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Applications Of Stochastic Differential Equations In Electronics

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Abstract. We argue that the stochastic differential calculus of Itô is the best tool for the analysis of noise in electronic circuits. We begin by showing how the nodal and mesh equations of ordinary circuit analysis can be extended to model the effects of thermal fluctuations. This leads to a systematic method for formulating Langevin equations for electronic circuits. These can then be transformed into ordinary differential equations, allowing the calculation of noise power without the need to explicitly solve the stochastic differential equations.

We have encountered a difficult unsolved problem of noise, which arises when our technique is applied to the standard noise model for a MOSFET. The resulting equations are singular and do not yield to the usual algebraic manipulations. We do not appear to have enough independent equations for the number of (apparently) independent variables. Why is it so?

Keywords: Circuit analysis, Electronics, Noise, Langevin equations, Stochastic Differential Equations
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INTRODUCTION

The traditional approach to thermal noise in circuits can be traced back to Johnson [1]. Over the years, a number of empirical techniques have been developed to estimate noise in circuits that filter the thermal noise. Our aim is to extend these results in a more systematic manner.

White noise cannot be constructed as an ordinary function of time, say $Z(t)$. It should be regarded a generalised function, or "distribution" of the type described by Schwartz [2, 3]. Since noise is a generalised function, it must appear inside an integration. Itô's differential equation

$$dX_t = \sigma(X_t)dB_t + \mu(X_t)dt \tag{1}$$

is really short hand for an integral equation

$$X_t - X_{t_0} = \int_{t_0}^{t} \sigma(X_\tau)dB_\tau + \int_{t_0}^{t} \mu(X_\tau)d\tau. \tag{2}$$

We can make a finite difference approximation for a very short time interval $\Delta t = t - t_0$:

$$\Delta X_t \approx \sigma(X_t)\Delta B_t + \mu(X_t)\Delta t. \tag{3}$$

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We can consider the behavior of a circuit over a short but finite time interval, Δt. This interval needs to be large enough to avoid problems with physical representation, such as infinite bandwidth, and yet short enough to represent the changing behaviour of the system under investigation. We shall work with infinitesimal increments, of the form of Equation 1, knowing that these are really short hand for integrals over finite time intervals of the form of Equation 2 and we can approximate the process in computer simulations using Equation 3.

We can re-write Kirchhoff's laws for infinitesimal intervals of time to provide a single simple systematic basis for combining noise signals and deterministic signals.

KIRCHHOFF'S CURRENT LAW

The infinitesimal version of Kirchhoff's Current Law (KCL) is written in terms of infinitesimal increments of electric flux (electric charge) flowing out of a node:

$$\sum_{\forall k} dQ_k = 0. \tag{4}$$

If all the contributions, dQ_k are deterministic, then we can integrate and get $\sum_{\forall k} I_k = 0$, which is the more usual form of the law. The proof of the infinitesimal form follows from one of Maxwell's equations [4], called Gauss's electric law: $\oint_A \underline{D} \cdot d\underline{A} = \int_v \rho \cdot dv = Q$. This can be reduced to: $\sum_{k=1}^{N} dQ_k = dQ$. The correct modelling of dQ depends the capacitance, C, of the surface, A, with respect to earth. The stored energy of the total charge in the node is: $U = \frac{1}{2C}Q^2$ If we assume that the system is at thermodynamic equilibrium then it is possible to show that $E\left[\frac{1}{2C}Q^2\right] = \frac{1}{2}kT$, which is simply the mean energy for each degree of freedom, in a classical thermodynamic system at temperature T. This implies that $E\left[Q^2\right] = kT{\cdot}C$. We regard the charge enclosed in a node as being a normally distributed random process, $Q \sim N(0, kT{\cdot}C)$ and we can write $\sum_{k=1}^{N} dQ_k = \sqrt{kTC}{\cdot}dB$. This differs from the homogeneous form of Kirchhoff's current law, in Equation 4. Kirchhoff's laws are ultimately statistical and there will be small fluctuations that can usually be ignored, as long as the circuit is well constructed and the capacitance of the node to earth, C is small. If this is not the case, then we would need to include the parasitic capacitance, C, in our model of the circuit. In the remainder of this paper, we will use the homogeneous, differential form of Kirchhoff's current law, Equation 4, where infinitesimal increments of charge are represented in the form, $dQ = Idt$ for deterministic currents, and also in the form, $dQ = i_n dB$ for noise currents.

KIRCHHOFF'S VOLTAGE LAW

The infinitesimal version of Kirchhoff's Voltage Law (KVL) is written in terms of infinitesimal increments of magnetic flux through a mesh:

$$\sum_{\forall k} d\Phi_k = 0. \tag{5}$$

If all the contributions, $d\Phi_k$, are deterministic then we can integrate and get $\sum_{\forall k} V_k = 0$, which is the more usual form of the law. This is equivalent to saying that electric fields are the product of a static potential field, $\underline{E} = -\nabla V$. The proof of the infinitesimal form of Kirchhoff's voltage law follows from one of Maxwell's equations [4], called Faraday's law: $\oint_\lambda \underline{E} \cdot d\underline{\lambda} = -\frac{\partial}{\partial t} \int_S \underline{B} \cdot d\underline{A} = -\frac{\partial \Phi}{\partial t}$. This can be reduced to: $\sum_{k=1}^{N} V_k dt = \sum_{k=1}^{N} d\Phi_k = -d\Phi$. The correct modelling of $d\Phi$ depends on the physical geometry of the current path. Specifically, we need to know the inductance, L, of the current path. The stored energy of the total charge in the node is: $U = \frac{1}{2L}\Phi^2$. If we assume that the system is at thermodynamic equilibrium with the surroundings, then we can show that $E\left[\Phi^2\right] = kT{\cdot}L$. We regard the magnetic flux linked by the contour, λ, as being a normally distributed random process, $\Phi{\sim}N(0, kT{\cdot}L)$ and we can write $\sum_{k=1}^{N} d\Phi_k = -\sqrt{kTL}{\cdot}dB$. This differs from the homogeneous from of Kirchhoff's voltage law, in Equation 5. Kirchhoff's laws are ultimately statistical and there will be small fluctuations, which can usually be ignored. If this is not the case, then we would need to include the parasitic inductance, L, in our model of the circuit. In the remainder of this paper, we will use the homogeneous, differential form of Kirchhoff's voltage law, Equation 5, where infinitesimal increments of magnetic flux are represented in the form $d\Phi = V dt$ for deterministic currents, and also in the form, $d\Phi = v_n dB$ for noise currents.

MODELS FOR RESISTORS

There are two possible linear models for thermal noise in a resistor, a Thévenin and a Norton model, which are shown in Figure 1. The Thévenin model, Figure 1(a), places

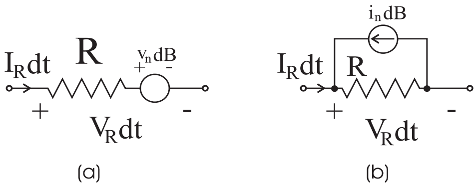

FIGURE 1. The Thévenin model, (a), and Norton model, (b), noise models for a resistor.

a noise voltage source in series with an ideal noiseless resistor. During an infinitesimal time interval, dt, the source contributes a magnetic flux of $v_n dB$ to all circuit meshes in which it sits. We can use the results of Johnson and Nyquist, $\Delta\left\langle V^2 \right\rangle = 4kTR{\cdot}\Delta f$, to estimate intensity of the noise voltage, $v_n = \sqrt{2kTR}$. The units of noise voltage are $\mathrm{V}/\sqrt{\mathrm{Hz}}$.

The use of the constant "2" rather than "4" is due to the convention that we use for frequency. It is consistent with widely used definitions of power spectral density and autocorrelation functions [5, 6, 7, 8, 9] and is sometimes called the two-sided power spectrum [9]. An interesting summary of the technicalities and history of spectral noise density can be found in [10].

The Norton model, Figure 1(b), places a noise current source in parallel with an ideal noiseless resistor. During an infinitesimal time interval, dt, the source contributes an electric flux, or charge, of $dQ_1 = i_n dB$ to the node into which the terminal flows. A corresponding opposite charge of $dQ_2 = -i_n dB$ is added at the other terminal. The corresponding intensity of noise current is $i_n = \sqrt{2kT/R}$. The units of noise current are $\mathrm{A}/\sqrt{\mathrm{Hz}}$.

We can write the increments of electric flux associated with a resistor as $dQ = i_n dB$. These increments of electric flux can be used directly in the differential form of KCL, Equation 4. We can write the increments of magnetic flux associated with a resistor as $d\Phi = v_n dB$. These increments of magnetic flux can be used directly in the differential form of KVL, Equation 5. We can also imagine other noise models for other devices. These will generally have different expressions for current and voltage noise intensities, v_n and i_n, but the calculations will generally be of the same form as for the resistor.

A functional mapping between electrical and mechanical systems

There is a widely used functional mapping between quantities in electrical and mechanical systems. These can be expressed in terms of the Hamiltonian function, $\mathscr{H}$, using Hamilton's canonical equations [11].

Variable	Electrical Quantity	Mechanical Quantity
Effort	Voltage, $-V = +\frac{\partial \Phi}{\partial t} = -\frac{\partial \mathscr{H}}{\partial Q}$	Force, $+F = +\frac{dp}{dt} = -\frac{\partial \mathscr{H}}{\partial x}$
Flow	Current, $I = \frac{+\partial Q}{\partial t} = +\frac{\partial \mathscr{H}}{\partial \Phi}$	Velocity, $v = +\frac{dx}{dt} = +\frac{\partial \mathscr{H}}{\partial p}$
Momentum	Magnetic flux,Φ	Momentum, p
Displacement	Electric flux, Q	Displacement, x

Further details of the mapping are described by Karnopp et al. [12]. There are differences of sign in the definitions of the effort variables. These are purely due to historical convention. The mapping applies to all equations of motion, including the Langevin equation.

Variable	Electrical Quantity	Mechanical Quantity
Viscous damping force	$-V = -R \cdot I$	$+F = -\alpha \cdot v$
Noise intensity	$E[{v_n}^2] = 2kT \cdot R$	$E[{f_n}^2] = 2kT \cdot \alpha$
Langevin Equation	$LdI + RIdt = v_n dB_t$	$mdv + \alpha v dt = f_n dB_t$

This demonstrates that the resulting equations for electrical circuits are Langevin equations in the literal sense.Established results for mechanical systems can be applied directly to electrical systems, by mapping solutions to the mechanical problems into the electrical domain. These considerations also suggest that we should use the generalised coordinates, Φ and Q to describe the dynamics of electrical systems, as an analogy of the use of p and x for mechanical systems.

CAPACITORS AND INDUCTORS

For a fixed capacitor, C, with terminal voltage, V, and terminal current, I, we require the following equivalent forms for increments of electric flux $dQ = CdV = Idt$. This allows the capacitor model to be used in Norton and Thévenin models. These increments of electric flux can be used directly in the differential form of KCL, Equation 4.

For a fixed inductor, L, with terminal voltage, V, and terminal current, I, we require the following equivalent forms for increments of magnetic flux $d\Phi = LdI = Vdt$. This allows the inductor model to be used in Norton and Thévenin models. These increments of magnetic flux can be used directly in the differential form of KVL, Equation 5.

A ONE-DIMENSIONAL EXAMPLE OF STOCHASTIC CIRCUIT ANALYSIS

We consider the well known case of a resistor in parallel with a capacitor [13]. This is shown in Figure 2. We expect to find a mean-square voltage across the capacitor of,

$$\left\langle V^2 \right\rangle = \frac{kT}{C}. \tag{6}$$

This is clear if we regard the voltage across the capacitor as a single degree of freedom and apply the principle of equipartition of energy, $\left\langle \frac{1}{2}CV^2 \right\rangle = \frac{1}{2}kT$. We represent the resistor using a Norton model of an noiseless resistor, R, in parallel with a noise source, denoted by $i_0 dB$ where $i_0 = \sqrt{2kT/R}$. We consider the nodal equation for the circuit. We require $\sum dQ = 0$. For the capacitor, we have $dQ = CdV$, where C is the capacitance and dV is an infinitesimal increment of voltage. For the resistor, we have $dQ = \frac{V}{R}dt$ and for the equivalent noise source in the resistor, we have $dQ = i_0 dB$, as discussed above. The nodal equation then becomes

$$CdV + \frac{1}{R}Vdt = i_0 dB, \tag{7}$$

which is the SDE for this system. It has the same form and plays exactly the same role as the Langevin equation in statistical physics [5, 14, 15, 16].

It is possible to evaluate the infinitesimal moments implied by a Langevin equation and to formulate a Partial Differential Equation (PDE), describing the probability density of an ensemble of solutions to the SDE in Equation 7. This PDE is known as the Fokker-Planck equation [5, 17, 16]. If we could solve the Fokker-Planck equation then we could apply operators to the solutions to calculate quantities of interest, such as the noise power as a function of time, $w(t)$. Unfortunately, this approach is cumbersome. Instead, we suggest the application of a method described by Gubner [18], where the SDE is transformed into a much simpler ODE in the noise power.

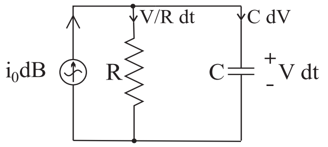

FIGURE 2. *RC* parallel circuit, with resistor, *R*, noise current source, $i_o dB$, and capacitor, *C*.

Equation 7 can be re-written as an SDE in the narrow sense: $dV = \frac{-1}{RC}Vdt + \frac{i_0}{C}dB$. This is of the form $dV = \alpha(t)Vdt + \beta(t)dB$ where $\alpha(t) = \alpha = -1/(RC)$ and $\beta(t) = \beta = +i_0/C = \sqrt{2kT/(RC^2)}$. If we define $\mu = E[V]$ and the variance as $w = E\left[(V-\mu)^2\right]$ then it can be shown [18] that $\frac{dw(t)}{dt} = 2\alpha(t)w(t) + \beta(t)^2$. We have now derived an Ordinary Differential Equation (ODE) in the variable of interest, noise power. If we consider the steady-state situation after all transients have decayed then we have $\frac{dw(t)}{dt} = 0$, which implies that $w = E\left[(V-\mu)^2\right] = -\frac{\beta(t)^2}{2\alpha(t)} = \frac{kT}{C}$, which is the classical result in Equation 6. We did not solve the SDE directly. We only used it to derive an ODE. We did not solve the ODE but only used it to derive an algebraic equation, which we then used to solve for the steady state value of the noise power. We believe that this simple and systematic method is general and should find wide application in the analysis of noise in circuits.

NOISE MODELS FOR THE JFET

The noise model that we use here, shown in Figure 3 is the one used by Abbott *et al.* [19] and is essentially a van der Ziel model [20, 9] with all noise sources referred to the input. For a JFET, the gate currents are limited by reverse biased PN junctions. We regard the gate leakage current as negligible and have not included it in the model.

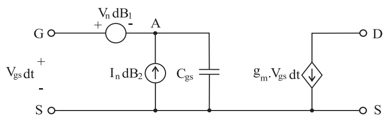

FIGURE 3. Noise model for a JFET. This is basically a van der Ziel model with all noise sources referred to the input. The gate to source capacitance is represented by C_{gs}. This model includes the standard noise-free small signal model for a JFET. The amplifying effect of the JFET is represented by the dependent current source, $g_m{\cdot}V_{gs}$. The noise is represented by two sources at the input, $V_n{\cdot}dB_1$ and $I_n{\cdot}dB_2$ where dB_1 and dB_2 are infinitesimal increments of Brownian motion.

ANALYSIS OF A SIMPLE JFET CIRCUIT

We use a very simple version of the Colpitts oscillator with a FET as the amplifying element. This is shown in Figure 4. In the Colpitts topology, the chain of capacitors, C_1 and C_2, allows for a feedback path with high impedance and voltage amplification. If we analyse the circuit in Figure 4, using small-signal technique, and insert the noise model from Figure 3, then we obtain the complete small signal noise model for the Colpitts oscillator, shown in Figure 5. If the circuit did not have noise sources then ordinary mesh and nodal analysis could be performed and we obtain a standard state-variable model [21]. With the presence of noise sources, we can still write down mesh equations. Kirchhoff's voltage law takes the form: $\sum d\Phi = \sum Vdt = 0$ where the contribution from a noise voltage source would be $d\Phi = V_n dB$. We obtain a mesh equation for the mesh

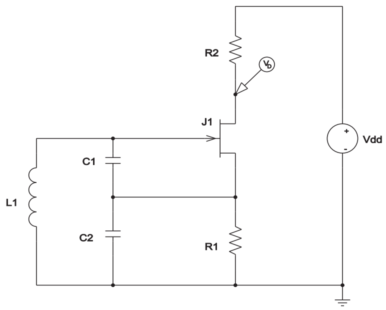

FIGURE 4. Standard, large-signal, schematic circuit diagram for a Colpitts oscillator using a JFET as the amplifying element. There are noise sources in R_1, R_2 and J_1.

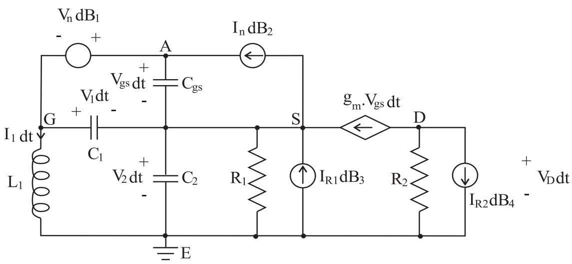

FIGURE 5. Small signal equivalent circuit of the Colpitts oscillator, including the van der Ziel noise model. The FET is mapped into the equivalent circuit with Source, S, Gate, G, Drain D. The earth is represented by E. The circuit has four energy storage elements, C_{gs}, C_1, C_2 and L_1. There are four corresponding state variables, V_{gs}, V_1, V_2 and I_1. Further analysis shows that only three of these are independent. There are four noise sources, $V_n{\cdot}dB_1$, $I_n{\cdot}dB_2$, $I_{R1}{\cdot}dB_3$ and $I_{R2}{\cdot}dB_4$. The nominal output is the drain voltage, V_D.

including the Gate G and the node A,

$$V_1\ dt - V_{gs}\ dt + V_n\ dB_1 = 0, \tag{8}$$

which indicates that the state-variables, V_1 and V_{gs} are not independent. We can obtain a mesh equation including L_1, C_1 and C_2,

$$dI_1 = \frac{1}{L_1}{\cdot}V_1\ dt + \frac{1}{L_1}{\cdot}V_2\ dt. \tag{9}$$

The current through the noise source, $V_n\ dB_1$ is undetermined but we can regard G and A as an enlarged node and write

$$I_1\ dt + C_1\ dV_1 + C_{gs}\ dV_{gs} - I_n\ dB_2 = 0. \tag{10}$$

Finally, the source is simply a very large node

$$C_1\ dV_1 - C_2\ dV_2 + C_{gs}\ dV_{gs} = \frac{V_2}{R_1}\ dt - g_m\ V_{gs}\ dt + I_n\ dB_2 - I_{R1}\ dB_3. \tag{11}$$

Equations 8, 9, 10 and 11 are the equations of state for this system.

The nominal output of the circuit is the drain voltage , V_D. This can be expressed with an output equation,

$$V_D\, dt = -R_2\, g_m\, V_{gs}\, dt - I_{R2}\, R_2\, dB_4. \tag{12}$$

This completes the formulation of the circuit. We could solve the resulting SDEs exactly if we could write these equations in matrix form:

$$d\underline{X_t} = \mathbf{A}\cdot\underline{X_t}dt + \mathbf{K}\underline{dB_t} \tag{13}$$

where $\mathbf{A}$ is a square state-transition matrix and $\exp(+\mathbf{A}t)$ is defined, $\underline{X_t}$ is a state vector containing the state-variables, $\{\ V_{gs},\ V_1,\ V_2,\ I_1\ \}$ and $\mathbf{K}$ is a matrix that combines the independent noise sources,$\{dB_1, dB_2, dB_3, dB_4\}$. In this case the matrix, $\mathbf{K}$ is square but this does not have to be the case. There will be an explicit solution of the form:

$$\underline{X_t} = \exp(+\mathbf{A}t)\cdot\left[\underline{X_0} + \exp(-\mathbf{A}t)\cdot\mathbf{K}\cdot\underline{B_t} + \int_0^t \exp(-\mathbf{A}s)\cdot\mathbf{A}\cdot\mathbf{K}\underline{B_s}ds\right]. \tag{14}$$

The derivation of this solution relies on the multi-dimensional form of Itô's lemma and may be found in Øksendal [22]. Of course, these equations can also be solved using numerical methods described in the literature, see for example [3, 23].

SUMMARY AND OPEN QUESTIONS

So far, we have constructed models for L, R and C. We have devised practical and consistent forms for Kirchhoff's laws. We have shown that the noise equations for electrical circuits are completely equivalent to the Langevin equations of statistical mechanics. We have shown that explicit solution of the SDEs is not always necessary, since we can often formulate much simpler ODEs in the variables of interest. Finally, we have made some progress on the formulation of a practical and consistent noise models for a JFET.

A cursory examination of the JFET in Figure 5 circuit would suggest that the state variables should be: $\{V_1, V_2, V_{gs}, I_1\}$. There "ought" to be 4 state variables. There seem to be 4 independent equations, 8, 9, 10 and 11, so we might expect that these equations could be written directly in the form of Equation 13. This turns out not to be the case, because Equation 8 is degenerate. It does not contain any explicit reference to dV_1 or dV_{gs} and we might expect that the noise components of these variables are not independent. We do not appear to have enough independent equations for the number of (apparently) independent variables.

Our unsolved problem of noise is that we cannot yet implement the noise model for the JFET until we can put Equations 8, 9, 10 and 11 into the form of Equation 13.

It seems that the solution must ultimately depend on a reduction of the number of state variables required to describe the circuit. We must somehow reduce V_1 and V_{gs} to a single variable. We would then have three truly independent variables in three truly independent equations, in the whole circuit, and the equations could be solved.

At first sight, this problem looks like it might be a simple oversight. Perhaps we "forgot to carry the one?" If the noise source, $v_n{\cdot}dB_1$ were a deterministic voltage source, say $V_3 \cdot dt$ then there would be no problem. Equation 8 would reduce to an ordinary mesh equation of the usual type, $V_1 - V_{gs} + V_3 = 0$, and the elimination of one of the state-variables, V_1 or V_{gs} would be trivial. Unfortunately, the circuit *does* contain a noise source, $v_n \cdot dB_1$, and the mesh equation does not reduce in the usual way. We cannot simply "cancel by dt," as it were.

We suspect that the required reduction will depend on the unique decomposition of semi-martingales as expounded by Doob [24]. The state-variables V_1 and V_{gs} cannot be independent because they do not have independent decompositions. There can only be "one noise current" in that mesh, or can there? We would welcome correspondence on this problem.

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