Simulation and properties of randomly switched control systems

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ABSTRACT

Power electronics has made great advances since the introduction of the thyristor in 1958. Even a casual study of consumer electronics, such as computer power supplies, reveals that switched mode power electronics have steadily replaced passive circuits. Switched mode circuits can accommodate higher power densities,[1] they are lighter, cheaper and easier to control. The use of microprocessors and microcontrollers can make switched mode circuits even more versatile. Unfortunately, there are some problems with switched mode circuits. The higher power densities handled by these circuits can cause catastrophic failure. Periodic switching can give rise to acoustic noise[2] or undesirable electromagnetic radiation. These problems can be reduced through the use of random switching policies.[3] One theoretical disadvantage of random switching policies is that the time averaged switched system is not strictly equivalent to the classical system with the same average parameters. The stability limits for the randomly switched and classical systems are different. This is a possible area for concern, given the high power densities and the possibility of catastrophic failure. In this paper we examine the stability of randomly switched control systems. We provide simulations, some analysis and derive some practical rules for stability. We show that some randomness can beneficial from the point of view of minimising the maximum power spectral density of the noise waveforms in the output current.

Keywords: stability, random, switched, converter, control

1. THE BUCK/BOOST REGULATOR AS AN EXAMPLE FOR ANALYSIS

In this paper we use a state variable approach to systems which is very general. Rather than discuss all systems purely in the abstract, we illustrate the important points using an example system. We have selected the buck/boost regulator for the following reasons:

- It is very commonly used. Applications are found with the data sheets of many of the commercially available integrated circuits, such as the LM78S40.
- Simple analysis can readily be found in the literature.[4,5]
- Since this regulator can "boost" voltages, it has interesting stability properties. It can appear to be unstable * if an inappropriate control rule is used.
- This regulator is composed from linear elements and can be readily formulated and analysed in state space.

The buck/boost regulator is basically a switched inductor circuit. The topology in Figure 1.

The regulator has two modes. In mode 1, called the "on" time, S1 is closed and S2 is open. In mode 2, called the "off" time, S1 is open and S2 is closed. We can denote the "on" time by ΔT_1 and the "off" time by ΔT_2. The use of the symbol "Δ" implies that the switching times are small compared with all of the time constants in the regulator.

In practice S1 is often a bipolar transistor and S2 is a diode.[4,5] The buck/boost circuit is an inverting regulator. The average D.C. values of V_s and Vc are opposite in sign.

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*The "unstable" responses are actually very large and long lived transient responses.

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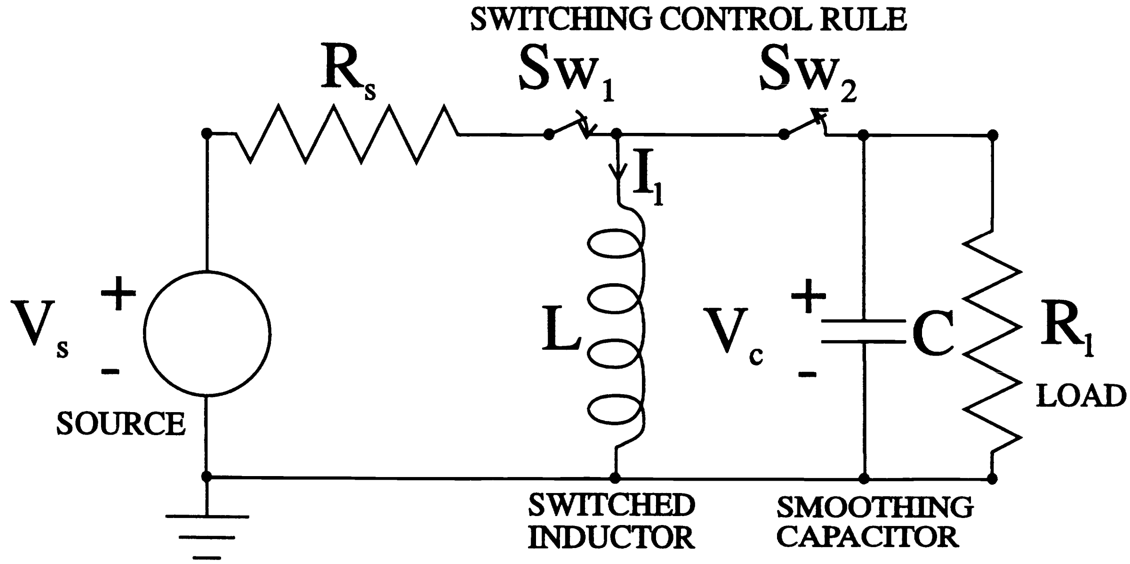

Figure 1. A schematic circuit for the buck/boost regulator.

2. NORMAL OPERATING CONDITIONS AND A SIMPLE APPROACH TO DESIGN

The natural state variables to use for this type of circuit are the capacitor voltages and the inductor currents. Together they characterise the total stored energy of the system. These variables are also preserved across switching boundaries.

If ΔT_1 and ΔT_2 are "small" then simulations show that the responses have a small triangular wave, or "ripple", superimposed on top of them. The rise and fall times of this superimposed wave are tied to the switching times, ΔT_1 and ΔT_2. If we consider the steady state D.C. case (after all transients have been attenuated) then we expect to get regular triangular waveforms as shown in figure 2. We can develop a piecewise linear model.

The symbol V_c is used here to denote the median value of the capacitor voltage and ΔV_c denotes the ripple voltage across the capacitor. The capacitor voltage is also equal to the output voltage, delivered to the load, R_l. Similarly, the symbol I_l is used to denote the median value of the inductor current and ΔI_l denotes the ripple current through the inductor. During the "on" time, the inductor current is also equal to the input source current. During the "off" time, the input source current is zero.

The simple application of nodal and mesh equations to the system, in both modes, leads to the following formulation:

during the "on" time:

$$I_l R_s + L \frac{\Delta I_l}{\Delta T_1} = V_s \tag{1}$$

$$C \frac{(-\Delta V_c)}{\Delta T_1} = \frac{V_c}{R_l} \tag{2}$$

and during the "off" time:

$$I_l - C \frac{\Delta V_c}{\Delta T_2} + \frac{V_c}{R_l} = 0 \tag{3}$$

$$-L \frac{\Delta I_l}{\Delta T_2} = V_c \ . \tag{4}$$

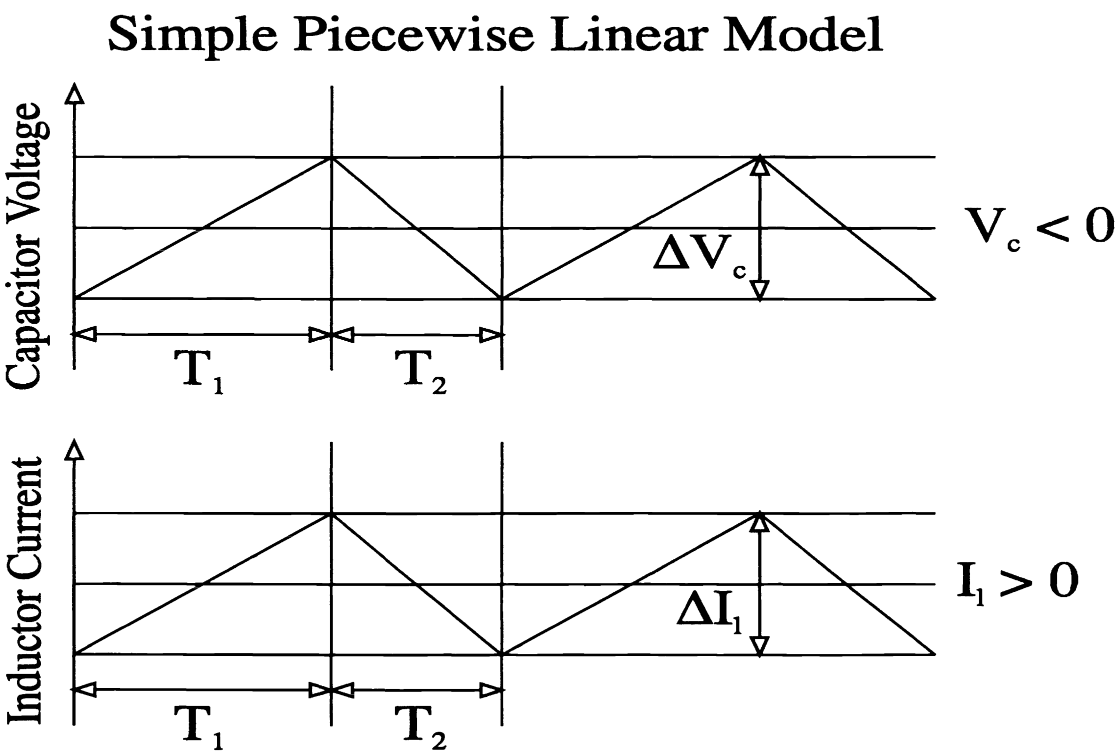

Figure 2. A schematic circuit for the buck/boost regulator.

We can eliminate terms involving ΔV_c and ΔI_l and write the equations in matrix form:

$$\begin{bmatrix} (\frac{-1}{R_l C}) & (\frac{-(1-d)}{C}) \\ (\frac{-(1-d)}{L}) & (\frac{-dR_s}{L}) \end{bmatrix} \begin{bmatrix} V_c \\ I_l \end{bmatrix} + \begin{bmatrix} 0 \\ (\frac{V_s d}{L}) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{5}$$

where d is the time averaged duty cycle, $\frac{\Delta T_1}{\Delta T_1 + \Delta T_2}$.

We can solve the equations algebraically and derive an expression for V_c in terms of the source voltage, V_s, the ratio of the switching times, $\frac{\Delta T_1}{\Delta T_2}$ and the ratio of the source and load impedances, $\frac{R_s}{R_l}$:

$$V_c = \frac{(-Vs)}{\frac{1}{\frac{\Delta T_1}{\Delta T_2}} + (\frac{R_s}{R_l})((\frac{\Delta T_1}{\Delta T_2}) + 1)}. \tag{6}$$

It is usually the case that $\frac{R_s}{R_l}$ is very small so we can write

$$V_c \approx (-V_s)(\frac{\Delta T_1}{\Delta T_2}) \tag{7}$$

so we can readily control the median output voltage by controlling the duty cycle of the switching control rule.

If the source impedance R_s is appreciable then V_c has an upper bound. This occurs when $(\frac{\Delta T_1}{\Delta T_2}) = \frac{1}{\sqrt{\frac{R_s}{R_l}}}$ and we get $\max|V_c| = \frac{(-V_s)}{2\sqrt{(\frac{R_s}{R_l})} + (\frac{R_s}{R_l})}$. If $(\frac{R_s}{R_l})$ is small then we can write $\max|V_c| \approx \frac{1}{2}\frac{V_s}{\sqrt{(\frac{R_s}{R_l})}}$. We can still control V_c within this range by altering the duty cycle.

We can readily calculate the amount of ripple, ΔV_c.

$$\Delta V_c = (-V_c)\frac{\Delta T_1}{R_l C} \ . \tag{8}$$

The designer must make some decision about the amount of ripple that can be tolerated. Equations 6 and 8 can be used to estimate appropriate values for C and the initial design value of $(\frac{\Delta T_1}{\Delta T_2})$.

We can also solve for the current, I_l and the ripple current ΔI_l.

$$I_l = \frac{V_s}{\frac{R_l}{(\frac{\Delta T_1}{\Delta T_2})((\frac{\Delta T_1}{\Delta T_2})+1)} + R_s} \ . \tag{9}$$

$$\Delta I_l = (-V_c)\frac{\Delta T_2}{L} = \frac{(-V_c)}{R_l}\frac{\Delta T_2}{(\frac{L}{R_l})} = I_{out}\frac{\Delta T_2}{(\frac{L}{R_l})} \tag{10}$$

Equations 9 and 10 can be used to select the inductance and the current rating of the inductor.

It was assumed that the regulator would be supplied by a source with the nominal parameters of $R_s = 0\ \Omega$ and $V_s = +10$ V. Using the simple approach, a system was designed to guarantee that $V_c = -10.0$ V and $\Delta V_c = 10$ mV. The percentage ripple in I_l was designed to be about the same at the percentage ripple in V_c. The selected values †. were: $C = 1500\mu$ F and $L = 68$ mH.

The simple, piecewise linear, model does allow some estimate of currents in the inductor and in the load and their rates of change. The main sources of EMI, in switch mode supplies, are the square switching waveforms, the charging current at the input and the current at the load.[6,7] If we are to estimate these quantities more accurately then we need a more realistic dynamical model. We also need to have a more accurate model to study the transient behaviour and the stability of these circuits. The most widely accepted and general approach to this problem is to use state variable techniques.[8–10]

3. A STATE VARIABLE FORMULATION OF A SWITCHED MODE CIRCUIT

If we apply infinitesimal differentials to the analysis of the buck/boost regulator, rather than finite differences, then Equations 1, 2, 3 and 4. can be rewritten using matrix notation.

For the "on" mode, we get:

$$\begin{bmatrix} \dot{V}_c \\ \dot{I}_l \end{bmatrix} = \begin{bmatrix} (\frac{-1}{R_l C}) & 0 \\ 0 & (\frac{-R_s}{L}) \end{bmatrix} \begin{bmatrix} V_c \\ I_l \end{bmatrix} + \begin{bmatrix} 0 \\ (\frac{V_s}{L}) \end{bmatrix} \tag{11}$$

For the "off" mode, we get:

$$\begin{bmatrix} \dot{V}_c \\ \dot{I}_l \end{bmatrix} = \begin{bmatrix} (\frac{-1}{R_l C}) & (\frac{-1}{C}) \\ (\frac{1}{L}) & 0 \end{bmatrix} \begin{bmatrix} V_c \\ I_l \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \ . \tag{12}$$

These can be written more concisely. For the "on" mode we get: $\dot{X} = A_1 X + U$. For the "off" mode we get: $\dot{X} = A_2 X + 0$. The column vector X contains the state variables. There are two different transition matrices, A_1 and A_2. The column vector, U represents the input from the source.

The state variables have been deliberately chosen in such a way that they are preserved across the switching boundaries. At these switching moments, the state transition matrices A_1 and A_2 are substituted for the matrix A.

†The actual numerical values are less important than the procedure used in the design.

4. STABILITY OF THE CIRCUIT IN EACH MODE

The system is linear in either mode and the stability can be readily analysed by examining the eigenvalues of the transition matrices. If we invoke the theory of the Laplace transform then these eigenvalues can be identified with the poles of the system in the s domain. The eigenvalues of A_1 are: $\frac{-1}{R_l C}$ and $\frac{-R_s}{L}$. The eigenvalues of A_2 are: $-\sigma + j\omega_d$ and $-\sigma - j\omega_d$, where $\zeta = \frac{1}{2}\frac{+\sqrt{\frac{L}{C}}}{R_l}$, $\omega_0 = \frac{1}{+\sqrt{LC}}$, $\sigma = +\omega_0\zeta$ and $\omega_d = \omega_0(+\sqrt{1-\zeta^2})$. The variable, ζ is called the damping factor and ω_d the damped frequency of oscillation, in radians per second. All of these eigenvalues have negative real parts and so the system is stable in either mode.

5. SOME SIMULATION RESULTS

The state space equations can be solved exactly using the theory of the Laplace transform.[10] The general solution is of the form:

$$x(t) = \exp(At)x(0) + \int_0^t \exp[A(t-\tau)]u(t)dt \ . \tag{13}$$

The exp() function refers to the matrix power series ‡ and the second term follows from the convolution property of the Laplace transform. If we perform the integration directly and apply Equation 13 to two adjacent time intervals ΔT_1 and ΔT_2 then we get an equation that describes the evolution of the system in the time domain:

$$\begin{aligned} X(t + \Delta T_1 + \Delta T_2) = {} & (\exp(A_2\Delta T_2) \cdot \exp(A_1\Delta T_1)) \cdot X(t) + \\ & (\Delta T_1) \cdot \exp(A_2\Delta T_2) \cdot \\ & (A_1\Delta T_1)^{-1} \cdot (\exp(A_1\Delta T_1) - I) \cdot U \end{aligned} \tag{14}$$

It is possible to simulate Equation 14 directly and quite accurately on a digital computer using mathematical software, such as Matlab. The quasi steady state case was described by the piecewise linear model. It is important to check that the state space model is consistent with this model. The simulation is shown in Figure 3.

This is consistent with what we expect from the simple model.

We wish to examine the way in which the system might respond to a sudden change in load, in the absence of any other changes §.

This was simulated and is shown in Figure 4.

Clearly this response is stable. This gives us some confidence that even very simple control laws, such as fixed duty cycle, may be useful. We would hope that the duty cycle and frequency could be altered by the control law at a rate which is slow in comparison with the switching operation.

Equations 6 and 7 suggest that we could make V_c arbitrarily large by changing the duty cycle. This suggests the following very simple control law: At each switching moment ΔT_2 is preserved and ΔT_1 is increased: $\Delta T_1 \leftarrow (1+\epsilon) \cdot \Delta T_1$. We chose $\epsilon = 0.0003$. The result is shown in Figure 5.

The result appears to be unstable. More detailed analysis ¶ suggests that this response is not "unstable" in the technical sense. The response should be bounded but it is not possible to simulate a long enough time interval to demonstrate this very clearly. The problem is that Equation 14 becomes numerically unstable for this simulation.

There are two fairly clear lessons to be drawn from this peculiar transient response.

1. Strict reliance on proofs of stability are often found in the literature.[11,12] These theorems are very formal and are ultimately true but they may not always be useful in practice. The response may become completely unsatisfactory ‖ long before it becomes "unstable" in the technical sense.

‡ $\exp(At) = \sum_{k=0}^{\infty} \frac{(At)^k}{k!}$

§ such as a change in time average duty cycle

¶ The eigenvalues of $(\exp(A_2\Delta T_2) \cdot \exp(A_1\Delta T_1))$ have absolute values of less than 1.

‖ The high power densities handled by these circuits can cause catastrophic failure.

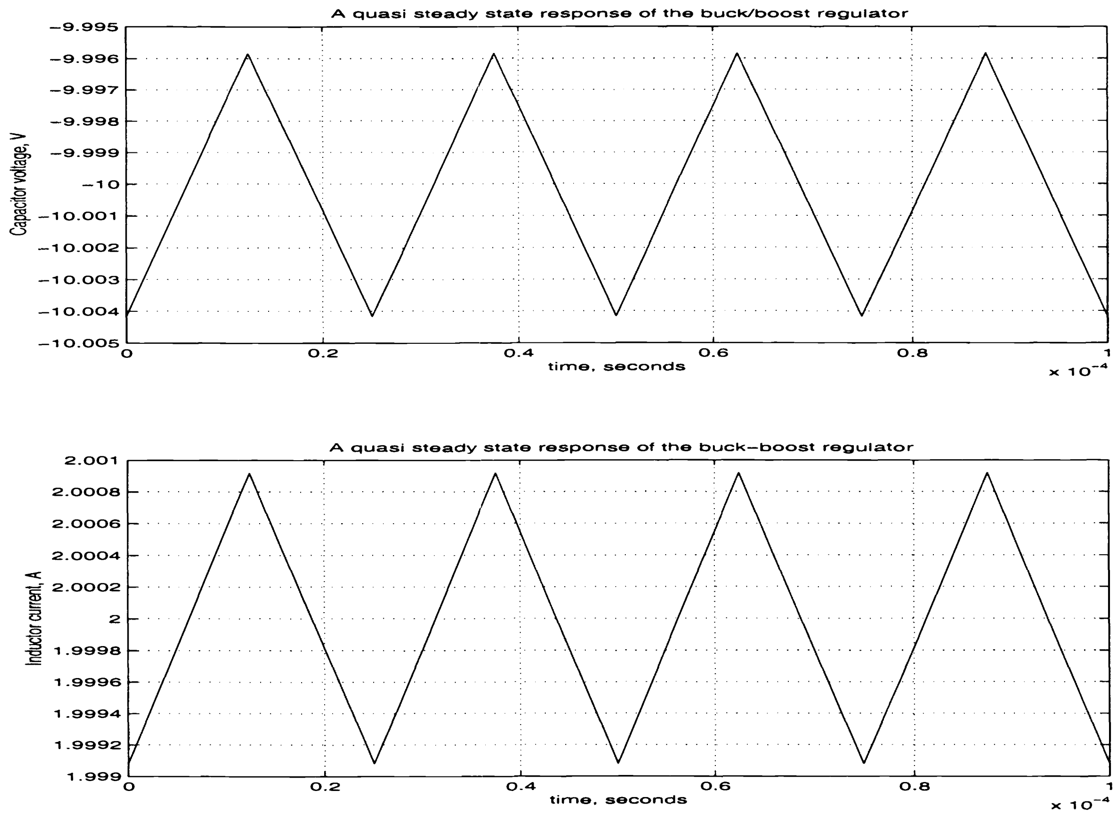

Figure 3. State space simulation of the quasi steady state case.

2. We need a very simple criterion to help us to stay out of danger. Equations 8 and 10 suggest that the switching times should be small compared with the time constants, $(R_l C)$ and (L/R_l). The definition of "small" should come from a specification of quality of the state variables. The requirement that the output ripple voltage be small is a much more stringent condition than the requirement that the output be stable.

In this paper we will only examine two measures of quality, RMS ripple or "noise" voltage in the output and the maximum power spectral density of the output current. The latter being a measure of EMC compliance.

6. THE STATE SPACE AVERAGED MODEL

The simulations in this paper were all achieved using a direct evaluation of the switched state space model. If we imagine a Boolean switching function, $q(t) \in \{0,1\}$ then we could write a single dynamical equation:

$$\dot{X} = (A_1 \cdot q(t) + A_2 \cdot (1 - q(t))) \cdot X + B \cdot V_s \tag{15}$$

from a pure mathematical point of view, the switching function, $q(t)$, could be arbitrarily complicated. It could have infinite bandwidth. In practice the fine structure of $q(t)$ does not matter very much since it is averaged by the plant. We can imagine a time averaged switching function: $d(t) = \frac{1}{T_s} \int_{t-\tau}^{t} q(\tau) \cdot d\tau$. In general, $d(t)$ would be a real function on the interval $[0, 1]$. We could then write:

$$\dot{X} = (A_1 \cdot d(t) + A_2 \cdot (1 - d(t))) \cdot X + B \cdot V_s = A_3 \cdot X + B \cdot V_s \tag{16}$$

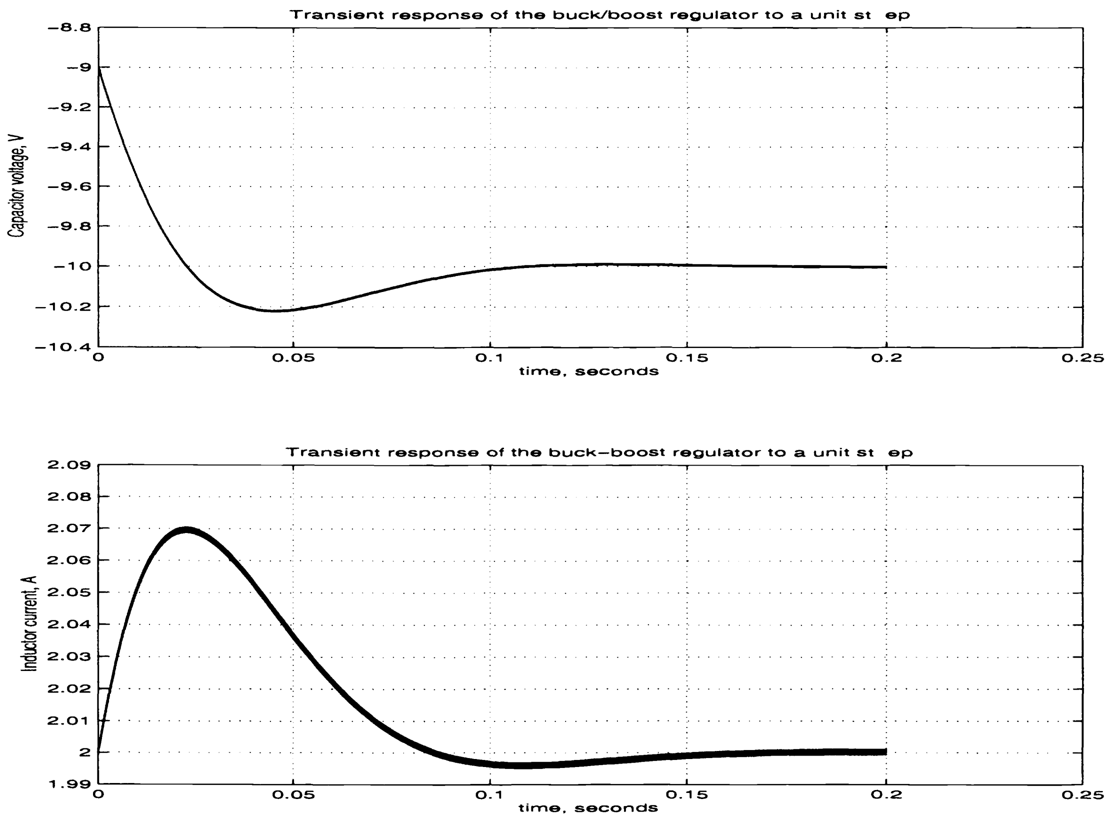

Figure 4. State space simulation of a transient response to a step change in the load.

This is called the state space averaged model.[13] It is a valid approximation ** as long as we choose an appropriate time frame, T_s. We could choose T_s to represent the last switching cycle, in which case we get $d(t) = \frac{\Delta T_1}{\Delta T_1 + \Delta T_2}$. We need to choose Ts to be small compared with the time constants of the plant but this still leaves us with considerable free choice for the values of T_s and $d(t)$.

The stability of the switched system can be established by examining the eigenvalues of the time averaged transition matrix, $A_3 = (A_1 \cdot d(t) + A_2 \cdot (1 - d(t)))$. For the buck boost regulator, the real parts of the eigenvalues are: $-\frac{1}{2}(\frac{1}{R_l C} + \frac{R_s}{L} d(t))$ which are always negative so the switched system is always stable as long as we switch quickly enough for the time averaged state space model to be valid.

7. SOME SIMULATIONS USING RANDOM VARIABLES IN THE CONTROL RULE

For the buck boost regulator we have a very simple stability result. The system is stable as long as the switching frequency high enough. We would like to use this freedom to improve the performance of the regulator.

We know that periodic switching can give rise undesirable electromagnetic radiation and that the EMC standards refer to maximum power spectral density.[7] Hui[3] has shown that the use of random switching functions can improve the spectral characteristics of power inverters. We propose an extremely simple random scheme for the variation of the switching frequency. We choose $T_s = (\rho \cdot \xi + (1 - \rho)) * T_0$, where ρ is a real number †† in the range $0 < \rho < 1$ and T_0 is the maximum switching time. The variable ξ is a uniform random variable in the range $0 < \xi < 1$. This scheme

**Strictly speaking, the averaging should occur in the time domain. We should average the matrix $(\exp(A_2 \Delta T_2) \cdot \exp(A_1 \Delta T_1))$ and compare this with an equivalent matrix $\exp(A_3(\Delta T_1 + \Delta T_2))$. The result is the same as averaging in state space as long as ΔT_2 and ΔT_1 are small.

††a degree of randomness or random factor

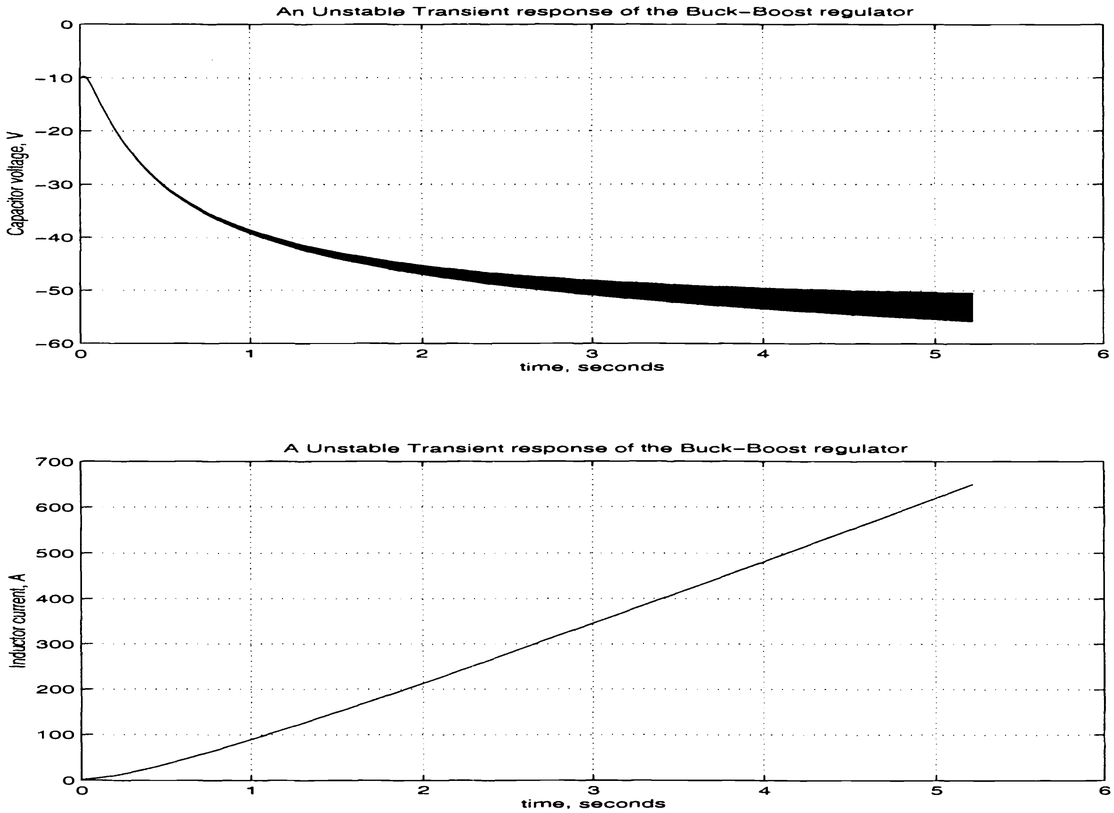

Figure 5. State space simulation of very large, almost unstable, transient response.

guarantees that the switching frequency always lies in the interval $(1/(\rho * T_0)) < Fs < (1/T_0)$. We can choose T_0 in order to satisfy the quality requirement for ripple voltage. This will also guarantee stability. We study the way in which RMS noise ripple voltage and maximum power spectral density vary in response to changes in ρ. The results are shown in Figure 6

The EMC performance shows a qualitative similarity to stochastic resonance.[14] The EMC performance improves as we increase the random factor, ρ, up to a value of about 0.5. It is possible to get a 5 to 10 dB improvement in performance. This is paid for by a small but steady increase in the RMS ripple voltage. Eventually the output becomes overwhelmed by noise and the EMC performance degrades.

The performance of this control law with $\rho = 0.5$ is shown in Figure 7.

8. SUMMARY AND CONCLUSIONS

The simple piecewise linear approach to design was found to be adequate and agreed quite well with the results from the more complex state space model. The switching times must be small in relation to the time constants of the system. [‡‡] If this condition is ever violated then the transient responses can be very large, even though the system is stable. The use of fast switching times allows the use of the time averaged state space model.

If we use fast switching times then we have some free choice of the switching frequency. We can use this freedom to improve EMC performance of the system by using some randomness in the control law. This can be done without compromising stability. The use of "too much" randomness eventually degrades the output of the regulator.

‡‡the absolute values of the eigenvalues of the state transition matrices

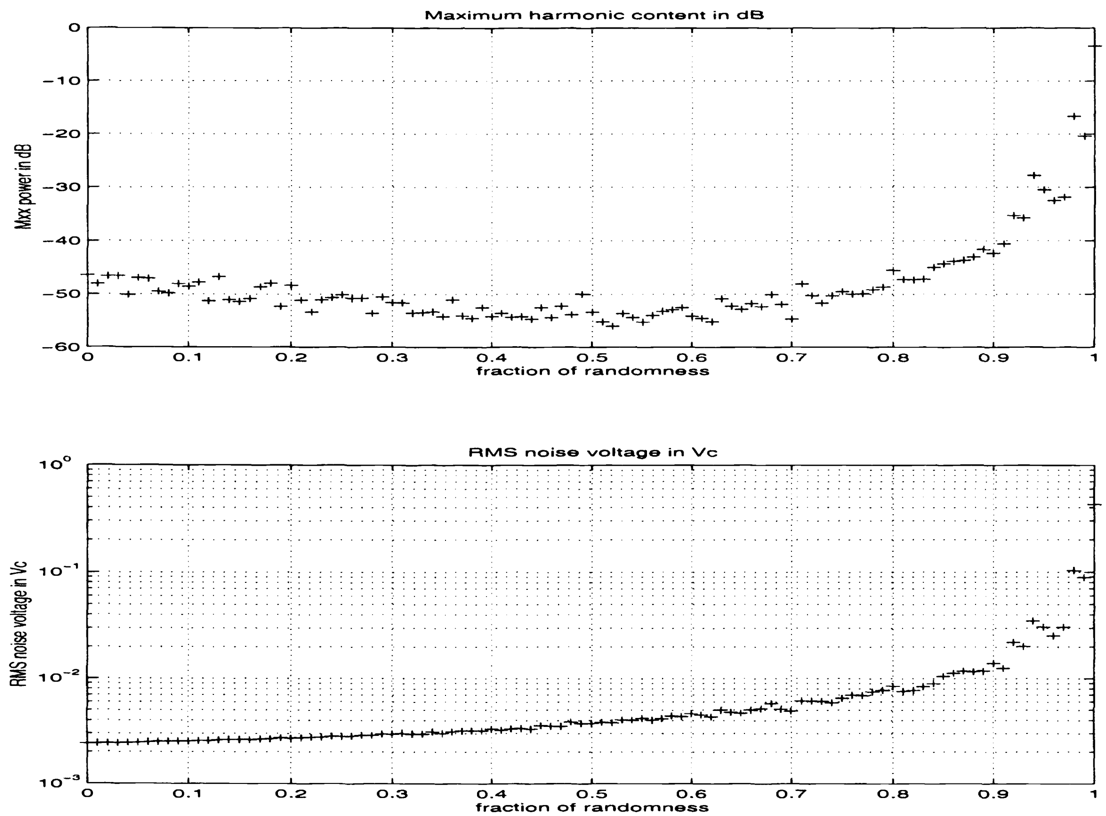

Figure 6. Variation of quality performance measures in response to a random factor, ρ.

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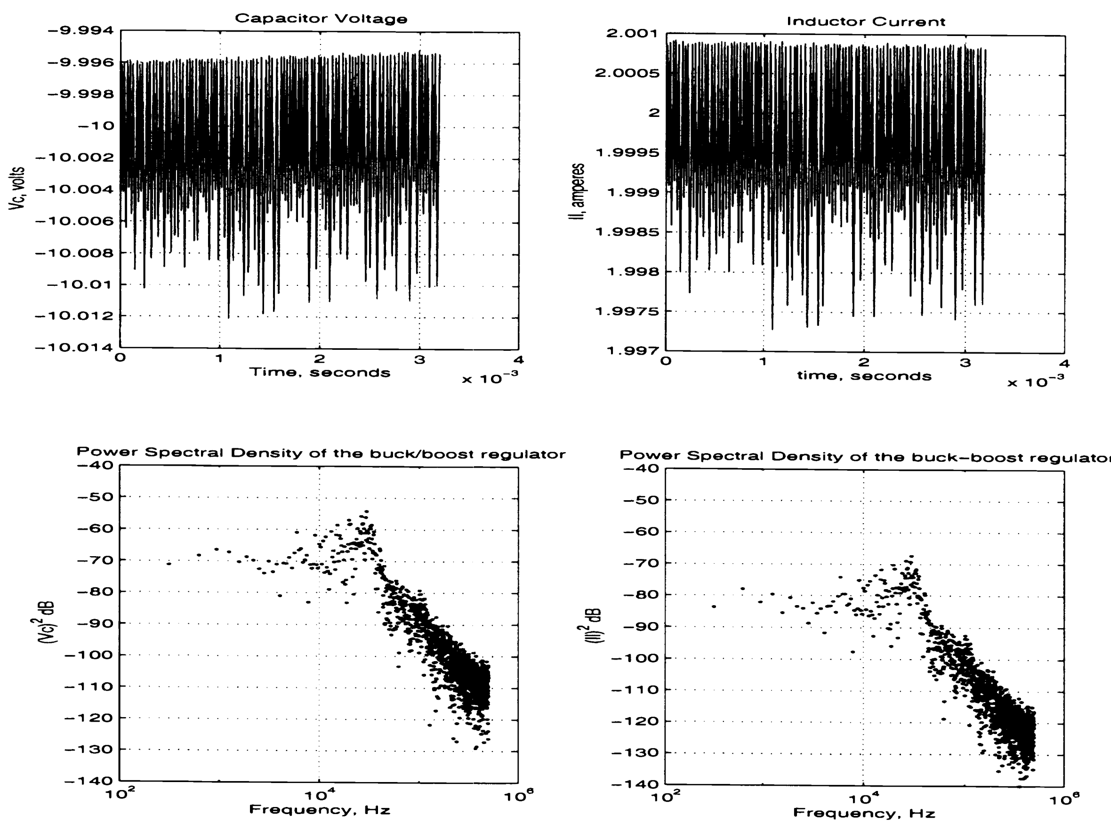

Figure 7. Transient behaviour and Spectral density of randomly switched response.