Detailed balance of the Feynman-micromotor

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ABSTRACT

One exciting implication of micromotors is that they can be powered by rectifying non-equilibrium thermal fluctuations or mechanical vibrations via the so-called Feynman-micromotor. An example of mechanical rectification is found in the batteryless wristwatch. The original concept was described in as early as 1912 by Smoluchowski and was later revisited in 1963 by Feynman, in the context of rectifying thermal fluctuations to obtain useful motion. It has been shown that, although rectification is impossible at equilibrium, it is possible for the Feynman-micromotor to perform work under non-equilibrium conditions (eg. in the presence of a thermal gradient). These concepts can now be realised by MEMS technology and may have exciting implications in biomedicine – where the Feynman-micromotor can be used to power a smart pill, for example. Previously, Feynman's analysis of the motor's efficiency has been shown to be flawed by Parrondo and Español. We now show there are further problems in Feynman's treatment of detailed balance. In order to design and understand this device correctly, the equations of detailed balance must be found. Feynman's approach was to use probabilities based on energies and we show that this is problematic. In this paper, we demonstrate corrected equations using level crossing probabilities instead. A potential application of the Feynman-micromotor is a batteryless nanopump that consists of a small MEMS chip that adheres to the skin of a patient and dispenses nanolitre quantities of medication. Either mechanical or thermal rectification via a Feynman-micromotor, as the power source, is open for possible investigation.

Keywords: Feynman-micromotor, ratchet & pawl, batteryless MEMS, non-equilibrium rectification of fluctuations, detailed balance.

1. INTRODUCTION

The Feynman-micromotor, also called the Feynman-Smoluchowski Engine (FSE), consists of a ratchet wheel connected to a set of vanes via an axle, as shown in Figure 1. As the air molecules randomly bombard the vanes, the ratchet oscillates. It would appear that the action of the ratchet & pawl 'rectifies' these oscillations and the system rotates in one direction, thus being able to perform useful work, in violation of the Second Law of Thermodynamics. In 1912, Smoluchowski was the first to correctly suggest that there is no net motion, at equilibrium, as fluctuations in the spring loaded pawl will occasionally allow the ratchet wheel to rotate in the opposite direction[1] – thus preserving detailed balance. Of course, for the non-equilibrium case, when energy is supplied into the system, there is net motion without violation of the Second Law.

The ratchet & pawl device was revisited, in 1963, by Feynman[2] in greater detail – detailed balance probabilities are given and engine efficiency calculations are explored. It is now well-known that Feynman's treatment was flawed, as he incorrectly applied the quasi-static assumption to the FSE, leading to an incorrect calculation of engine efficiency.[3] This paper now also questions Feynman's treatment of the detailed balance. Although Smoluchowski & Feynman only saw the FSE as a 'thought experiment,' the FSE is no longer hypothetical as the so-called Feynman-micromotor[4] has been fabricated using MEMS technology and has inspired the 'Brownian ratchet' concept. Hence there is renewed interest in the FSE, and correct analysis is now of importance.

We begin our discussion by performing a detailed balance, using Feynman's method based on energy probabilities, to highlight the problems. Then we offer a solution by adopting a different approach, based on crossing rates.

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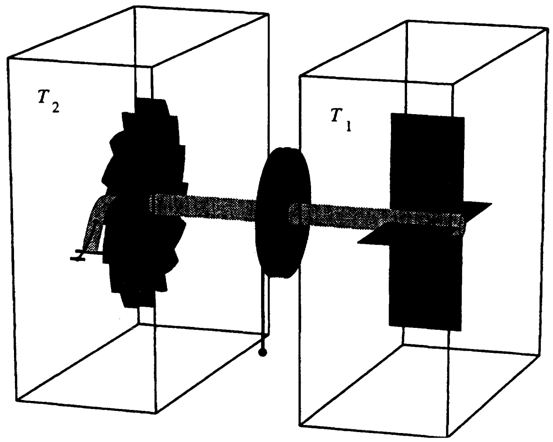

Figure 1. Rotary ratchet and pawl system as a rectifier

2. FEYNMAN'S APPROACH

Feynman begins by calling the threshold energy that the ratchet wheel needs to rotate clockwise (CW) one notch passed the pawl, ϵ. He then states the probability of the ratchet wheel attaining ϵ is $e^{-\epsilon/kT}$. Also he states that this is the same probability required for the pawl to fluctuate enough to disengage, thus allowing the ratchet to rotate counterclockwise (CCW).

Without discussion, Feyman implicitly identifies these probabilities as the *same* probabilities required for CW and CCW rotation. Thus he concludes that the system is balanced and there is no net rotation on average. Of course, his final conclusion is correct, as we cannot allow a violation of the Second Law of Thermodynamics. However, one question is the leap in logic from probabilities to do with pawl and ratchet states, to probabilities of CW and CCW rotation.

The real situation is much more complex. For instance, when the pawl is disengaged, the ratchet wheel can rotate in either direction! Also when pawl is engaged, the ratchet wheel may attain the energy ϵ, but in the wrong (CCW) direction, and thus will be dissipated as heat.

These arguments demonstrate that in order to fully understand the FSE, a detailed balance from first principles is required. But let us now use Feynman's approach, to examine the detailed balance to further highlight the difficulties. Now, let us consider the CW and CCW directions separately.

CW Rotation:
As before, let the required energy threshold for the ratchet wheel to rotate one notch passed the pawl be ϵ. In general we can say that $\epsilon = \epsilon_r + \epsilon_p$, where ϵ_r is supplied by the ratchet wheel fluctuation trying to move passed the pawl, and ϵ_p is supplied by the pawl fluctuation trying to (partially) disengage. Now the probability of attaining ϵ_r is $e^{-\epsilon_r/kT}$ and attaining ϵ_p is $e^{-\epsilon_p/kT}$. But note that when the ratchet wheel gets a 'kick' of energy equal to ϵ_r there is a chance of $\frac{1}{2}$ that the kick would be in the CW direction. Similarly, the pawl can fluctuate upwards (to escape the ratchet teeth) or downwards (to dig into the ratchet teeth) and the chance of attaining ϵ_p in the upwards direction will be $\frac{1}{2}e^{-\epsilon_p/kT}$. Therefore, the probability of CW rotation is, $P(\mathrm{CW}) = \frac{1}{2}e^{-\epsilon_p/kT}\frac{1}{2}e^{-\epsilon_r/kT} = \frac{1}{4}e^{(-\epsilon_p-\epsilon_r)/kT} = \frac{1}{4}e^{-\epsilon/kT}$.

CCW Rotation:
In this case, we require an energy ϵ from the pawl alone to disengage from the ratchet wheel. When the the pawl is disengaged, there is a chance of $\frac{1}{2}$ that the ratchet wheel will rotate in the CCW direction. Hence, $P(\mathrm{CCW}) = \frac{1}{2}e^{-\epsilon/kT}\frac{1}{2} = \frac{1}{4}e^{-\epsilon/kT}$.

Therefore, $P(\mathrm{CW}) = P(\mathrm{CCW})$ and we have detailed balance. But do we? When calculating $P(\mathrm{CW})$, we ignored the case when ϵ_p acts in the direction to dig the pawl deeper into the ratchet teeth – in this case the ratchet must attain $\epsilon_p + \epsilon$ for CW rotation. However, if we alter the probabilities to reflect this, we apparently lose detailed balance.

An even more serious flaw is as follows. Now,

$$p(E_p > \epsilon) = e^{-\epsilon/kT} \text{ and } p(E_r > \epsilon) = e^{-\epsilon/kT}$$

therefore,

$$p(E_p) = kTe^{-E_p/kT} \text{ and } p(E_r) = kTe^{-E_r/kT}.$$

So if $E = E_p + E_r$, then $p(E) = p(E_p) \otimes p(E_r)$ or,

$$p(E) = \int_0^E kTe^{-E_p/kT} kTe^{-(E-E_p)/kT} dE_p = \alpha^2 E e^{-\alpha E}$$

and so

$$p(E > \epsilon) = \int_W^\infty \alpha^2 E e^{-E/kT} dE = (1 + \alpha\epsilon) e^{-\epsilon/kT}.$$

For CW rotation, the requirement is for $\epsilon_r + \epsilon_p > \epsilon$. Hence we have that $P(\text{CW})$ is always unequal to $P(\text{CCW})$, which is clearly not allowed. The question then arises, where is the flaw and what is the correct approach?

3. SIMULATION RESULTS

The objective of the simulation was to find the *simplest* description of the ratchet & pawl that gave detailed balance. This enables us to concentrate analytical methods on the essential system and ignore unnecessary levels of model sophistication. The system shown in Figure 2 was simulated using the following parameters:
X = horizontal pitch of ratchet tooth = 1,
Y = vertical height of ratchet tooth = 1,
y_o = rest position of pawl = 0.5,
x_o = rest position of the ratchet = 0.5,
m_p = mass of pawl = 0,
m_r = mass of ratchet = kT,
λ_p = spring constant of spring connected to pawl = kT,
λ_r = spring constant of spring connected to ratchet = 0,
d_p = damper constant for damper connected to pawl = 1,
d_r = damper constant for damper connected to ratchet = 1.

The simulation in Figure 3 shows a number of interesting phenomena. The first at time $t = 1.2$ is where the vertical part of the ratchet has collided with the pawl and bounced back, the second is at $t = 1.9$ (and elsewhere) where the pawl has collided with the ratchet and is carried along with it, the third is at $t = 6.5$ and $t = 9.1$ where the ratchet has slipped a tooth in the preferred direction, and the fourth is at $t = 8.3$ where the ratchet has slipped back a tooth because the pawl was clear of the ratchet at that time.

Due to the interaction between the two parts of the system, it seemed likely that the energies are not totally uncorrelated and that the energy associated with each may not be $\frac{1}{2}kT$. The simulation gave confidence of detailed balance with a very simple system based on assuming elastic (lossless) collisions between ratchet and pawl. Two types of collision were included: (1) the pawl striking the ramp section of the ratchet, and (2) the vertical part of the ratchet striking the pawl, and it is these collisions which form the interaction between the ratchet and the pawl. For simplicity a linear ratchet was considered, Figure 2, but equivalent results apply to the more familiar rotary ratchet in Figure 1. The dampers provided the source of thermal energy to the system and are the only dissipative elements in the system. The pawl moves vertically under the influence of the spring and damper, whereas the ratchet is able to move horizontally under the influence of a second damper.

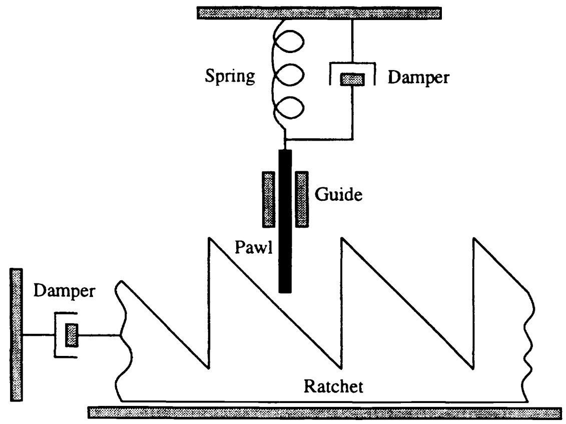

Figure 2. Linearized ratchet and pawl system

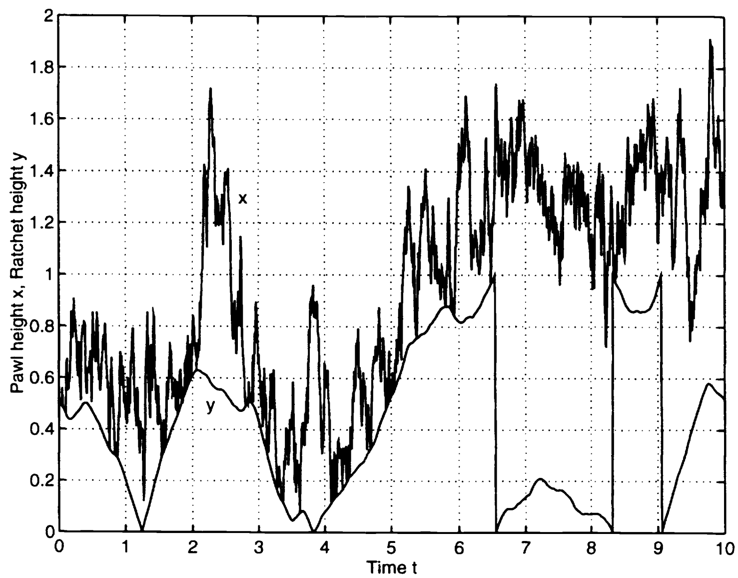

Figure 3. Example simulation result

4. BALANCE OF CROSSING RATES

In seeking an alternative method for analytically obtaining detailed balance, we turned to the idea of calculating crossing rates[5] of the pawl over the top of a ratchet tooth. For simplicity, a linear ratchet is considered as in Figure 2 – equivalent results apply to the more familiar rotary ratchet. We define
y = position of pawl above bottom of ratchet teeth,
x = position of ratchet,
X = horizontal pitch of ratchet tooth,
Y = vertical height of ratchet tooth,
y_o = rest position of pawl,
x_o = rest position of the ratchet,
m_p = mass of pawl,
m_r = mass of ratchet,

λ_p = spring constant of spring connected to pawl,
λ_r = spring constant of spring connected to ratchet,
d_p = damper constant for damper connected to pawl,
d_r = damper constant for damper connected to ratchet.

We take $x = 0$ to correspond to when the bottom of the ratchet tooth is opposite the pawl. There is a constraint that $y/Y \geq x/X$, for $0 < x < X$ since the pawl cannot be below the ratchet tooth. Now, the ratchet slips one tooth to the right (normal ratchet action) if x crosses the value X in the positive direction. Similarly, a slip to the left occurs when x crosses the value 0 in the negative direction (abnormal ratchet action).

The Hamiltonian of the system is:

$$H = \frac{1}{2}\lambda_p (x - x_o)^2 + \frac{1}{2} m_p \dot{x}^2 + \frac{1}{2}\lambda_r (y - y_o)^2 + \frac{1}{2} m_r \dot{y}^2$$

where $\dot{x} = dx/dt$ and $\dot{y} = dy/dt$. Notice this does not correspond exactly to Figure 2 – we have added a second spring to the ratchet that can be discarded at the end of the analysis. This is necessary to provide a constraint in x, to enable the integrals that follow.

The steady state joint probability density function of these variables is given by the Gibbs relation:

$$p(x, \dot{x}, y, \dot{y}) = \frac{1}{Z_o} e^{-H/kT}$$

where Z_o is a normalizing constant called the partition function.

The crossing rates of x at some level x_1, within Transition State Theory,[5] are given by:

$$\nu^+ = \int_0^\infty d\dot{x} \int_{y_1}^\infty dy \int_{-\infty}^\infty d\dot{y}\, \dot{x}\, p(x, \dot{x}, y, \dot{y}) \tag{1}$$

$$\nu^- = \int_{-\infty}^0 d\dot{x} \int_{y_1}^\infty dy \int_{-\infty}^\infty d\dot{y}\, \dot{x}\, p(x, \dot{x}, y, \dot{y}) \tag{2}$$

where y_1 is the ratchet height when $x = x_1$.

For the above probability density function $\nu^+ = \nu^- = \nu$ and is given by:

$$\nu = \frac{2\pi}{Z_o} \sqrt{\frac{(kT)^3}{\lambda_p\, m_p\, m_r}}\, Q\left(\frac{(y_1 - y_o)\sqrt{\lambda_p}}{\sqrt{kT}}\right) e^{-\lambda_r (x_1 - x_o)^2/kT}$$

and $Q(x)$ is the Gaussian error probability function.

Now for a right slip across $x = X$ we must have $y_1 = Y$ and hence:

$$\nu_{right} = \frac{2\pi}{Z_o} \sqrt{\frac{(kT)^3}{\lambda_p\, m_p\, m_r}}\, Q\left(\frac{(Y - y_o)\sqrt{\lambda_p}}{\sqrt{kT}}\right) e^{-\lambda_r (X - x_o)^2/kT}$$

whereas for a left slip across $x = 0$ we have:

$$\nu_{left} = \frac{2\pi}{Z_o} \sqrt{\frac{(kT)^3}{\lambda_p\, m_p\, m_r}}\, Q\left(\frac{(Y - y_o)\sqrt{\lambda_p}}{\sqrt{kT}}\right) e^{-\lambda_r (-x_o)^2/kT}$$

since we must still have $y_1 = Y$ for this slip to occur.

If we remove the extraneous ratchet spring, by letting $\lambda_r \to 0$, then we have: $\nu_{left} = \nu_{right}$ – thus detailed balance is preserved. Note that even with the ratchet spring included, balance occurs if $x_o = X/2$ as might be expected.

The dampers do not explicitly appear in any of the above analysis, but will affect the nature of the fluctuations.

5. CONCLUSIONS

The analysis presented shows that there is no paradox associated with the FSE, but questions Feynman's approach. Note that it was necessary to initially include the spring connected to the ratchet in order to properly treat the constraint between the ratchet and pawl positions, but this spring was later discarded. We have shown that, if we assume the validity of Transition State Theory, a crossing rate analysis does indeed lead to detailed balance. It is hoped this analysis will assist in the development of the design equations for future batteryless MEMS devices.

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