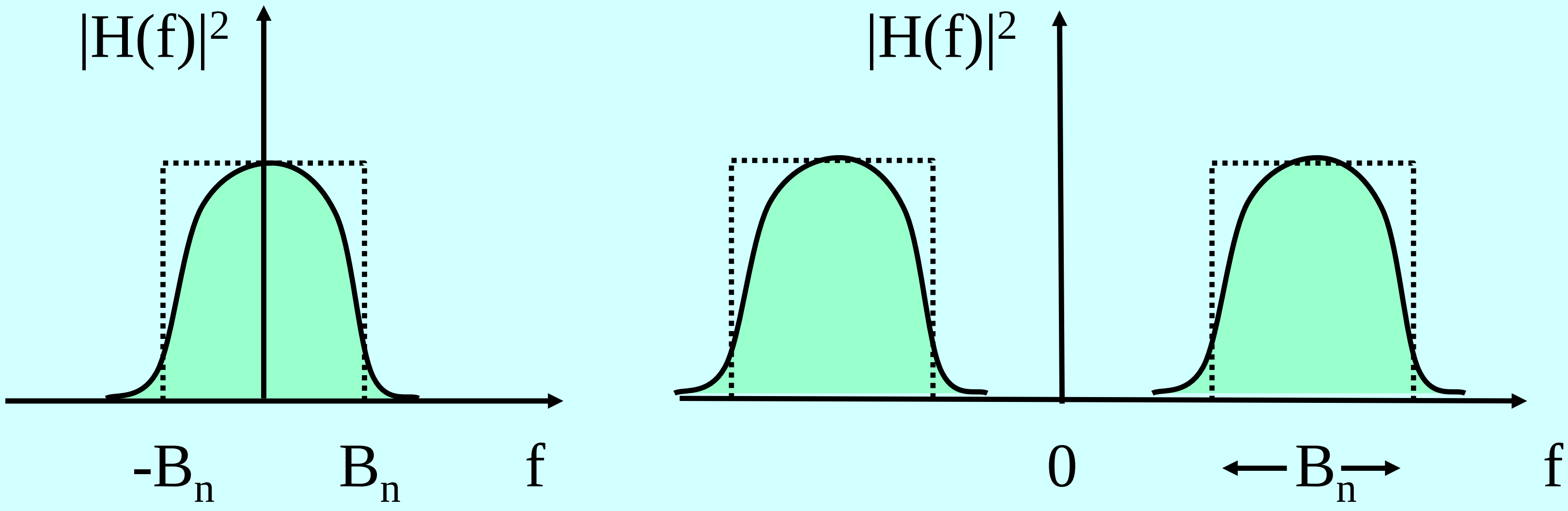
$$B_n = \frac{1}{|H_o|^2} \int_{0}^{\infty} |H(f)|^2 \, df$$

B_n is called the ***noise bandwidth*** in Hz, and is the bandwidth of the rectangular filter which has the same mean square noise at its output.

Note that ***bandwidth is a positive frequency concept*** (and does not include negative frequencies), so the integration is from 0 to ∞.

H_o is the maximum passband gain.

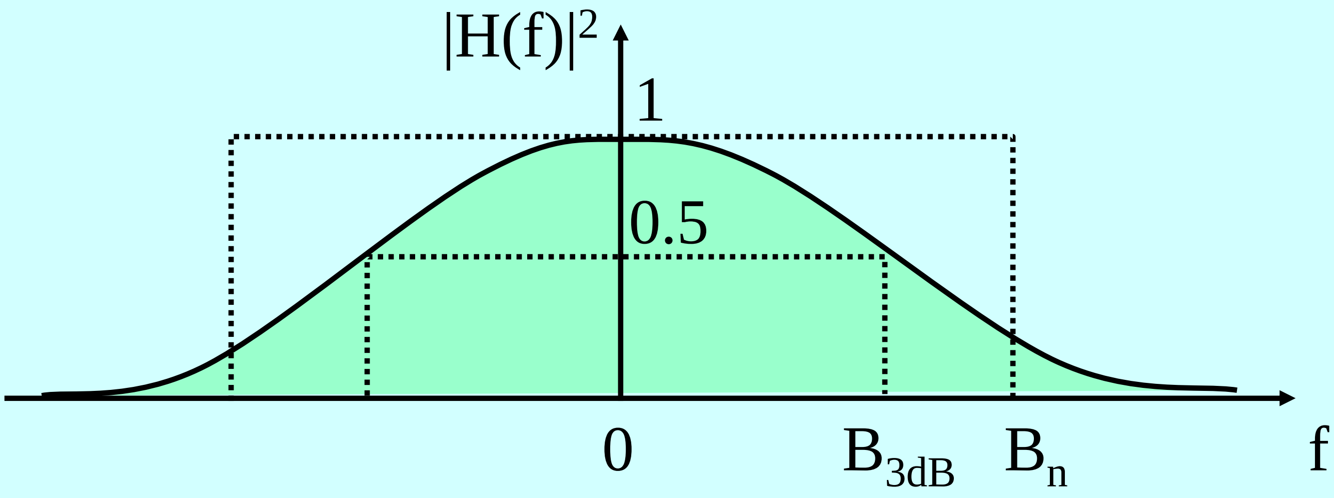
Noise bandwidth of low pass and band pass filters. Note that bandwidth is always measured at positive frequencies.



Exercise: Show that for a low pass RC filter:

$$H(f) = \frac{1}{1 + j2\pi f RC}$$

$$B_n = \frac{1}{4RC} = \frac{\pi}{2} B_{3dB}$$

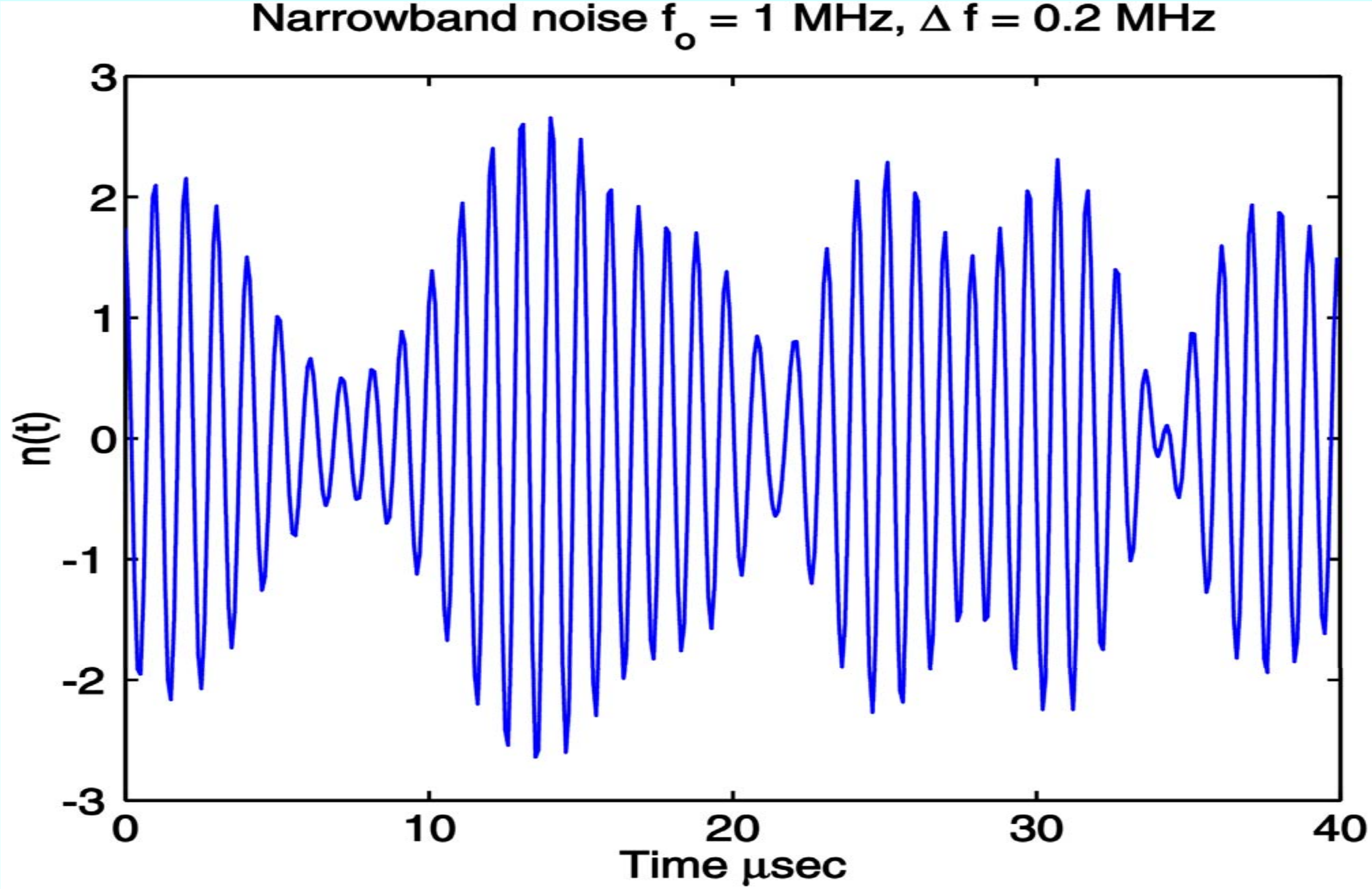


4.9 Narrowband Noise

This looks like a sinewave with random varying amplitude and phase.

$$n(t) = n_c(t)\cos(2\pi f_o t) - n_s(t)\sin(2\pi f_o t)$$
$$= r(t)\cos[2\pi f_o t + \theta(t)]$$

The envelope $r(t)$ and phase $\theta(t)$ [and hence the in-phase and quadrature components $n_c(t)$ and $n_s(t)$] vary at a rate comparable to the **bandwidth**.


Narrowband noise f_o = 1 MHz, Δ f = 0.2 MHz
n(t)
3
2
1
0
-1
-2
-3
0
10
20
30
40
Time µsec

The **analytic signal** for narrowband noise is

$$n^{+}(t) = [n_c(t) + jn_s(t)]e^{j2\pi f_o t}$$

and note that $n(t) = \mathrm{Re}\{n^{+}(t)\}$. The **noise phasor** is

$$\tilde{n}(t) = n_c(t) + jn_s(t)$$

When we demodulate signals, it will be $n_c(t)$ or $n_s(t)$ which will be of interest, so we need to find their power spectral densities.

Since $n^+(t)$ is obtained from $n(t)$ by filtering with $H(f) = 2u(f)$, the power spectrum of $n^+(t)$ is:

$$S_{n^+n^+}(f) = 4u(f)S_{nn}(f) = 4S_{nn}^{(+)}(f)$$

(Don't confuse $S_{nn}^{(+)}(f)$ with the analytic signal). The complex conjugate of $n^+(t)$ is denoted $n^-(t)$ and has only negative frequencies, so its power spectrum is:

$$S_{n^-n^-}(f) = 4u(-f)S_{nn}(f) = 4S_{nn}^{(-)}(f)$$

Also $n^+(t)$ and $n^-(t)$ are uncorrelated, because they have no common frequency components.

Now $\tilde{n}(t) = n_c(t) + jn_s(t) = n^+(t)e^{-j2\pi f_o t}$ so we have:

$$n_c(t) = \operatorname{Re}\left\{n^+(t)e^{-j2\pi f_o t}\right\}$$

$$= \frac{1}{2}\left\{n^+(t)e^{-j2\pi f_o t} + n^-(t)e^{j2\pi f_o t}\right\}$$

Similarly :

$$n_s(t) = \operatorname{Im}\left\{n^+(t)e^{-j2\pi f_o t}\right\}$$

$$= \frac{1}{2j}\left\{n^+(t)e^{-j2\pi f_o t} - n^-(t)e^{j2\pi f_o t}\right\}$$

Since $n^+(t)$ and $n^-(t)$ are uncorrelated, we can add their power spectral densities:

$$S_{n_c n_c}(f) = \frac{1}{4}\left\{ S_{n^+ n^+}(f + f_o) + S_{n^- n^-}(f - f_o) \right\}$$

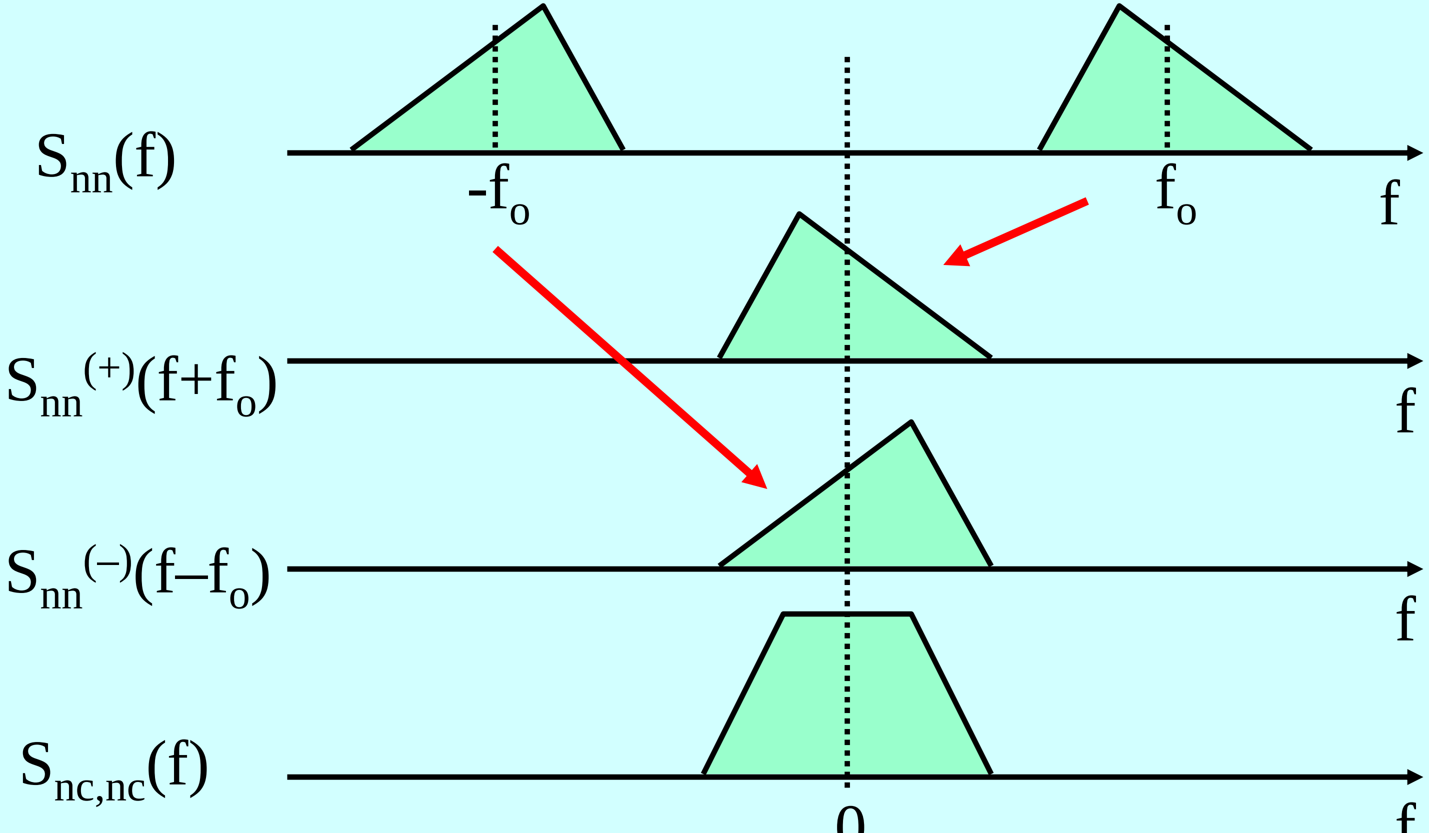
$$S_{n_c n_c}(f) = S_{nn}^{(+)}(f + f_o) + S_{nn}^{(-)}(f - f_o)$$

$$S_{n_s n_s}(f) = S_{nn}^{(+)}(f + f_o) + S_{nn}^{(-)}(f - f_o)$$

$$S_{n_c n_s}(f) = j\left\{ S_{nn}^{(+)}(f + f_o) - S_{nn}^{(-)}(f - f_o) \right\}$$

Exercise: Prove the last relation.

While this looks messy, it is simply the sum of the negative frequency part of $S_{nn}(f)$ shifted up by f_o and the positive part shifted down by f_o.



If the power spectrum $S_{nn}(f)$ is symmetrical about f_o, then the cross power spectrum of $n_c(t)$ and $n_s(t)$ disappears. It is usually not of interest anyway.

If the $S_{nn}(f) = N_o/2$, then the power spectral densities of $n_c(t)$ and $n_s(t)$ are both N_o (at all frequencies of interest $|f| < f_o$). This will be important when we deal with the effect of noise on modulated signals.

$$S_{nn}(f) = \frac{N_o}{2}\left\{ \mathrm{rect}\left(\frac{f - f_o}{B}\right) + \mathrm{rect}\left(\frac{f + f_o}{B}\right)\right\}$$

$$S_{n_c n_c}(f) = S_{n_s n_s}(f) = N_o \,\mathrm{rect}\left(\frac{f}{B}\right)$$

Note that the RF noise n(t) has a power spectral density $N_o/2$ and bandwidth B, but $n_c(t)$ has a power spectral density N_o [which is double that of n(t)], and a bandwidth B/2 [which is half that of n(t)] and similarly for $n_s(t)$.

Exercises: You are expected to attempt the following exercises in Proakis & Salehi. Completion of these exercises is part of the course. Solutions will be available later.

4.10

4.44 (Part 4 is $\sigma_x^2 \neq \sigma_y^2$)

4.48

4.50

4.56