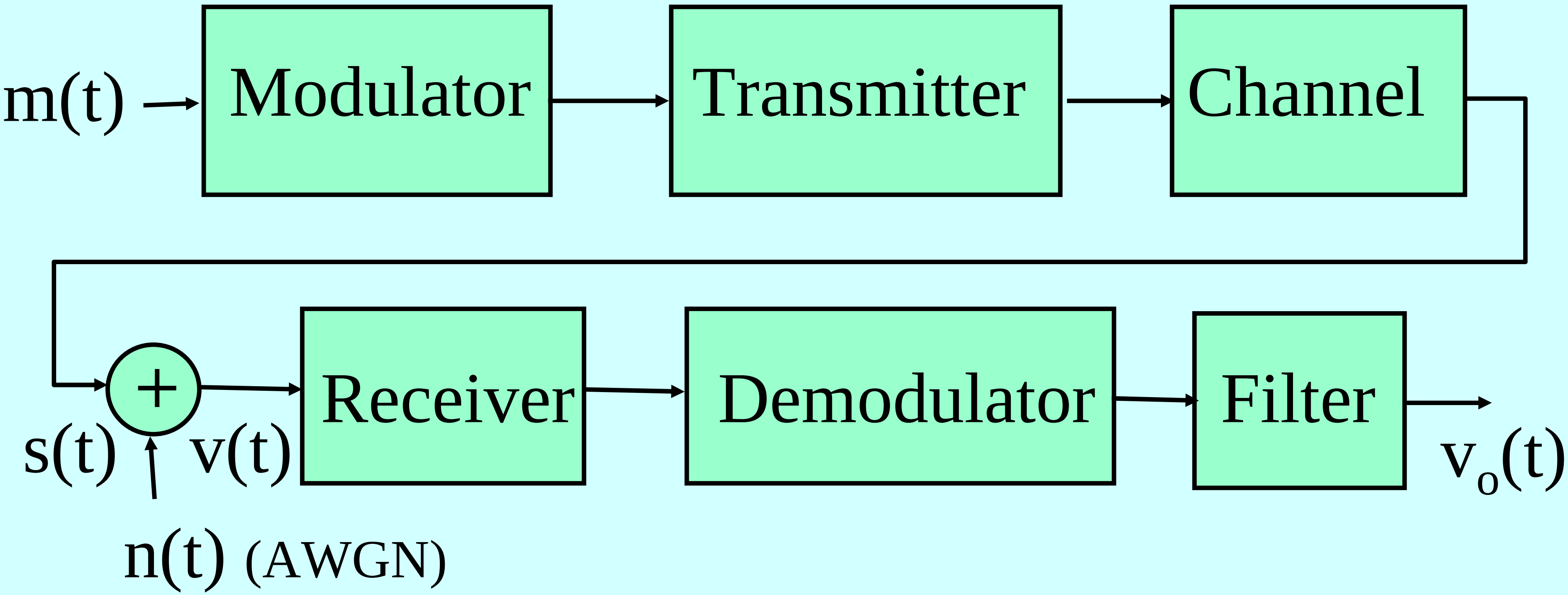
Section 5: Effect of Noise on Analog Systems

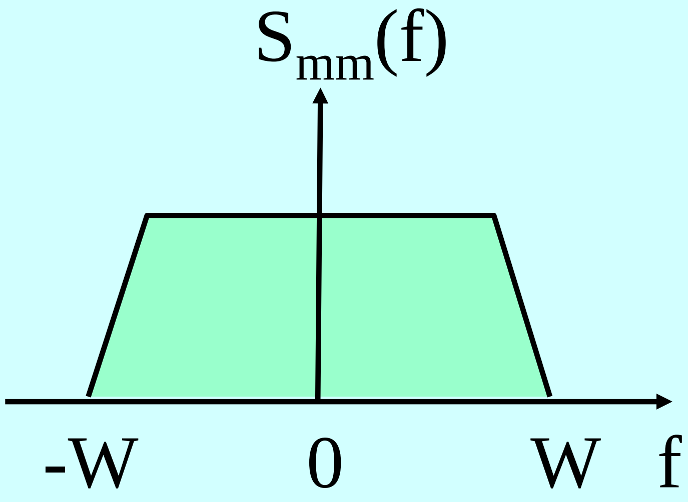
Contents

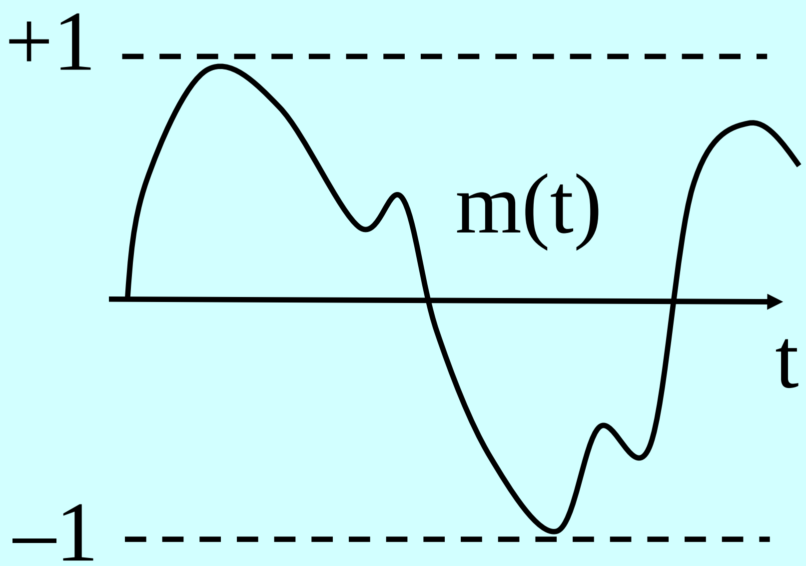
5. Effect of Noise on Analog Systems

An analog communication system is subject to additive noise as shown below.

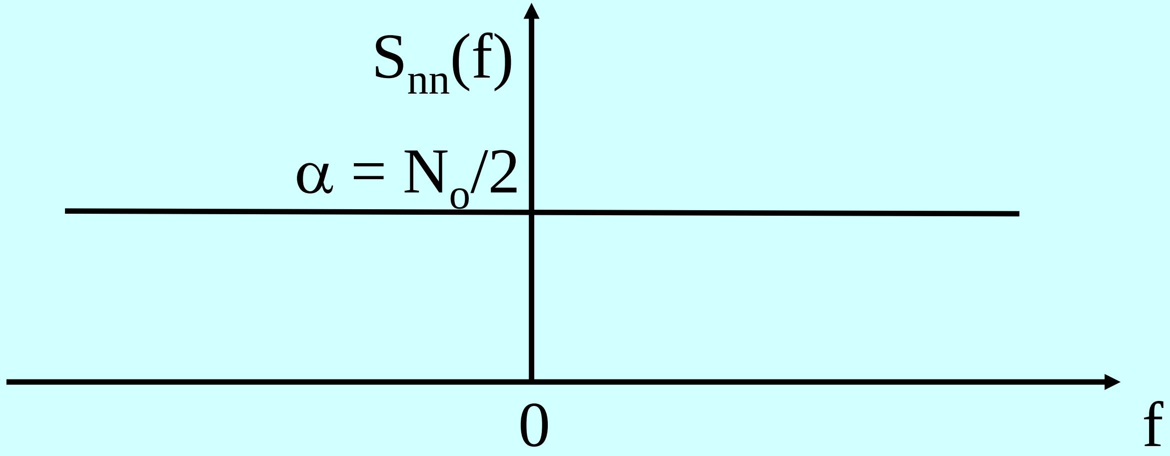


In practice the message signal m(t) will be a low pass ***random signal*** of bandwidth W, so we need to consider its power spectral density $S_{mm}(f)$. As before, we require $-1 \le m(t) \le 1$.





The noise n(t) will be assumed to be **additive white Gaussian noise** (AWGN). Its spectral density is $\mathbf{S_{nn}(f) = \alpha = N_o/2}$. Note that although N_o is the 'single sided' noise spectral density, in all our working we will use double sided power spectral densities. We will assume n(t) includes the effects of receiver noise as well.



Signal to noise ratio

In assessing the performance of analog communication systems, we will be concerned with ***signal to noise ratio*** (SNR). Since amplifier gains do not affect SNR, they will usually be ignored. However if the actual signal levels are of interest, then the amplifier gains must be included.

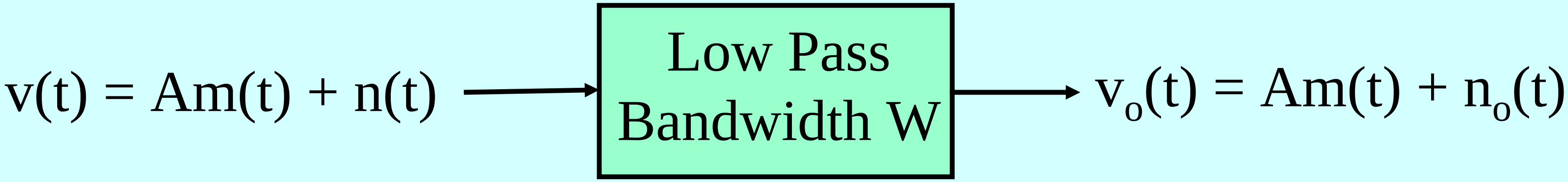
Note that the SNR is often expressed in **decibels (dB) = 10 $\log_{10}$(power ratio)** and **(power ratio) = $10^{(0.1*dB)}$**. Do not use dB in any formulae.

We will compare the performance of various systems on the basis of the **output SNR** obtained compared with that of a baseband system for the same ***average received signal power*** and the same ***noise power spectral density***. (In practice it might be better to compare on the basis of ***peak power***, since this is the limiting factor in transmitter design).

Of course, in many situations a baseband system is not a viable alternative (eg. radio broadcasting), but it serves as a useful comparison basis.

5.1 Baseband Transmission

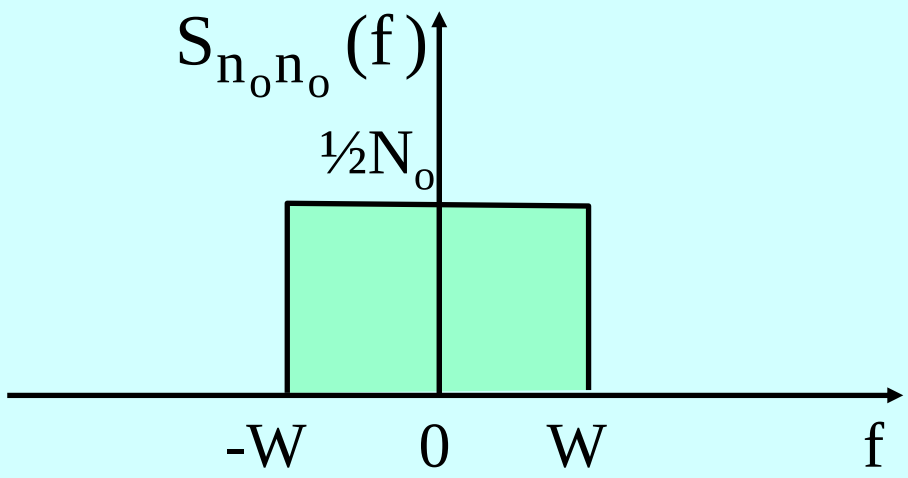
No modulator is used, so the receiver simply consists of a low pass filter of bandwidth **W** to remove extraneous noise components, while not affecting the signal s(t) = Am(t).



We will assume the low pass filter is ideal (which is not true in practice), but we can approximate this very closely.

The (average) received signal power is $P_r = A^2\langle m^2 \rangle$. The output noise $n_o(t)$ has a power spectral density as shown below, and the output noise power is:

$$\left\langle n_o^2(t) \right\rangle = \int_{-\infty}^{\infty} S_{n_o n_o}(f)\,df = N_o W$$



Hence for a baseband system we have:

$$v(t) = s(t) + n(t) = Am(t) + n(t)$$

$$v_o(t) = s_o(t) + n_o(t) = Am(t) + n_o(t)$$

$$SNR_o = \frac{\left\langle s_o^2(t) \right\rangle}{\left\langle n_o^2(t) \right\rangle}$$

$$SNR_o = \frac{A^2 \left\langle m^2(t) \right\rangle}{N_o W} = \frac{P_r}{N_o W}$$

5.2 Amplitude Modulation

In an amplitude modulation (AM) system the received signal v(t) = s(t) + n(t) is given by:

$$v(t) = A[1 + a\,m(t)]\cos(2\pi f_o\, t) + n(t)$$

where A is the carrier amplitude and ‘a’ is the ***modulation index*** $0 \le a \le 1$.

The RF and IF amplifiers amplify the signal and have a bandwidth of **$B \geq 2W$**. It will be assumed that they have no effect on the signal components, but bandlimit the noise reaching the demodulator.

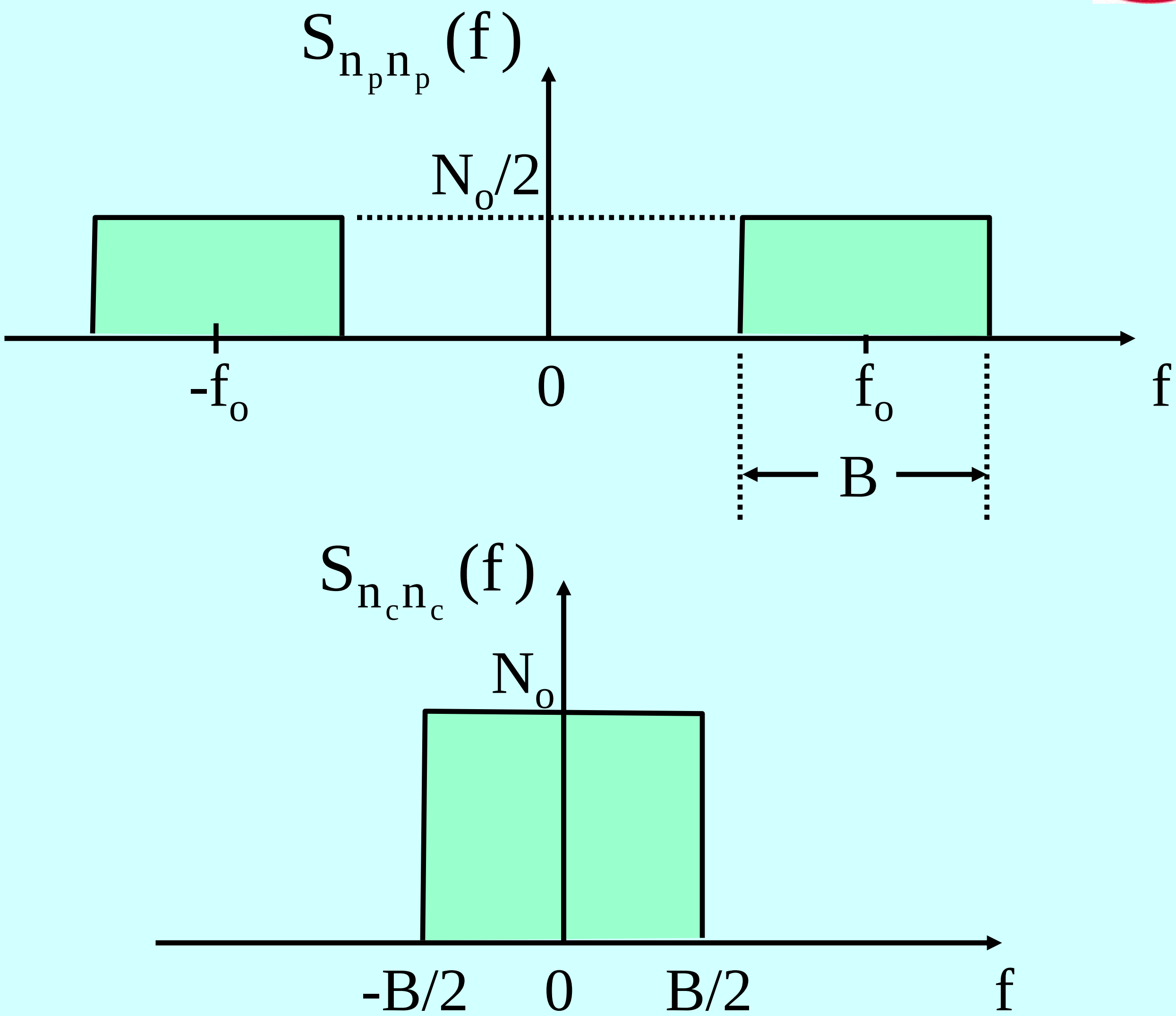
Just prior to the demodulator we have the ***pre-detection signal*** **$v_p(t)$** given by (ignoring gains):

$$v_p(t) = A[1 + am(t)]\cos(2\pi f_o t) + n_p(t)$$

where $n_p(t)$ is ***bandlimited white noise***. To draw a phasor diagram, we need to express this in phasor form.

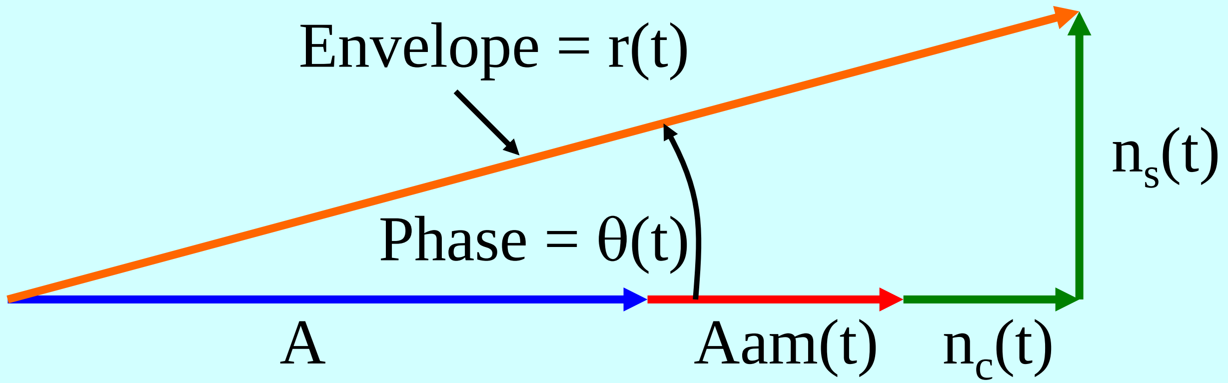
$$
\begin{aligned}
n_p(t) &= n_c(t)\cos(2\pi f_o t) - n_s(t)\sin(2\pi f_o t) \\
&= \mathrm{Re}\left\{\left(n_c(t) + jn_s(t)\right)e^{j2\pi f_o t}\right\}
\end{aligned}
$$

$$
\begin{aligned}
S_{n_c n_c}(f) &= S_{n_s n_s}(f) \\
&= S_{n_p n_p}(f + f_o) + S_{n_p n_p}(f - f_o) \\
&= N_o \quad ; |f| < B/2
\end{aligned}
$$



$S_{n_p n_p}(f)$
$N_o/2$
$-f_o$
0
f_o
f
B
$S_{n_c n_c}(f)$
N_o
-B/2
0
B/2
f

The phasor diagram of the pre-detection signal $v_p(t)$ is:



Note that the noise causes phase modulation, and the envelope is not linearly related to the message signal.

The ***envelope detector*** measures r(t).

$$
\begin{aligned}
v_d(t) &= r(t) \\
&= \sqrt{\left(A + Aa\,m(t) + n_c(t)\right)^2 + n_s^2(t)} \\
&\approx A + Aa\,m(t) + n_c(t)
\end{aligned}
$$

The approximation is valid if $n_c(t)$ and $n_s(t)$ are small compared to A. We ignore the DC component and pass the AC components to the post-detection low pass filter.

The output of the post-detection filter (which we assume is an ideal low pass filter of bandwidth W except that it does not pass DC) is:

$$v_o(t) = s_o(t) + n_o(t)$$

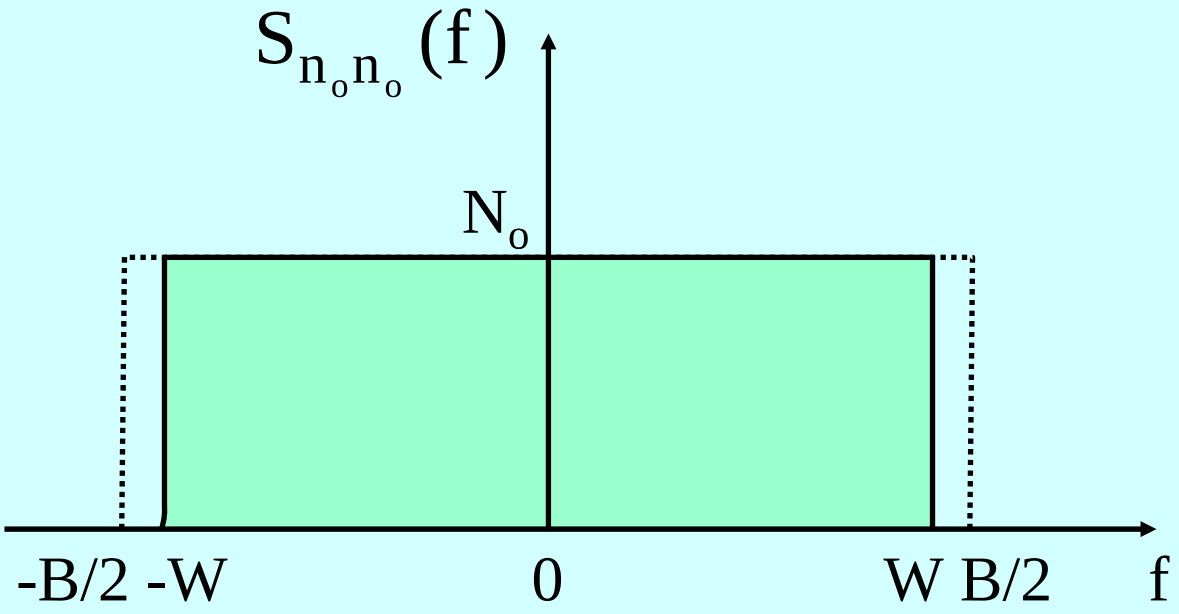
$$\text{where } \; s_o(t) = Aa\,m(t)$$

The DC component A is often used for automatic gain control (AGC) of the RF and IF amplifiers, so that the voltage presented to the demodulator is relatively independent of the received signal level.

The output signal and noise powers are:

$$\langle s_o^2(t) \rangle = A^2 a^2 \langle m^2(t) \rangle$$

$$\langle n_o^2(t) \rangle = 2 N_o W$$



Hence the output signal to noise ratio for AM is:

$$SNR_{am} = \frac{\left\langle s_o^2(t) \right\rangle}{\left\langle n_o^2(t) \right\rangle} = \frac{A^2 a^2 \left\langle m^2 \right\rangle}{2N_o W}$$

For comparison purposes, we need to express this in terms of the **average received power P_r**.

$$P_r = \left\langle s^2(t) \right\rangle = \left\langle \left[A(1 + am(t))\cos(\omega_o t)\right]^2 \right\rangle$$

$$= \frac{1}{2} A^2 \left(1 + a^2 \left\langle m^2(t) \right\rangle \right)$$

Hence we obtain:

$$SNR_{am} = \frac{A^2 a^2 \langle m^2 \rangle}{2N_o W} = \left(\frac{a^2 \langle m^2 \rangle}{1 + a^2 \langle m^2 \rangle} \right) \left(\frac{P_r}{N_o W} \right)$$

where $P_r = \frac{A^2}{2}\left(1 + a^2 \langle m^2 \rangle \right)$

and we note that P_r/N_oW is the output SNR for a baseband system. AM does not perform as well in comparison.

Comparing AM with a baseband system on the basis of average power, its performance is rather poor, since the factor multiplying P_r/N_oW can be quite small. While for a sinewave with 100% modulation this factor is 0.3333, for $\langle m^2 \rangle = 0.1$ we have 0.0909.

Exercise: Do the comparison of these systems on the basis of peak power. (Answer: SNR_{am} is 0.25 of that for a baseband system for 100% sinewave modulation and the same peak power).

The ***pre-detection signal to noise ratio*** is given by:

$$SNR_p = \frac{P_r}{\left\langle n_p^2 \right\rangle} = \frac{P_r}{N_o B} = \frac{A^2(1 + a^2 < m^2 >)}{2N_o B}$$

The envelope detector linear approximation gives the correct result provided this is greater than about 5 dB.

A synchronous demodulator will extract the message signal **linearly** even with large noise. The SNR calculations are the same as before, except they are valid even for small values of SNR_p.

$$v_p(t) = A\{1 + am(t)\}\cos(\omega_o t)$$
$$+n_c(t)\cos(\omega_o t) - n_s(t)\sin(\omega_o t)$$

$$v_d(t) = 2v_p(t)\cos(\omega_o t)$$
$$= A + A\,a\,m(t) + n_c(t) + \text{high frequency terms}$$

The high frequency terms are at a frequency near $2f_o$ and are removed by the post-detection filter, and need concern us no further.

This detector is ***linear***, and produces a better result than the envelope detector if $n_c(t)$ and $n_s(t)$ are large, and also correctly demodulates if $a > 1$ (as in colour TV for instance). The envelope detector is non-linear for large noise, and produces distortion of the signal and also extra noise.

5.3 Double Sideband Suppressed Carrier

If we omit the carrier component of AM, we obtain the DSBSC signal below. The modulation index "a" now has no meaning and is omitted. To demodulate this signal we must use synchronous demodulation, because an envelope detector will not retrieve the modulation m(t). The bandwidth required is the same as for AM, viz. **$B \geq 2W$**.

$$v(t) = A\,m(t)\cos(2\pi f_o t) + n(t)$$

$$v_p(t) = A\,m(t)\cos(2\pi f_o t) + n_p(t)$$

$$v_d(t) = A\,m(t) + n_c(t)$$

$$v_o(t) = A\,m(t) + n_o(t)$$

$$P_r = \frac{1}{2} A^2 \left\langle m^2(t) \right\rangle$$

$$SNR_{dsbsc} = \frac{A^2 \left\langle m^2 \right\rangle}{2 N_o W} = \frac{P_r}{N_o W}$$

Hence this gives the same SNR as a baseband system for the same received power (and noise spectral density).

The ***pre-detection signal to noise ratio*** is given by:

$$SNR_p = \frac{P_r}{\langle n_p^2 \rangle} = \frac{P_r}{N_o B} = \frac{A^2 < m^2 >}{2 N_o B}$$

The synchronous detector is linear and the output SNR result is correct even if $SNR_p < 0$ dB.

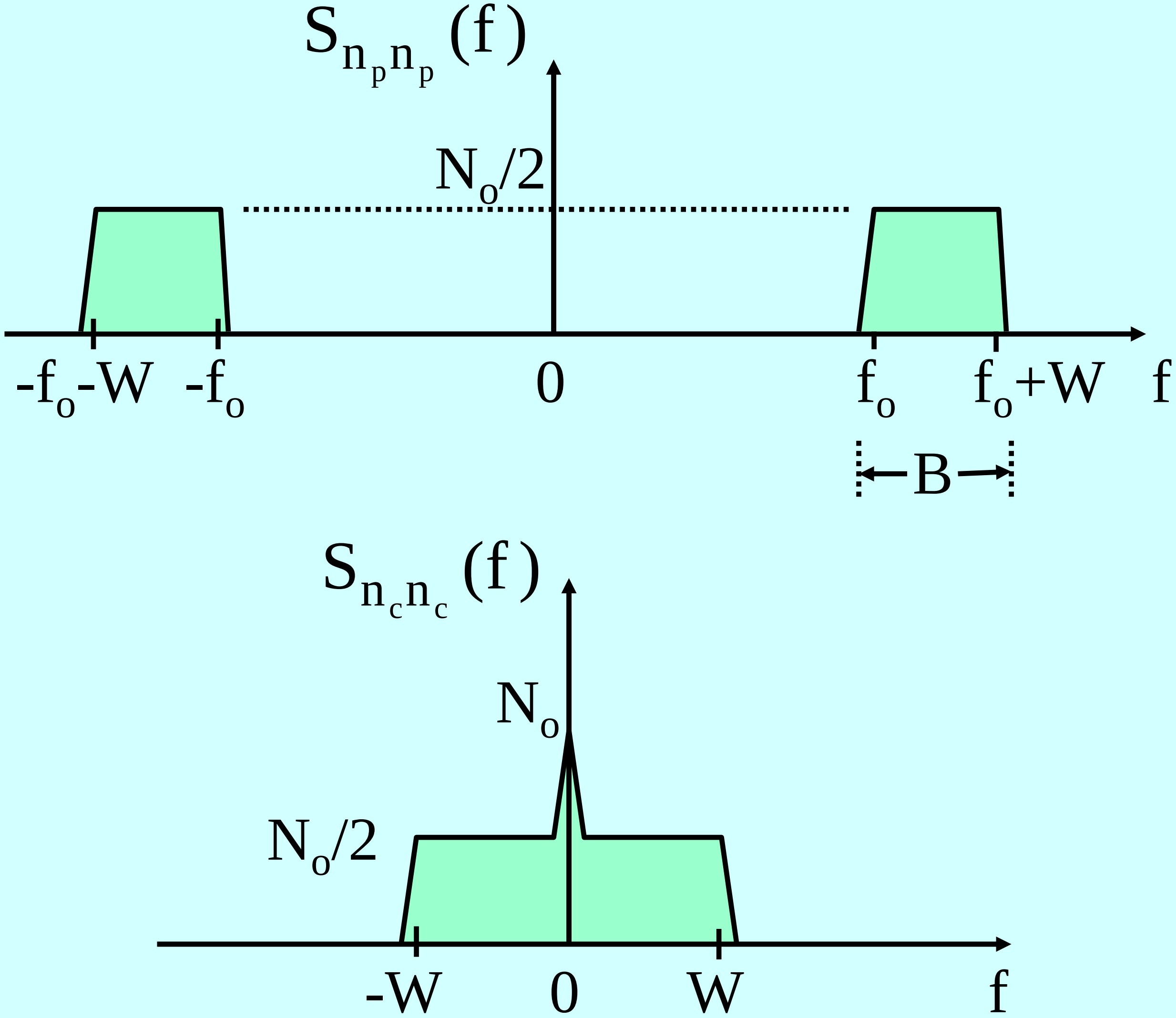
5.4 Single Sideband Suppressed Carrier

For an upper sideband system:

$$v(t) = A\left\{m(t)\cos(2\pi f_o t) - \hat{m}(t)\sin(2\pi f_o t)\right\} + n(t)$$

$$v_p(t) = A\left\{m(t)\cos(2\pi f_o t) - \hat{m}(t)\sin(2\pi f_o t)\right\} + n_p(t)$$

The RF and IF filters only pass frequencies from f_o to f_o + W, so the bandwidth **B ≥ W**. Note that as a consequence, the power spectrum of $n_p(t)$ is **not** the same as for AM and DSBSC.


$S_{n_p n_p}(f)$
$N_o/2$
$-f_o-W$
$-f_o$
0
f_o
f_o+W
f
B
$S_{n_c n_c}(f)$
N_o
$N_o/2$
-W
0
W
f

Note that $n_c(t)$ has a power spectral density of $N_o/2$. Synchronous demodulation gives:

$$v_d(t) = A\,m(t) + n_c(t)$$

$$v_o(t) = A\,m(t) + n_o(t)$$

$$P_r = \tfrac{1}{2}A^2\left\langle m^2(t)\right\rangle + \tfrac{1}{2}A^2\left\langle \hat{m}^2(t)\right\rangle$$

$$= A^2\left\langle m^2(t)\right\rangle$$

$$SNR_{ssbsc} = \frac{A^2\left\langle m^2\right\rangle}{N_o W} = \frac{P_r}{N_o W}$$

Hence, SSBSC has the same SNR performance as DSBSC (and a baseband system), the advantage being that it only requires half the bandwidth of AM or DSBSC (ie. B = W).

The ***pre-detection signal to noise ratio*** is given by:

$$SNR_p = \frac{P_r}{\langle n_p^2 \rangle} = \frac{P_r}{N_o B} = \frac{A^2 < m^2 >}{N_o B}$$

5.5 Carrier Phase Estimation

To use a synchronous detector in DSBSC, it is necessary to generate a local carrier signal. However, the signal does not contain any component at the carrier frequency, so we have to resort to non-linear processing.

We can generate a double frequency component by squaring the received signal. To reduce the noise components we do this after RF and IF filtering.

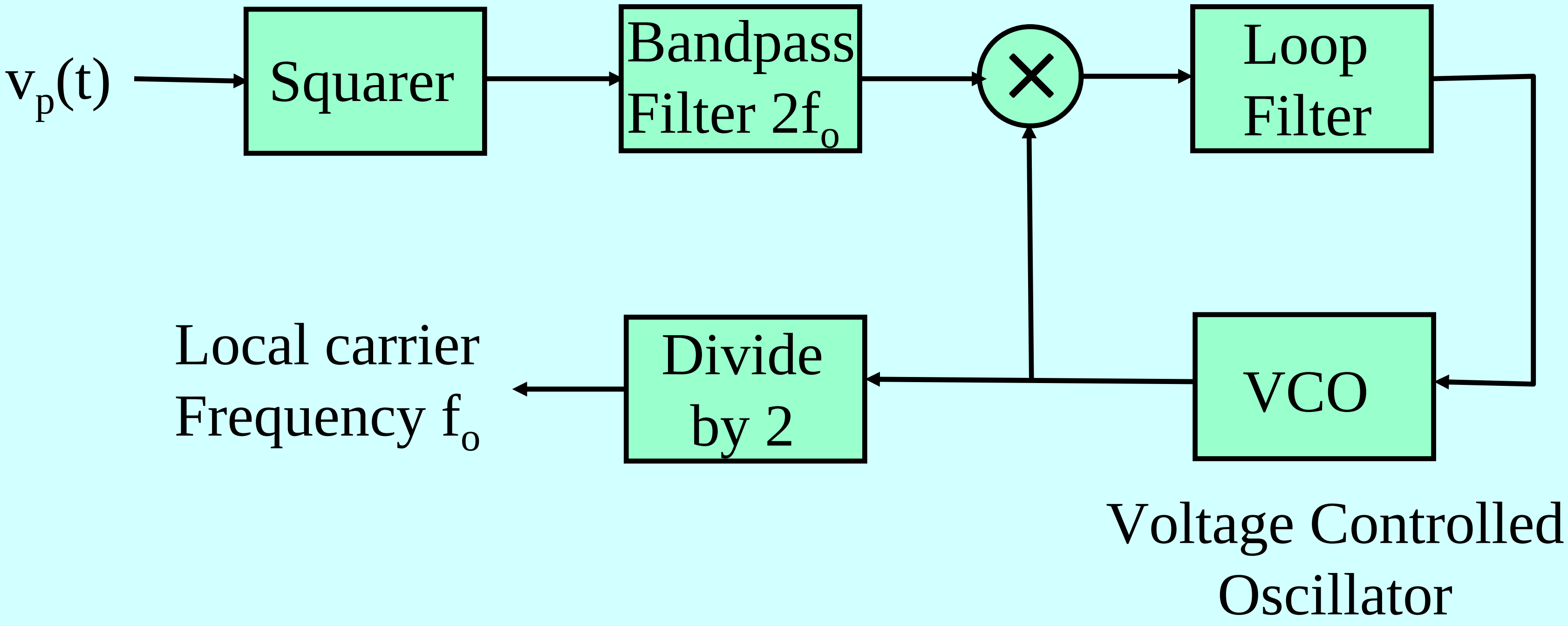
$$v_p(t) = A\,m(t)\cos(2\pi f_o t) + n_p(t)$$

$$v_p^2(t) = A^2 m^2(t)\cos^2(2\pi f_o t) + \text{noise terms}$$

$$= \frac{A^2}{2} m^2(t) + \frac{A^2}{2} m^2(t)\cos(4\pi f_o t) + \text{noise terms}$$

The second term is the one of interest. Because $m^2(t) > 0$, this term contains a component at double the carrier frequency (actually it will be double the IF frequency at this point). We can filter this signal to obtain a double carrier frequency signal and then divide this by two.

A possible carrier recovery system is shown below. The prefilter removes the extraneous components from the squarer, and the ***phase locked loop*** extracts the double frequency carrier.



If the component of interest from the squarer is $v_1(t) = C_1 \cos(2\omega_o t)$ and the VCO signal is $v_2(t) = C_2 \sin(2\omega_o t+\phi)$, then the output of the multiplier is:

$$v_3(t) = v_1(t)\, v_2(t)$$
$$= C_1 C_2 \cos(2\omega_o t)\sin(2\omega_o t + \phi)$$
$$= \frac{1}{2} C_1 C_2 \sin(\phi) + \frac{1}{2} C_1 C_2 \sin(4\omega_o t + \phi)$$

The $4\omega_o$ term is rejected by the loop filter and if ϕ is small the first term is approximately $\frac{1}{2} C_1 C_2 \phi$ and the phase locked loop will force $\phi = 0$.

The closed loop response of the phase locked loop is designed to have a narrow bandwidth to provide a jitter free carrier reference.

The sign ambiguity when we divide the frequency by two does not matter since for a synchronous detector either $\pm\cos(\omega_o t)$ can be used.

For details on the phase locked loop design, see Proakis.

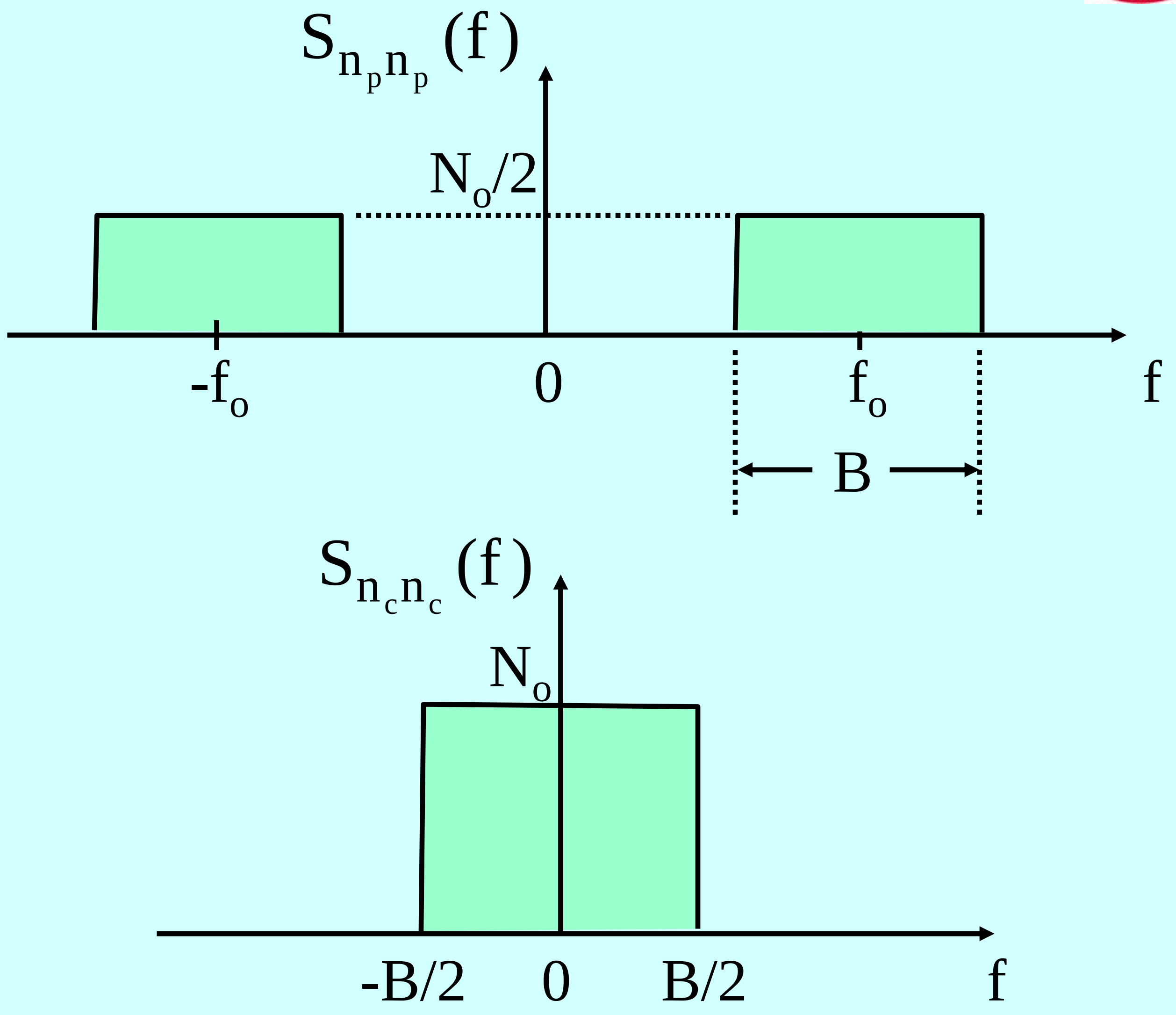
5.6 Frequency Modulation

The received signal in a ***frequency modulation*** (FM) system has the form:

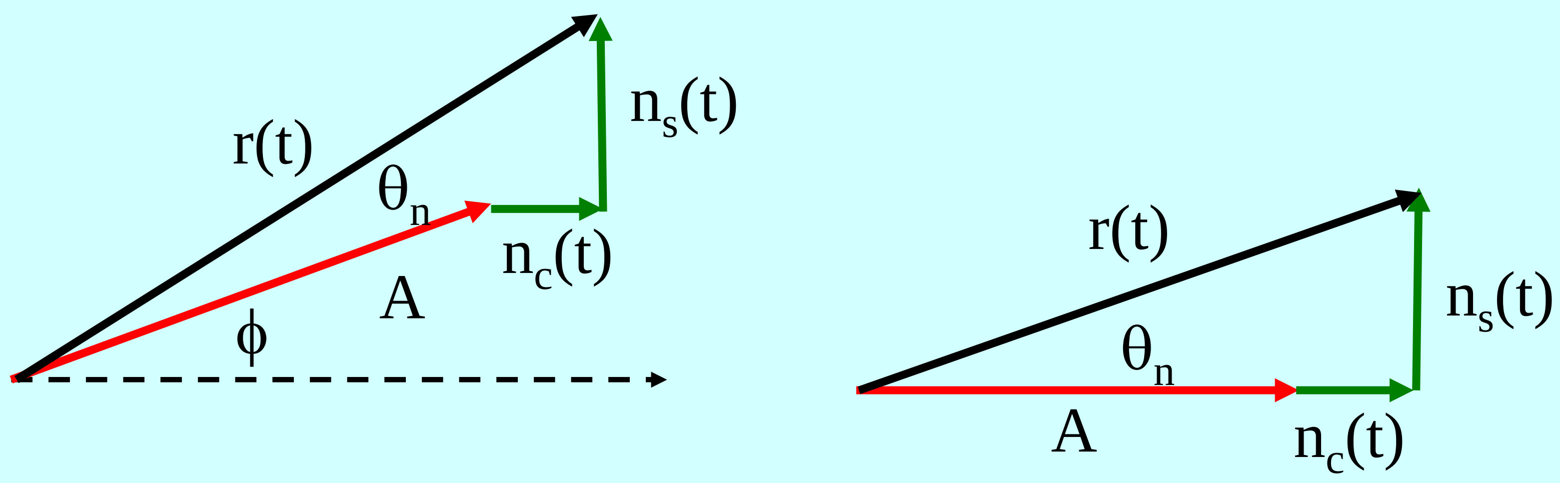
$$v(t) = A\cos\left(2\pi f_o\, t + 2\pi f_d \int m(\tau)\, d\tau\right) + n(t)$$

$$v_p(t) = A\cos\left(2\pi(f_o\, t + 2\pi f_d \int m(\tau)\, d\tau\right) + n_p(t)$$

In this case the RF and IF amplifiers have a bandwidth of approximately $B = 2(f_d + W)$, which is usually >> 2W.


$S_{n_p n_p}(f)$
$N_o/2$
$-f_o$
0
f_o
f
B
$S_{n_c n_c}(f)$
N_o
-B/2
0
B/2
f

The phasor diagram of the predetection signal is as shown below, with $\phi(t) = 2\pi f_d \int m(\tau)\,d\tau$.



With Modulation

No Modulation

The frequency detector produces a voltage proportional to the instantaneous frequency deviation from f_o.

$$v_d(t) = \frac{1}{2\pi}\frac{d}{dt}\left(\phi(t) + \theta_n(t)\right)$$

$$= f_d\, m(t) + \frac{1}{2\pi}\frac{d\theta_n(t)}{dt}$$

Proakis analyses the noise in FM with modulation present, which is somewhat complicated.

We will compute θ_n for the case where:

(1) $n_c(t)$ and $n_s(t)$ are both << A
(2) there is no modulation.

The first assumption requires that the **pre-detection** signal to noise ratio be greater than about **10 dB*** and it can be shown that the modulation has a negligible effect on the output noise. [* Previously I have used 12 dB, but this is a bit conservative].

$$SNR_p = \frac{P_r}{N_o B} = \frac{A^2}{2N_o B} \geq 10 \ (10 \text{ dB})$$

$$\theta_n(t) \approx \frac{n_s(t)}{A}$$

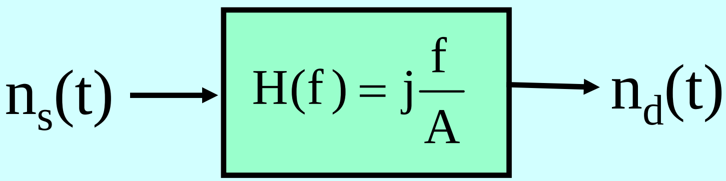
$$n_d(t) = \frac{1}{2\pi}\frac{d\theta_n}{dt} = \frac{1}{2\pi A}\frac{dn_s(t)}{dt}$$

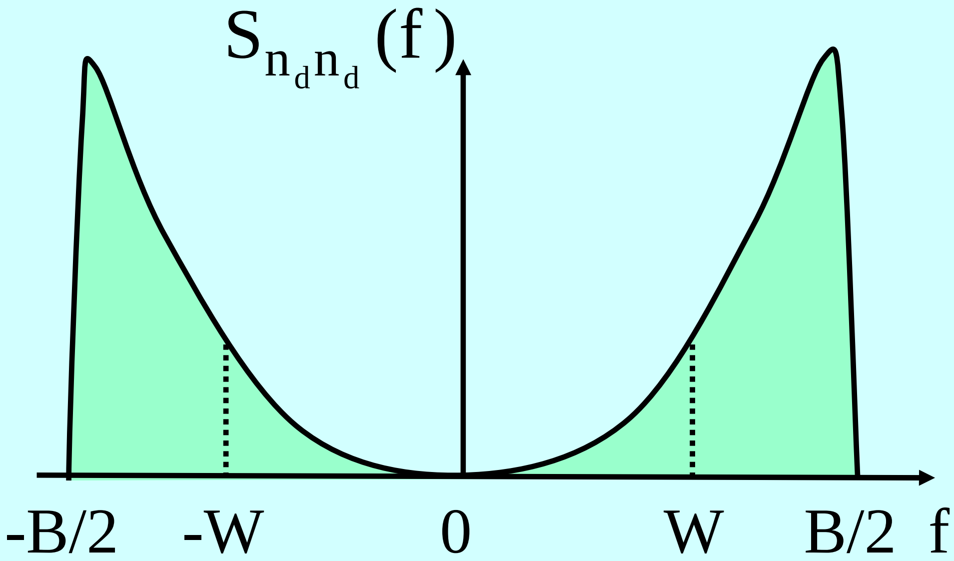
$$S_{n_d,n_d}(f) = S_{n_s,n_s}(f)\left|H(f)\right|^2$$

$$= \frac{N_o f^2}{A^2}$$

$$\text{since} \quad S_{n_s,n_s}(f) = N_o \quad \text{for} \quad -B/2 \le f \le B/2$$

The transfer function H(f) connecting $n_s(t)$ and $n_d(t)$ is H(f) = jf/A.





$$\left\langle n_o^2(t) \right\rangle = \int_{-W}^{W} \frac{N_o f^2}{A^2} df = \frac{2 N_o W^3}{3A^2}$$

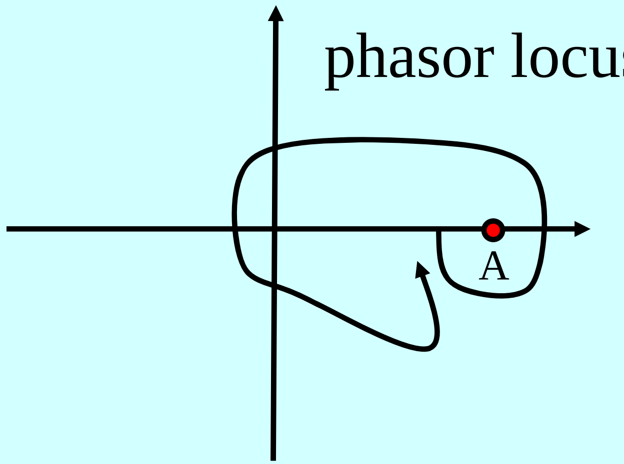
$$\left\langle s_o^2(t) \right\rangle = f_d^2 \left\langle m^2(t) \right\rangle$$

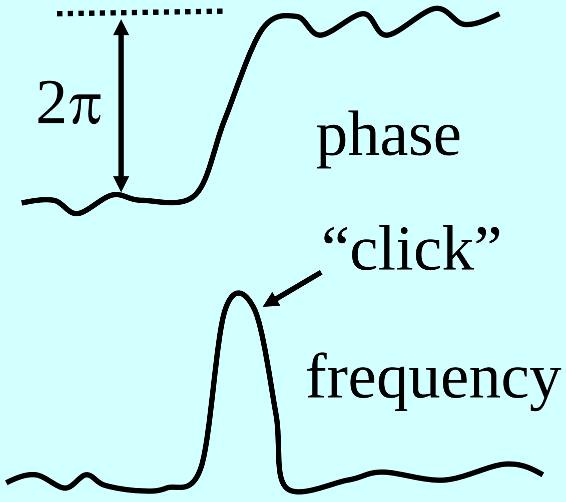
$$SNR_{fm} = \frac{3A^2 f_d^2 \left\langle m^2 \right\rangle}{2 N_o W^3} = 3\left\langle m^2 \right\rangle \left(\frac{f_d}{W} \right)^2 \left(\frac{P_r}{N_o W} \right)$$

Hence SNR_{fm} is much greater than that of a baseband system if $\beta = f_d/W >> 1$ (ie. for wideband FM). However, as we increase β we require more bandwidth, and if **SNR_p** falls below **10 dB** the output SNR falls rapidly, and the system is said to be **below threshold**. The critical value of P_r/N_oW at threshold is:

$$\frac{P_r}{N_o W}(th) = \left(\frac{P_r}{N_o B}\right)\left(\frac{B}{W}\right) = 10\frac{B}{W}$$

The threshold in FM is caused by the noise phasor encircling the origin. The 2π jump in phase is converted into an impulse by the frequency detector. As SNR_p falls below **10 dB**, the number of impulses increases rapidly, and so does the output noise.





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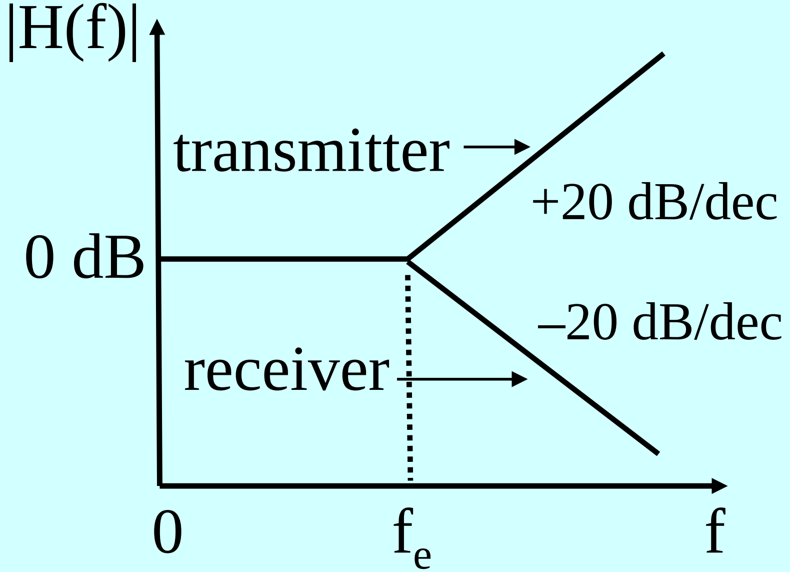
For an FM system with modulation index β, the bandwidth required is B = 2W(β + 1), so **at threshold** where $P_r/N_oB = 10$, we have $P_r/(N_oW) = 20(\beta + 1)$. [This is consistent with Proakis].

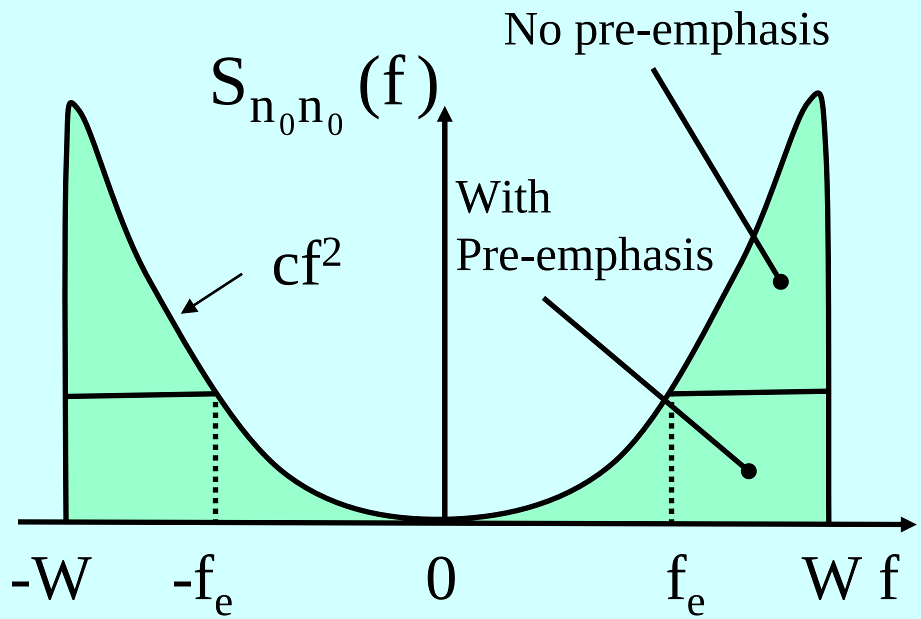
For β <<1, $P_r/(N_oW)$ (th) = 20 (**13.0 dB**)

For β = 2, $P_r/(N_oW)$ (th) = 60 (**17.8 dB**)

For β = 5, $P_r/(N_oW)$ (th) = 120 (**20.8 dB**).

Most FM systems use ***pre-emphasis***. At the transmitter, high frequencies are boosted and this is compensated by a de-emphasis in the receiver. There is no nett effect on modulation, but the noise is substantially reduced.





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Without pre-emphasis:

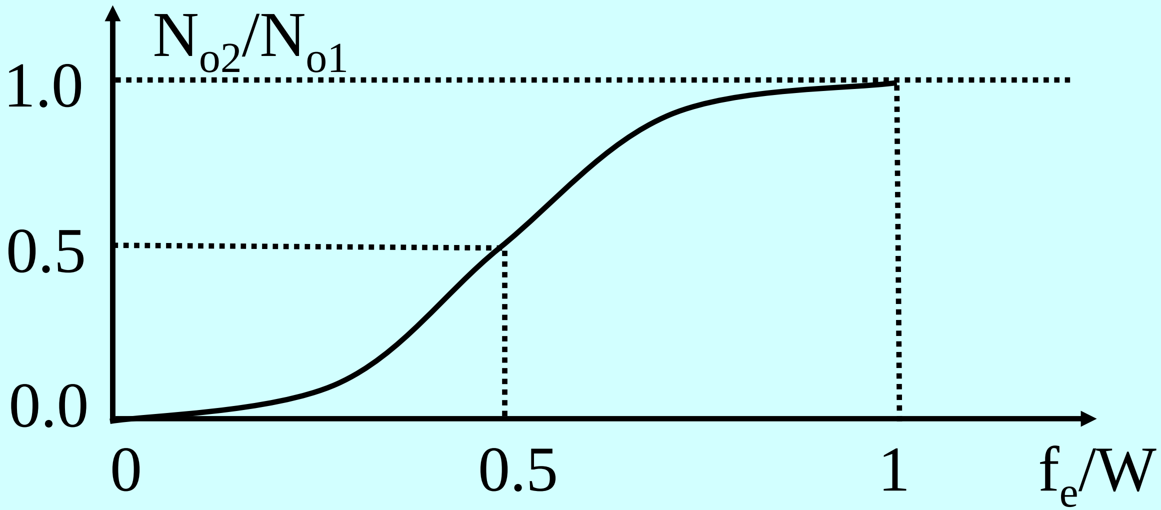
$$N_{o1} = \int_{-W}^{W} cf^2 df = \frac{2}{3} cW^3$$

With pre-emphasis:

$$N_{o2} \approx \int_{-f_e}^{f_e} cf^2 df + 2cf_e^2 (W - f_e) = 2cf_e^2 W - \frac{4}{3} cf_e^3$$

$$\frac{N_{o2}}{N_{o1}} = 3\left(\frac{f_e}{W}\right)^2 - 2\left(\frac{f_e}{W}\right)^3$$

For $f_e << W$ substantial gains can be made. However, f_e must be high enough that the amplitude of the signal is not significantly increased, otherwise the pre-emphasised signal will have to be reduced in amplitude (at the transmitter).



Example: In Australia, broadcast FM uses β =5 and a pre-emphasis time constant of 50 us, which corresponds to f_e = 3.18 kHz. For a baseband bandwidth of 15 kHz, this gives an improvement of 9.4 dB.

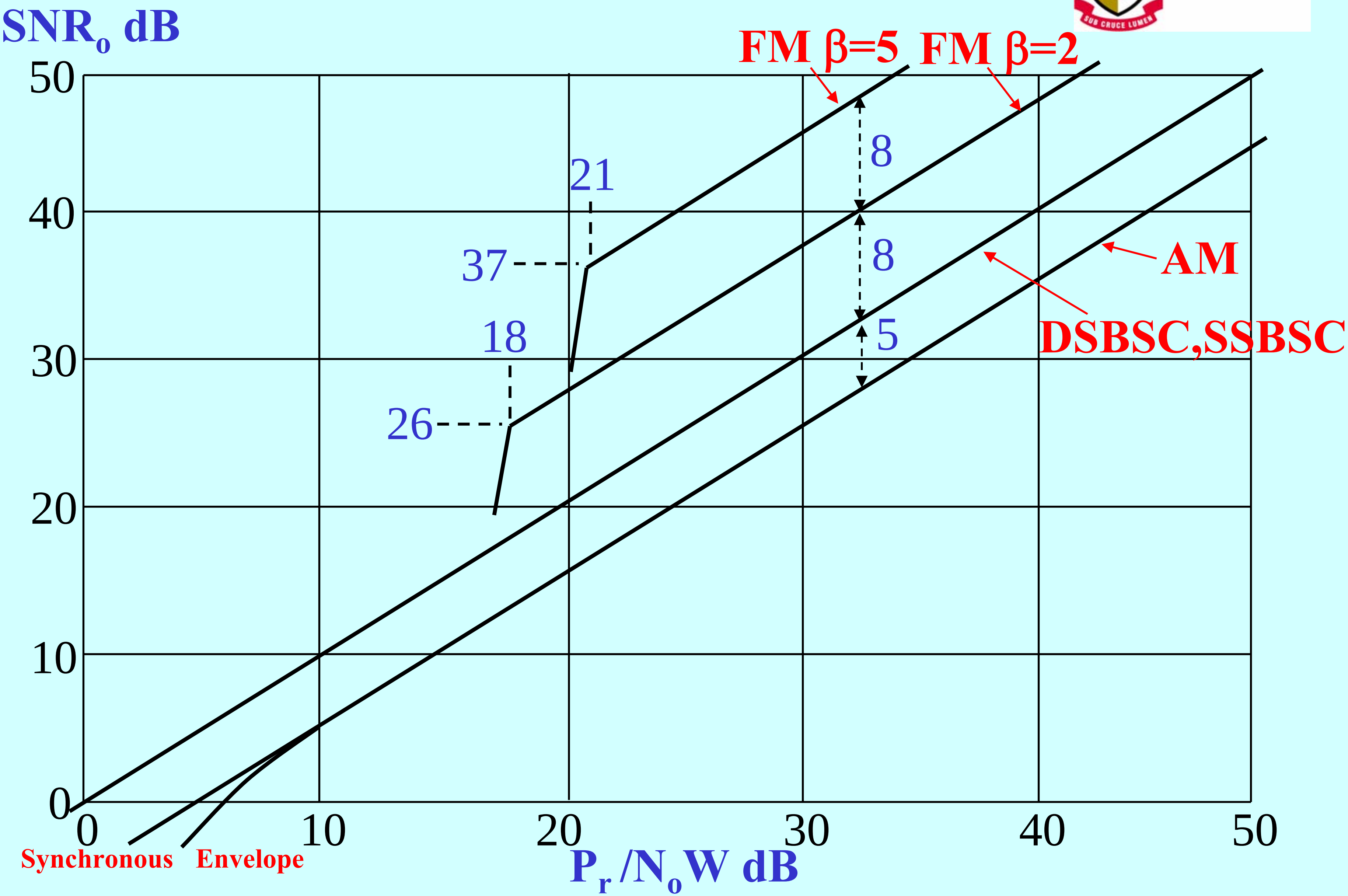
Exercise:

(i) Do the necessary calculations to verify this.
(ii) The above analysis is approximate. Try an exact analysis using a de-emphasis filter $H(f) = 1/(1+jf/f_e)$, it gives essentially the same answer.

5.7 Comparison of Analog Modulation Systems

To compare the various analog modulation systems we will plot the output SNR against P_r/N_oW for 100% sinusoidal modulation. In general the output SNR for a typical speech or music signal will not be as high.

The SNR for FM is that without pre-emphasis. Also FM is usually transmitted in a stereo format, and the SNR for this is lower than for monaural transmission.


SNRo dB
FM β=5
FM β=2
50
40
30
20
10
0
21
37
8
8
5
18
26
AM
DSBSC,SSBSC
0
10
20
30
40
50
Synchronous
Envelope
Pr /NoW dB

Exercises: You are expected to attempt the following exercises in Proakis & Salehi. Completion of these exercises is part of the course. Solutions will be available later.

5.4
5.5
5.9 (1)
5.10
5.11