Neural mechanisms for analog to digital conversion

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ABSTRACT

Consider an array of threshold devices, such as neurons or comparators, where each device receives the same input signal, but is subject to independent additive noise. When the output from each device is summed to give an overall output, the system acts as a noisy Analog to Digital Converter (ADC). Recently, such a system was analyzed in terms of information theory, and it was shown that under certain conditions the transmitted information through the array is maximized for non-zero noise. Such a phenomenon where noise can be of benefit in a nonlinear system is termed Stochastic Resonance (SR). The effect in the array of threshold devices was termed Suprathreshold Stochastic Resonance (SSR) to distinguish it from conventional forms of SR, in which usually a signal needs to be subthreshold for the effect to occur. In this paper we investigate the efficiency of the analog to digital conversion when the system acts like an array of simple neurons, by analyzing the average distortion incurred and comparing this distortion to that of a conventional flash ADC.

Keywords: stochastic resonance, suprathreshold stochastic resonance, analog to digital converter, ADC, neuron, noise, information theory, distortion, dithering

1. INTRODUCTION

We analyze two different methods for analog to digital conversion in noisy conditions. We use a general model that consists of an array of N comparators (see Fig. 1) which each receive the same input signal, x. The i-th device is subject to continuously valued *iid* additive noise, η_i ($n = 1, .., N$) and the output from each, y_i, is unity if the input signal plus the noise is greater than its threshold, θ_i and zero otherwise. All outputs are summed to give the overall output signal, y, which is an encoding of the input. Hence, y is a discrete signal taking on integer values from 0 to N and can be considered as the number of devices that are currently "on" and thus

$$y = \frac{1}{2}\sum_{i=1}^{N} \text{sign}[x + \eta_i - \theta_i] + \frac{N}{2}. \tag{1}$$

For the sake of generality, we assume that the input signal consists of uncorrelated samples of a random signal. Such an array can model various devices such as flash analog to digital converters[1] subject to threshold noise, DIMUS (Digital Multibeam Steering) sonar arrays in the "on target" position[2] or a summing network of N FitzHugh-Nagumo neurons.[3]

Stocks has analyzed this system using Shannon information theory.[4,5] For the case of all thresholds set equal to the input signal mean (which is the situation if we wish to consider our model to be a population of simple neurons) it was shown that the input-output mutual information has a maximum for nonzero noise. This phenomenon was termed Suprathreshold Stochastic Resonance (SSR). Stochastic Resonance (SR) is the term used to describe the phenomenon of the output of a nonlinear system being optimized by a nonzero value of internal or input noise. It has been observed in many nonlinear systems including electronic circuits, ion channels, neurons and ring lasers (see [6,7] for extensive reviews). In many of these systems, the main nonlinearity

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is a step-function or threshold, and it has been noted that for a system consisting of a single threshold, SR only occurs for subthreshold signals. By contrast, SSR occurs for arbitrary signal magnitude, due to the presence of more than one threshold and therefore more than two output states.

Previous work has commented on the similarity between SR in simple threshold based systems and dithering.[8–10] We note that SSR is also similar to dithering in ADC's, but with the crucial difference being that dithering is the term used for the deliberate addition of noise to the *input signal* in order to modify the shape of the output noise spectrum, whereas in the SSR model, the noise is assumed to be an inherent, unavoidable part of the system, and is *added independently at each threshold*.

Several application areas exploiting the use of SSR have been suggested, including cochlear implant encoding based on SSR,[11] and artificial insect-vision based motion detection devices.[12] Our research is focused on investigating these and other applications which could incorporate SSR to improve performance when compared to conventional means. In previous work, we showed the that the qualitative behavior of SSR can be seen using other measures, including the average distortion (error variance),[13] and correlation coefficient.[14] The average distortion is a particularly important measure to study if we wish the output of the system to be an approximation of the input, which is the case for an ADC. In such context, the mutual information measures the number of bits on average that the system encodes for each input sample and when SSR occurs, the optimal encoding occurs for the minimizing value of noise intensity. However this does not provide any details on how to decode for that optimal value of noise. The average distortion measures how well the output from the array is decoded and in this paper we examine several methods for decoding the output from the SSR array and compare the average distortion incurred by each method to the average distortion in a conventional ADC.

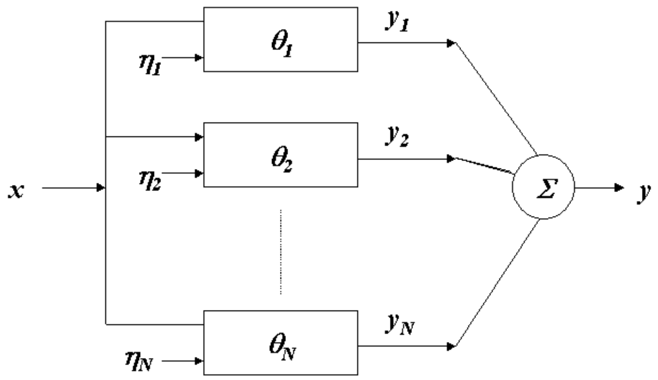

Figure 1. Array of N comparators. Each comparator receives the same input signal, x, and is subject to independent additive noise, η_i. The output from comparator is unity if the sum of the signal and noise is greater than that comparator's threshold, θ_i, and zero otherwise. The overall output, y, is the sum of the individual comparator outputs.

2. CONVENTIONAL ANALOG TO DIGITAL CONVERSION

2.1. Uniformly spaced quantization

The conventional model for analog to digital conversion is as follows.[15] Suppose we have a discrete time input signal, x, which is a random variable. If x has an unbounded amplitude then it is necessary to clip it so that its probability density function is in the range $(-m_p, m_p)$. This amplitude range is then divided uniformly into $N+1$ intervals of width $\Delta\theta = 2m_p/(N+1)$ by N thresholds, or partition boundary points. Hence the threshold values required are

$$\theta_i = -m_p \left(1 - \frac{2i}{N+1}\right) \quad i = 1, ..N.$$

The output of this ADC, z, (the values of which are known as the reproduction points), is usually then given by the midpoint value of the interval number, $y \in 0, .., N$, in which an input sample lies, so that the output is a digital approximation to the input. Since y is equivalent to the number of thresholds that are greater than the input sample amplitude, i.e. the number of comparators "on", we can write

$$z = \frac{2m_p}{N+1}y - \frac{Nm_p}{N+1} \quad y = 0, .., N, \tag{2}$$

where y can be expressed by (1). Thus, z can be considered as a linear normalization of y. More conventionally, we can view the mapping of x to y as an encoder, and the mapping of y to z as a decoder.[16] The encoder is nonlinear and consists of a partitioning of the input signal, x, and the decoder is linear and specifies a set of output values, or reproduction points, for a given code y. It has been postulated that this same operation takes place in populations of neurons, where there is evidence to suggest that the coding of neural representations is nonlinear and the decoding is linear.[17]

Assuming there is no noise on the thresholds, then the output is deterministic for a given input sample, and the only output noise is caused by the lossy quantization operation. This noise, which is the difference $n_q = z - x$, is known as the quantization noise and is bounded by $\pm 0.5\Delta\theta$. The signal degradation caused by this noise is known as the distortion, and can be quantified by its statistical moments. The most commonly used measures are the error variance, $\mathrm{var}[n_q^2]$, and the mean absolute error, $\mathrm{E}[|n_q|]$. Note that if the mean error is zero (which is the case for uniform noise) the error variance is also the mean square error, $\mathrm{E}[n_q^2]$, and is often known as the average distortion.

If we make the assumption that the probability density function of n_q is uniform (this is only strictly true for a uniformly distributed signal, however if $\Delta\theta$ is small (i.e. N is reasonably large) and the signal probability density function is reasonably smooth, then this is a reasonable approximation[16]), then it is straightforward to show that the average distortion is $(\Delta\theta)^2/12$, i.e.

$$\mathrm{E}[n_q^2] = \frac{m_p^2}{3(N+1)^2}, \tag{3}$$

and that therefore the output Signal to Noise Ratio (SNR), for a signal with a mean of zero and a mean square value of S is

$$\mathrm{SNR} = 3(N+1)^2 \frac{S}{m_p^2}. \tag{4}$$

2.2. Dithering

Notice that the output SNR of (4) is a linear function of the signal power. Ideally, however, we would like the the SNR to be constant for all signal powers, which requires that the average distortion is independent of the signal. Such a goal can be achieved by the use of dithering.[18] Non-subtractive dithering involves the addition of an independent random signal to the input signal prior to being input to the ADC, and can render desired statistical moments of the error, n_q, independent of x.[19] If the dither is then subtracted at the output, it is possible to ensure that n_q is entirely statistically independent of x.[20] We note that when dithering is used, the coding and encoding performed by the uniform quantizer ADC becomes non-deterministic for any given input sample, which is also the case for the neuron like system in which SSR occurs. However, as mentioned, the difference is that noise is added independently at each comparator for the SSR array, whereas with dithering the noise is added to the signal.

2.3. Companding

The above discussion of the conventional ADC is restricted to the most commonly used model of analog to digital conversion, i.e. the uniform quantizer. In reality, it is often required that the quantization partitions are not uniformly spaced, and that the output values, i.e. reproduction points, are not the midpoint of the partition. An important aspect of the design of an ADC is the selection of the optimal partitions, i.e. threshold values, and reproduction points for a given signal distribution. Usually the optimization criteria is to minimize the average distortion for a fixed N. Various algorithms exist for doing this, such as the Lloyd algorithm.[16]

However, when only a uniform quantizer is available, it is still possible to reduce the average distortion for non uniform signals by the use of companding. Companding comprises three steps; the nonlinear transformation of the input signal to be between $\pm m_p$ (compression), an analog to digital conversion of this compressed signal by the uniform quantizer, and then the expansion of the digital signal using the inverse operation to the nonlinear compression.[15] This procedure is equivalent to a nonlinear quantization.

3. THE MODEL FOR SSR

Let the threshold value of all comparators be $\theta_i = 0$. Therefore all comparators are identical, apart from being subject to independent noise. We shall refer to this configuration of Fig. 1 as the "SSR array". Let the signal, x, and the noise at each comparator η_i, have mean zero. Note that as with the ADC of Sec. 2.1, y can be expressed by (1).

3.1. Encoding for SSR

The SSR array provides a stochastic mapping of the input signal to the output code, y. Therefore the key function describing this mapping is the conditional probability distribution of the output given the input. Following the notation of Stocks,[4] we let $P_{1|x}$ be the probability of any device being "on", (that is, the probability that the signal plus noise exceeds the threshold $\theta = 0$, given the input signal x) and have

$$P_{1|x} = \int_{-x}^{\infty} R(\eta)d\eta = 1 - F_R(-x).$$

where F_R is the cumulative distribution function of the noise. The probability distribution of the output given the input is an $N+1$ state discrete probability distribution, and is given by the binomial distribution, i.e. $P(n|x) = C_n^N P_{1|x}^n (1 - P_{1|x})^{N-n}$. This probability mass function defines the encoding of any given value of the input signal, x, to the encoded output set $y \in \{0, .., N\}$ in the SSR array. Hence, like the dithered uniform quantizer ADC, the encoding for any given input sample is non-deterministic.

3.2. Decoding for SSR

In previous work, we performed a linear decoding of the output of the SSR array to $\hat{y}$. It is straightforward to show that if x has an even probability density function, then $\mathrm{E}[y] = N/2$. Since x has mean zero, we also wish $\hat{y}$ to have zero mean. We also wish $\hat{y}$ to be bounded by $\pm c$. Therefore we have

$$\hat{y} = \frac{2c}{N} y - c, \tag{5}$$

where c is chosen from knowledge of the amplitude dynamic range of the input signal. If we let $c = N m_p/(N+1)$ then z of Eqn. (2) and $\hat{y}$ take on the same output values and our decoding scheme for the SSR array is identical to that of the conventional uniform ADC.

We wish to explore how well the quantization and encoding performed by the SSR array followed by decoding to $\hat{y}$ compares with a conventional uniformly quantized ADC signal. We do this by examining the average distortion for both cases.

4. UNIFORM INPUT SIGNAL

4.1. Noiseless ADC

We examine the particular case of a zero mean, uniformly distributed input signal between $-\sigma_x/2$ and $\sigma_x/2$. This gives $m_p = \sigma_x/2$ and a signal variance of $\sigma_x^2/12$.

For the SSR array let the independent noise in each device be uniformly distributed between $-\sigma_r/2$ and $\sigma_r/2$, with zero mean. The noise variance is $\sigma_r^2/12$. Stocks has derived an exact formula for the mutual information between input and output for this case and showed that for large N the optimal value of σ_r is exactly σ_x.[21] We note that regardless of the decoding we use, the mutual information remains the same, since this measure is solely dependent on the input-output joint probability function and not the actual output reproduction points.

For an undithered ADC the average distortion is given by (3), which in this case with $m_p = \sigma_x/2$ is

$$\mathrm{E}[n_q^2] = \frac{\sigma_x^2}{12(N+1)^2}.$$

For the SSR array we have previously derived an analytical expression for the average distortion.[13] Denoting the quantization error for the SSR array as $\varepsilon = \hat{y} - x$ we have

$$\mathrm{Var}[\varepsilon] = \begin{cases} c^2\left(\frac{3N-2\sigma(N-1)}{3N}\right) + c\sigma_x\left(\frac{\sigma^2-3}{6}\right) + \frac{\sigma_x^2}{12} & \sigma \leq 1 \\ \frac{4c^2(N-1+3\sigma^2)-4cN\sigma_x\sigma+\sigma^2\sigma_x^2 N}{12N\sigma^2} & \sigma \geq 1, \end{cases} \tag{6}$$

where $\sigma = \sigma_x/\sigma_r$. This has a minimum at $\sigma_m = 2(N-1)c/(N\sigma_x)$. If we put $c = Nm_p/(N+1)$ then $\sigma_m = (N-1)/(N+1)$, which gives the minimum variance as

$$\begin{aligned} \mathrm{Var}[\varepsilon]_m &= \frac{\sigma_x^2}{12(N+1)^2}\left(N(N+1)\sigma_m^2 - N(N-1)(2\sigma_m - 1) + 1\right) \\ &= \mathrm{E}[n_q^2]\left(\frac{2N^2-N+1}{N+1}\right) \\ &> \mathrm{E}[n_q^2]. \end{aligned}$$

Thus, the distortion for the SSR array of size N can never be as small as that for the undithered uniform quantizing ADC of size N, and in fact for large N, the distortion approaches $2N$ times as much. We wish examine how much larger an N is required in the SSR array to provide the same minimum distortion as the ADC. We have plotted the average distortion for the SSR array given by (6) for various values of N in Fig. 2. The SSR array requires $N = 31$ to achieve the same performance of an ADC with $N = 3$, and this performance is only achieved at the minimizing value of noise. Thus, the performance of a 2 bit ADC can only be reached with a 5 bit SSR array.

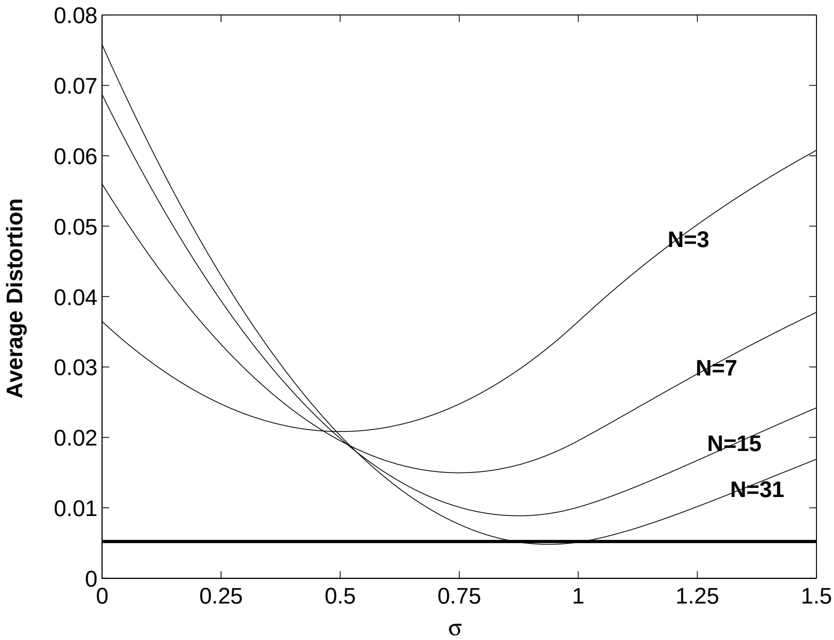

Figure 2. Plot of the average distortion for $N = 3$ against the ratio of noise standard deviation to signal standard deviation, σ, for the ADC (thick solid line), and the average distortion for the SSR array for various number of threshold devices, N.

4.2. Noisy ADC

Now we assume that the same noise present in the SSR array is present on the thresholds in the ADC. In this case the only difference between the two is the threshold settings. We do not have an exact expression for the average distortion in the noisy ADC but can obtain it by digital simulation. This is achieved by generating an input signal, x, of M samples, as well as N independent noise signals each of M samples, and the corresponding output signals (z for the dithered ADC and $\hat{y}$ for the SSR array) calculated from this signal by use of (1), (2) and (5). The average distortion is then easily obtained from these signals, with increasing accuracy for increasing M. An example of this simulated signal of length $M = 20$ for $N = 15$ is shown in Fig. 3. It can be seen visually that both the SSR array and ADC outputs give close approximations to the input signal.

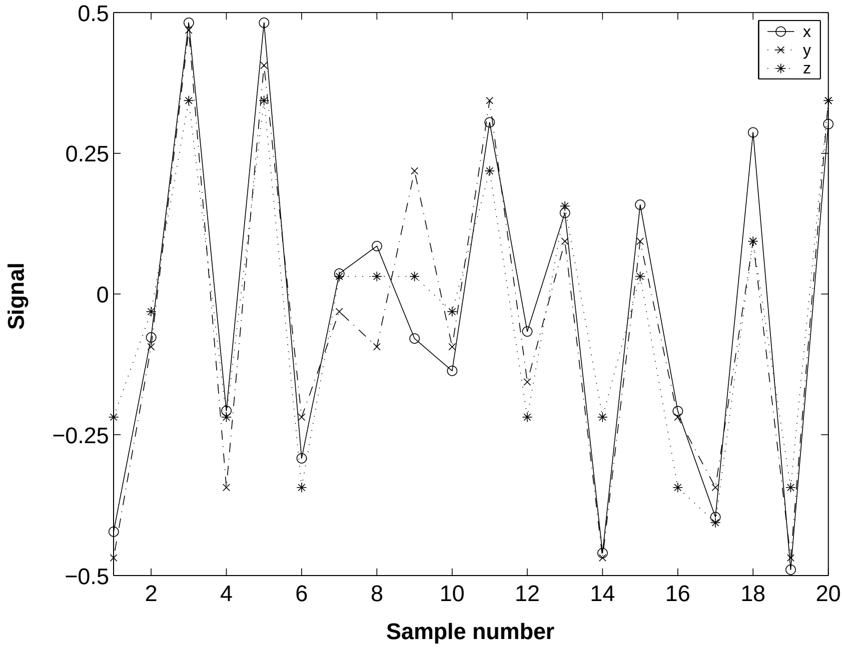

Figure 3. Plot of M samples of the signals x, z and $\hat{y}$ for $N = 15$ and $\sigma = 1$ for both the ADC and the SSR array. We have used $\sigma_x = 1$ and $c = N/(2N + 2)$.

Results for the average distortion for the noisy ADC with $N = 3$ and the SSR array for $N = 2$ and $N = 3$ are shown in Fig. 4. It can be seen that unlike with the noiseless ADC, the SSR array can obtain better performance than the noisy ADC when both have the same value of N. Fig. 4 shows that this occurs for a range of σ between about 0.6 and 1.2.

5. GAUSSIAN INPUT SIGNAL

Recall that for the case of non-uniform signal densities it is still possible to achieve low ADC distortion with the uniform quantizer by the use of companding. We now briefly present a comparison between the distortion through the uniform ADC with a companded Gaussian input signal and the distortion through the SSR array when the input signal is Gaussian and the internal noise is also Gaussian.

5.1. SSR array

If the Gaussian input signal has zero mean and variance σ_x^2 then it has probability density function

$$P(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} \exp\left(-\frac{x^2}{2\sigma_x^2}\right).$$

If the independent noise in each device is Gaussian with zero mean and variance σ_r^2, then it has probability density function

$$\frac{1}{\sqrt{2\pi\sigma_r^2}}\exp\left(-\frac{\eta^2}{2\sigma_r^2}\right).$$

We have previously[13] derived an expression for the average distortion for the case of the linear decoding of (5),

$$\mathrm{var}[\varepsilon] = c^2\left(\frac{1}{N}+\frac{2(N-1)\arcsin\left(\frac{1}{1+\sigma^2}\right)}{N\pi}\right) - 2c\sigma_x\sqrt{\frac{2}{\pi(1+\sigma^2)}}+\sigma_x^2.$$

The distortion for $N = 7$ and $m_p = \sigma_p$ is shown in Fig. 5.

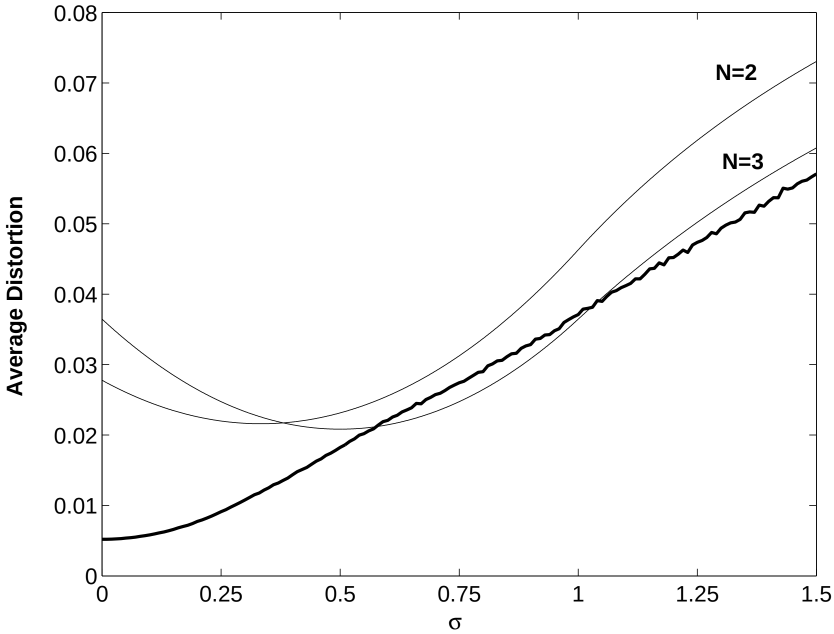

Figure 4. Plot of the average distortion for $N = 3$ against the ratio of noise standard deviation to signal standard deviation, σ, for the ADC with noisy thresholds (thick solid line), and the average distortion for the SSR array for $N = 2$ and $N = 3$. The SSR array outperforms the noisy ADC for $N = 3$ between about $\sigma = 0.6$ and $\sigma = 1.2$.

5.2. Companded Gaussian signal

Several standard companding formulae exist.[16] We use the μ-law characteristic which is given by

$$G_\mu(x) = m_p\frac{\ln(1+\frac{\mu|x|}{m_p})}{\ln(1+\mu)}\mathrm{sign}(x),$$

where μ is a control parameter determining the amount of compression. Fig. 5 shows the result of companding a Gaussian input signal with $\mu = 1$ and $m_p = \sigma_p$ for $N = 7$ when the ADC is subject to additive Gaussian noise at each threshold.

We can see that the SSR array outperforms the ADC once σ increases above about 0.4. This indicates another potential use of the SSR array in analog to digital conversion, i.e. the appropriate selection of a noise distribution at each threshold could provide the effect of companding, without the need for nonlinear signal transformations at the input and output. In conditions where the noise is already present, not only are different threshold levels, i.e. reference voltages not required, but compression and expansion of the input signal is also provided by the noise.

6. CONCLUSIONS

Our analysis has shown that analog to digital conversion can be achieved by an array of comparators with thresholds equal to the signal mean via the use of additive noise at each threshold. Such an array provides a crude model of a population of neurons and indicates a method by which neurons may be able to encode data. We have seen however, that many more comparators are required in such a configuration than a conventional flash ADC to obtain the same output distortion. This disadvantage in size however is partially offset by a reduction in complexity. The flash ADC requires a different reference voltage at each threshold whereas the SSR array, or a population of neurons, has all thresholds equal and therefore the desired output can be obtained using identical device, provided each threshold is subject to *iid* noise.

Furthermore, we have shown that if noise is inherently present at the thresholds in an ADC, then better performance can be achieved by making use of this noise by way of the SSR effect than to attempt to use conventional ADC threshold settings, without needing to increase the number N. This is particularly the case for nonuniform input signals, where conventional uniform ADCs require companding of the input signal. With an appropriate internal additive noise distribution, the SSR effect provides a companding without the need for nonlinear signal transformations. It is envisioned that such circumstances may occur in nanoscale devices where it may be very difficult to set precise reference voltages and conditions are very noisy. In such a case the achievement of analog to digital conversion could be achieved using the SSR effect.

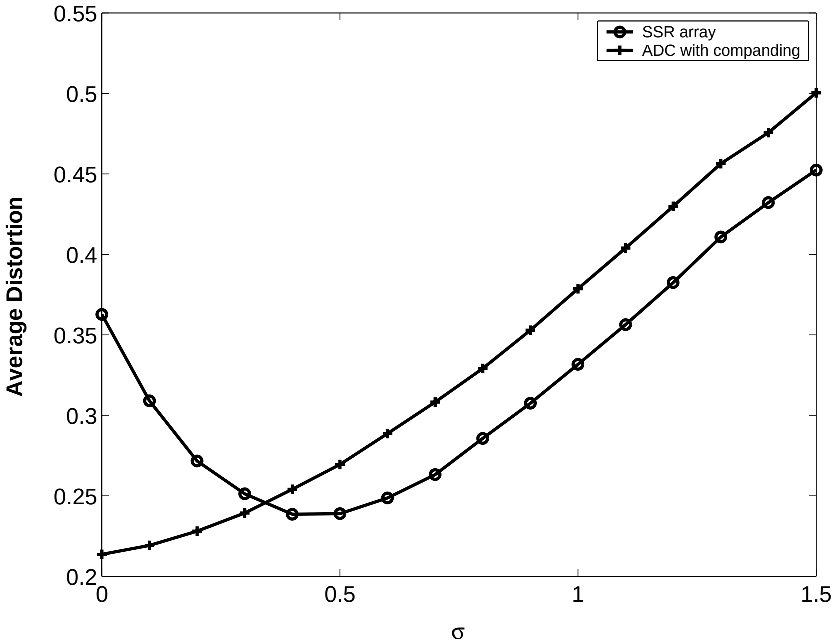

Figure 5. Plot of the average distortion for $N = 7$ against σ for the SSR array with Gaussian signal and noise, and a uniformly quantized ADC subject to Gaussian noise at each comparator and companded Gaussian signal with compression control paramater $\mu = 1$.

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