Decoherence in quantum games

Adrian P. Flitney and Derek Abbott

Centre for Biomedical Engineering and
Department of Electrical and Electronic Engineering,
The University of Adelaide, Adelaide, SA 5005, Australia

ABSTRACT

A protocol for considering decoherence in quantum games is presented. Results for general two-player, two-strategy quantum games subject to decoherence are derived and some specific examples are given. Decoherence in other types of quantum games is also considered. As expected, the advantage that a quantum player achieves over a player restricted to classical strategies is diminished for increasing decoherence but only vanishes in the limit of maximum decoherence when the results of the game are randomized.

Keywords: Quantum games, decoherence, bit flip errors

1. INTRODUCTION

Game theory has long been commonly used in economics, the social sciences and biology to model decision making situations where the outcomes are contingent upon the interacting strategies of two or more agents with conflicting, or at best, self-interested motives. There is now increasing interest in applying game-theoretic techniques in physics.[1] With the enthusiasm for quantum computation there has been a surge of interest in the discipline of quantum information[2] that has lead to the creation of a new field combining game theory and quantum mechanics: quantum game theory.[3] By replacing classical probabilities with quantum amplitudes and allowing the players to employ superposition, entanglement and interference, quantum game theory produces new ideas from classical two-player[4–9] and multi-player settings.[10–13] Quantum prisoners' dilemma has been realized on a two qubit nuclear magnetic resonance machine.[14] A review of quantum games is given by Flitney and Abbott.[15]

Decoherence can be defined as non-unitary dynamics resulting from the coupling of the system with the environment. In any realistic quantum computer, interaction with the environment cannot be entirely eliminated. Such interaction can destroy the special features of quantum computation. Quantum computing in the presence of noise is possible with the use of quantum error correction[16] or decoherence free subspaces.[17] These techniques work by encoding the logical qubits in a number of physical qubits. Quantum error correction is successful provided the error rate is low enough, while decoherence free subspaces control certain types of decoherence. Both have the disadvantage of expanding the number of qubits required for a calculation. Without such measures, noise and decoherence in quantum games needs to be considered, an area little studied. Johnson has considered a quantum game corrupted by noisy input.[18] Above a certain level of noise it was found that the quantum effects impede the players to such an a degree that they were better off playing the classical game. Chen *et al* have discussed decoherence in quantum prisoners' dilemma.[19] Decoherence was found to have no effect on the Nash equilibrium in this model. The current work considers general quantum games in the presence of decoherence. The paper is organized as follows. Section 2 outlines our model for introducing decoherence into quantum games, section 3 presents some specific results from this model for two-player, two-strategy quantum games, section 4 gives an example of decoherence in another quantum game and section 5 presents concluding remarks.

Further author information: (Send correspondence to A. P. Flitney)
A. P. Flitney: E-mail: aflitney@eleceng.adelaide.edu
D. Abbott: E-mail: dabbott@eleceng.adelaide.edu

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2. QUANTUM GAMES WITH DECOHERENCE

The process of quantizing a game with two pure strategies proceeds as follows. In the classical game the possible actions of a player can be encoded by a bit. This is replaced by a qubit in the quantum case. The computational basis states $|0\rangle$ and $|1\rangle$ represent the classical pure strategies, with the players' qubits initially prepared in the $|0\rangle$ state. The players' moves are unitary operators, or more generally, completely positive, trace-preserving maps, drawn from a set of strategies S, acting on their qubits. Interaction between the players' qubits is necessary for the quantum game to give something new. Eisert *et al* produced interesting new features by introducing entanglement.[4] The final state of an N-player quantum game in this model is computed by

$$|\psi_f\rangle = \hat{J}^\dagger (\hat{M}_1 \otimes \hat{M}_2 \otimes \ldots \otimes \hat{M}_N) \, \hat{J}|\psi_0\rangle \,, \tag{1}$$

where $|\psi_0\rangle = |00\ldots0\rangle$ represents the initial state of the N qubits, $\hat{J}$ ($\hat{J}^\dagger$) is an operator that entangles (disentangles) the players' qubits, and $\hat{M}_k$, $k = 1, \ldots, N$, represents the move of player k. A measurement over the computational basis is taken on ψ_f and the payoffs are subsequently determined using the payoff matrix of the classical game. The two classical pure strategies are the identity and the bit flip operator. The classical game is made a subset of the quantum one by requiring that $\hat{J}$ commute with the direct product of N classical moves. Games with more than two classical pure strategies are catered for by replacing the qubits by qunits (n level quantum systems) or, equivalently, by associating with each player a number of qubits.

It is most convenient to use the density matrix notation for the state of the system and the operator sum representation for the quantum operators. Decoherence can take many forms including dephasing, which randomizes the relative phases of the quantum states, and dissipation, that modifies the populations of the quantum states. Pure dephasing of a qubit can be expressed as

$$a|0\rangle \, + \, b|1\rangle \rightarrow a|0\rangle \, + \, b\,e^{i\phi}|1\rangle. \tag{2}$$

If we assume that the phase kick ϕ is a random variable with a Gaussian distribution of mean zero and variance 2λ, then the density matrix obtained after averaging over all values of ϕ is[2]

$$\begin{pmatrix} |a|^2 & a\bar{b} \\ \bar{a}b & |b|^2 \end{pmatrix} \rightarrow \begin{pmatrix} |a|^2 & a\bar{b}\,e^{-\lambda} \\ \bar{a}b\,e^{-\lambda} & |b|^2 \end{pmatrix}. \tag{3}$$

Over time, the random phase kicks cause an exponential decay of the off-diagonal elements of the density matrix.

In the operator sum representation, the act of making a measurement with probability p in the $\{|0\rangle, |1\rangle\}$ basis on a qubit ρ is

$$\rho \, \rightarrow \, \sqrt{1-p}\,\hat{I}\,\rho\,\sqrt{1-p}\,\hat{I} \, + \, \sum_{j=0}^{1} \mathcal{E}_j\,\rho\,\mathcal{E}_j^\dagger, \tag{4}$$

where $\mathcal{E}_0 = \sqrt{p}|0\rangle\langle0|$ and $\mathcal{E}_1 = \sqrt{p}|1\rangle\langle1|$. An extension to N qubits is achieved by the addition of further $\mathcal{E}_j$'s:

$$\rho \, \rightarrow \, \sqrt{1-p}\,\hat{I}\,\rho\,\sqrt{1-p}\,\hat{I} \, + \, \sum_{j_1,\ldots,j_N=0}^{1} \mathcal{E}_{j_1\ldots j_N}\,\rho\,\mathcal{E}_{j_1\ldots j_N}^\dagger \tag{5}$$

where $\mathcal{E}_{j_1\ldots j_N} = \sqrt{p}|j_1\ldots j_N\rangle\langle j_1\ldots j_N|$. This process also leads to the decay of the off-diagonal elements of ρ. By identifying $1 - p = e^{-\lambda}$, the measurement process has the same results as pure dephasing.

Independently of the particular model used, a quantum game with decoherence can be described in the
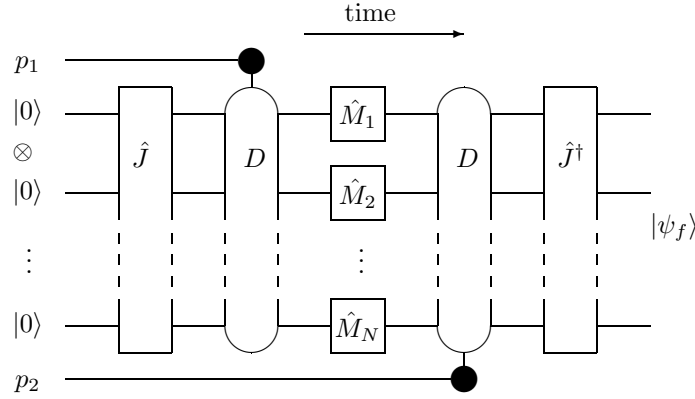
Figure 1. The flow of information in an N-person quantum game with decoherence, where M_k is the move of the kth player and $\hat{J}$ ($\hat{J}^\dagger$) is an entangling (dis-entangling) gate. The central horizontal lines are the players' qubits and the top and bottom lines are classical random bits with a probability p_1 or p_2, respectively, of being 1. D is some form of decoherence controlled by the classical bits.

following manner

$$\rho_i \equiv \rho_0 = |\psi_0\rangle\langle\psi_0| \qquad \text{(initial state)} \tag{6}$$
$$\rho_1 = \hat{J}\rho_0\hat{J}^\dagger \qquad \text{(entanglement)}$$
$$\rho_2 = D(\rho_1, p_1) \qquad \text{(partial decoherence)}$$
$$\rho_3 = (\Pi_{k=1}^N \hat{M}_k)\,\rho_2\,(\Pi_{k=1}^N \hat{M}_k)^\dagger \qquad \text{(players' moves)}$$
$$\rho_4 = D(\rho_3, p_2) \qquad \text{(partial decoherence)}$$
$$\rho_5 = \hat{J}^\dagger\rho_4\hat{J} \qquad \text{(dis-entanglement)},$$

to produce the final state $\rho_f \equiv \rho_5$ upon which a measurement is taken. The function $D(\rho, p)$ is a completely positive map that applies some form of decoherence to the state ρ controlled by the probability p. The products in the expression for ρ_3 are direct products. The scheme is shown in figure 1. The expectation value of the payoff for the kth player is

$$\langle \$^k \rangle = \sum_\alpha \hat{P}_\alpha\, \rho_f\, \hat{P}_\alpha^\dagger\, \$_\alpha^k, \tag{7}$$

where $P_\alpha = |\alpha\rangle\langle\alpha|$ is the projector onto the state $|\alpha\rangle$, $\$_\alpha^k$ is the payoff to the kth player when the final state is $|\alpha\rangle$, and the summation is taken over $\alpha = j_1 j_2 \ldots j_N$, $j_i = 0, 1$.

3. RESULTS FOR 2×2 QUANTUM GAMES

Let $S = \{\hat{U}(\theta, \alpha, \beta) : 0 \leq \theta \leq \pi, -\pi \leq \alpha, \beta \leq \pi\}$ be the set of pure quantum strategies, where

$$\hat{U}(\theta, \alpha, \beta) = \begin{pmatrix} e^{i\alpha}\cos(\theta/2) & ie^{i\beta}\sin(\theta/2) \\ ie^{-i\beta}\sin(\theta/2) & e^{-i\alpha}\cos(\theta/2) \end{pmatrix} \tag{8}$$

is an SU(2) operator. The move of the kth player is $\hat{U}(\theta_k, \alpha_k, \beta_k)$. The classical moves are $\hat{I} \equiv \hat{U}(0, 0, 0)$ and $\hat{F} \equiv \hat{U}(\pi, 0, 0)$. Entanglement is achieved by

$$\hat{J} = \frac{1}{\sqrt{2}}(\hat{I}^{\otimes N} + i\sigma_x^{\otimes N}). \tag{9}$$

Operators from the set $S_{\text{cl}} = \{\hat{U}(\theta, 0, 0) : 0 \leq \theta \leq \pi\}$ are equivalent to classical mixed strategies since when all players use these strategies the quantum game reduces to the classical one. There is some arbitrariness about

representation of the operators. Different representations will only lead to a different overall phase in the final state and this has no physical significance.

After choosing Eq. (5) to represent the function D in Eq. (6), we are now in a position to write down the results of decoherence in a 2×2 quantum game. Using the subscripts A and B to indicate the parameters of the two traditional protagonists Alice and Bob, respectively, and writing $c_k \equiv \cos(\theta_k/2)$ and $s_k \equiv \sin(\theta_k/2)$ for $k = A,B$, the expectation value of a player's payoff is

$$
\begin{aligned}
\langle \$ \rangle = {} & \frac{1}{2}(c_A^2 c_B^2 + s_A^2 s_B^2)(\$_{00} + \$_{11}) \ + \ \frac{1}{2}(c_A^2 s_B^2 + s_A^2 c_B^2)(\$_{01} + \$_{10}) \\
& + \frac{1}{2}(1 - p_1)(1 - p_2)\{[c_A^2 c_B^2 \cos(2\alpha_A + 2\alpha_B) - s_A^2 s_B^2 \cos(2\beta_A + 2\beta_B)](\$_{00} - \$_{11}) \\
& + [c_A^2 s_B^2 \cos(2\alpha_A - 2\beta_B) - s_A^2 c_B^2 \cos(2\alpha_B - 2\beta_A)](\$_{01} - \$_{10})\} \\
& + \frac{1}{4}\sin\theta_A \sin\theta_B \ [(1 - p_1)\sin(\alpha_A + \alpha_B - \beta_A - \beta_B)(-\$_{00} + \$_{01} + \$_{10} - \$_{11}) \\
& + (1 - p_2)\sin(\alpha_A + \alpha_B + \beta_A + \beta_B)(\$_{00} + \$_{11}) \ + \ (1 - p_2)\sin(\alpha_A - \alpha_B + \beta_A - \beta_B)(\$_{10} - \$_{01})],
\end{aligned}
\tag{10}
$$

where a superscript A or B is added to the \$'s to the give the payoffs to the two protagonists. Setting $p_1 = p_2 = 0$ gives the 2×2 quantum games studied in the literature. If in addition, $\alpha_k = \beta_k = 0$, $k = A,B$, a 2×2 classical game results with the mixing between the two classical pure strategies $\hat{I}$ and $\hat{F}$ being determined by θ_A and θ_B for Alice and Bob, respectively. Maximum decoherence with $p_1 = p_2 = 1$ gives a result where the quantum phases α_k and β_k are not relevant:

$$
\langle \$ \rangle = \frac{x}{2}(\$_{00} + \$_{11}) \ + \ \frac{1 - x}{2}(\$_{01} + \$_{10}),
\tag{11}
$$

where $x = c_A^2 c_B^2 + s_A^2 s_B^2$. However, this is not equivalent to the classical game as one might expect.

One way of measuring the "quantum-ness" of the game is to consider the known advantage of a player having access to the full set of quantum strategies S over a player who is limited to the classical set S_{cl}.[4,20] If we restrict Alice to $\alpha_A = \beta_B = 0$, then,

$$
\begin{aligned}
\langle \$ \rangle = {} & \frac{x}{2}(\$_{00} + \$_{11}) \ + \ \frac{1 - x}{2}(\$_{01} + \$_{10}) \\
& + \frac{1}{2}(1 - p_1)(1 - p_2)\left\{ c_B^2 \cos 2\alpha_B \ [c_A^2(\$_{00} - \$_{11}) + s_A^2(\$_{10} - \$_{01})] \right. \\
& \left. - s_B^2 \cos 2\beta_B \ [c_A^2(\$_{10} - \$_{01}) + s_A^2(\$_{00} - \$_{11})] \right\} \\
& + \frac{1}{4}\sin\theta_A \sin\theta_B \ [(1 - p_1)\sin(\alpha_B - \beta_B)(-\$_{00} + \$_{01} + \$_{10} - \$_{11}) \\
& + (1 - p_2)(\sin(\alpha_B + \beta_B)(\$_{00} + \$_{01} - \$_{10} + \$_{11})].
\end{aligned}
\tag{12}
$$

For prisoners' dilemma, the standard payoff matrix is

$$
\begin{array}{cc|cc}
& & \multicolumn{2}{c}{\text{Bob :}} \\
& & \text{cooperation}(C) & \text{defection}(D) \\
\hline
\text{Alice :} & C & (3,3) & (0,5) \\
& D & (5,0) & (1,1)
\end{array}
\tag{13}
$$

where the numbers in parentheses represent payoffs to Alice and Bob, respectively. The classical pure strategies are cooperation (C) and defection (D). Defecting gives a better payoff regardless of the other player's strategy, so it is a dominant strategy, and mutual defection is the Nash equilibrium. The well known dilemma arises from the fact that both players would be better off with mutual cooperation, if this could be engineered. With the payoffs of Eq. (13), the best Bob can do from Eq. (12) is to select $\alpha_B = \pi/2$ and $\beta_B = 0$. Bob's choice of θ_B will depend on Alice's choice of θ_A. He can do no better than $\theta_B = \pi/2$ if he is ignorant of Alice's
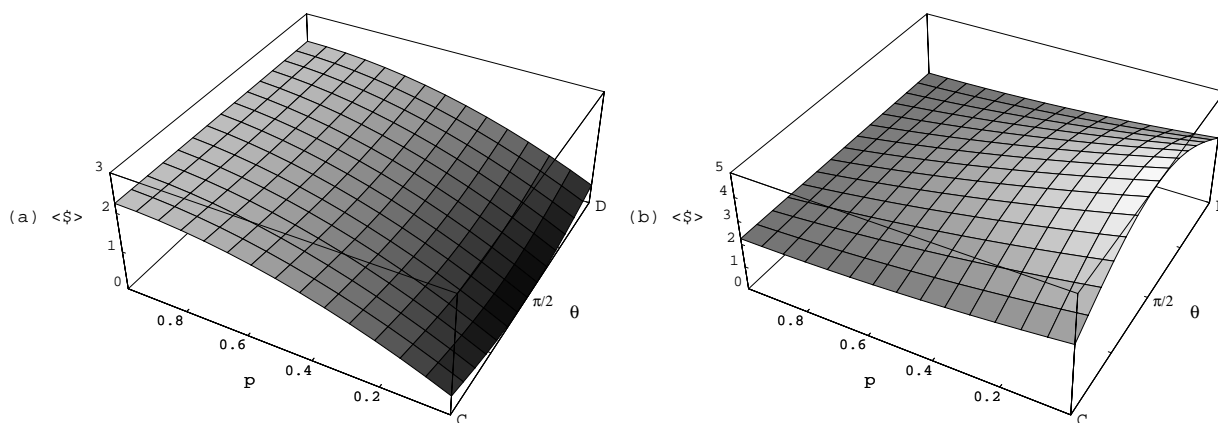
Figure 2. Payoffs for (a) Alice and (b) Bob in quantum prisoners' dilemma as a function of decoherence probability p and Alice's strategy θ (being a measure of the mixing between cooperation (C) and defection (D) with $\theta = 0$ giving C and $\theta = \pi$ giving D), when Bob plays the optimum quantum strategy and Alice is restricted to classical strategies. The decoherence goes from the unperturbed quantum game at $p = 0$ to maximum decoherence at $p = 1$.

strategy*. Figure 2 shows Alice and Bob's payoffs as a function of decoherence probability $p = p_1 = p_2$ and Alice's strategy $\theta \equiv \theta_A$ when Bob selects his optimal strategy.

The standard payoff matrix for the game of chicken is

$$
\begin{array}{cc|cc}
 & & \multicolumn{2}{c}{\text{Bob :}} \\
 & & \text{cooperation}(C) & \text{defection}(D) \\
\hline
\text{Alice :} & C & (3,3) & (1,4) \\
 & D & (4,1) & (0,0)
\end{array}
\tag{14}
$$

There is no dominant strategy. Both CD and DC are Nash equilibria, with the former preferred by Bob and the latter by Alice. Again there is a dilemma since the Pareto optimal results CC is different from both Nash equilibria. Bob's payoff is again optimized by $\alpha_B = 0$, $\beta_B = \pi/2$ and $\theta_B = \pi/2$. Figure 3 shows the payoffs as a function of decoherence probability p and Alice's strategy θ.

One form of the payoff matrix for the battle of the sexes is

$$
\begin{array}{cc|cc}
 & & \multicolumn{2}{c}{\text{Bob :}} \\
 & & \text{opera}(O) & \text{television}(T) \\
\hline
\text{Alice :} & O & (2,1) & (0,0) \\
 & T & (0,0) & (1,2)
\end{array}
\tag{15}
$$

Here the two protagonist must decide on an evenings entertainment. Alice prefers opera (O) and Bob television (T), but their primary concern is that they do an activity together. In the absence of communication there is a coordination problem. A quantum Bob maximizes his payoff in a competition with a classical Alice by choosing $\alpha_B = -\pi/2$, $\beta_B = 0$ and $\theta_B = \pi/2$. Figure 4 shows the resulting payoffs for Alice and Bob as a function of decoherence probability p and Alice's strategy θ.

The optimal strategy for Alice in the three games considered is $\theta = \pi$ (or 0) for prisoners' dilemma, or $\theta = \pi/2$ for chicken and the battle of the sexes. Figure 5 shows the expectation value of the payoffs to Alice and Bob as a function of the decoherence probability p for each of the games when Alice chooses her optimal classical strategy. In all cases considered, Bob outscores Alice and performs better than his classical Nash equilibrium result provided $p < 1$.[†] The advantage of having access to quantum strategies decreases as p increases but

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*See Flitney and Abbott[20] for details of quantum versus classical players.

[†]or the poorer of his two Nash equilibria in the case of chicken or the battle of the sexes
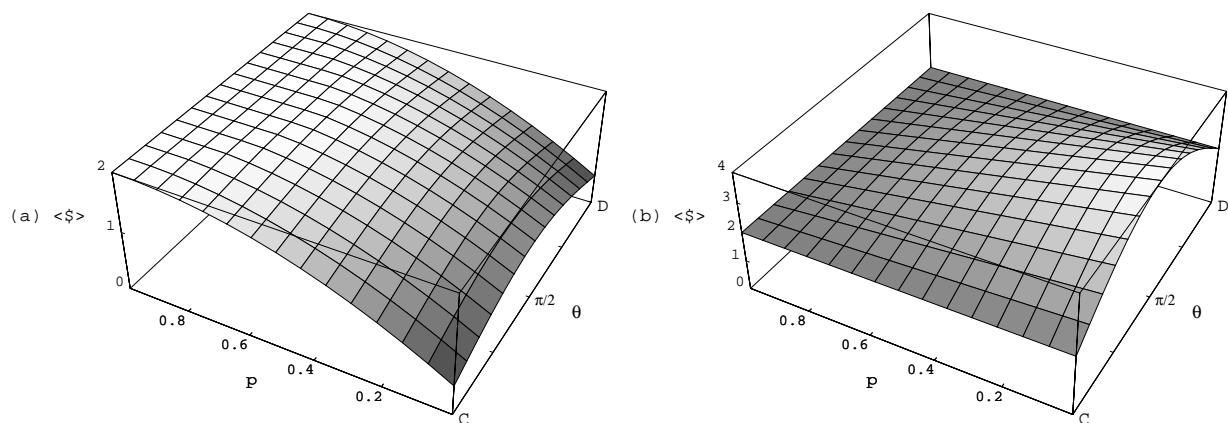
Figure 3. Payoffs for (a) Alice and (b) Bob in quantum chicken as a function of decoherence probability p and Alice's strategy θ, when Bob plays the optimum quantum strategy and Alice is restricted to a classical mixed strategy.
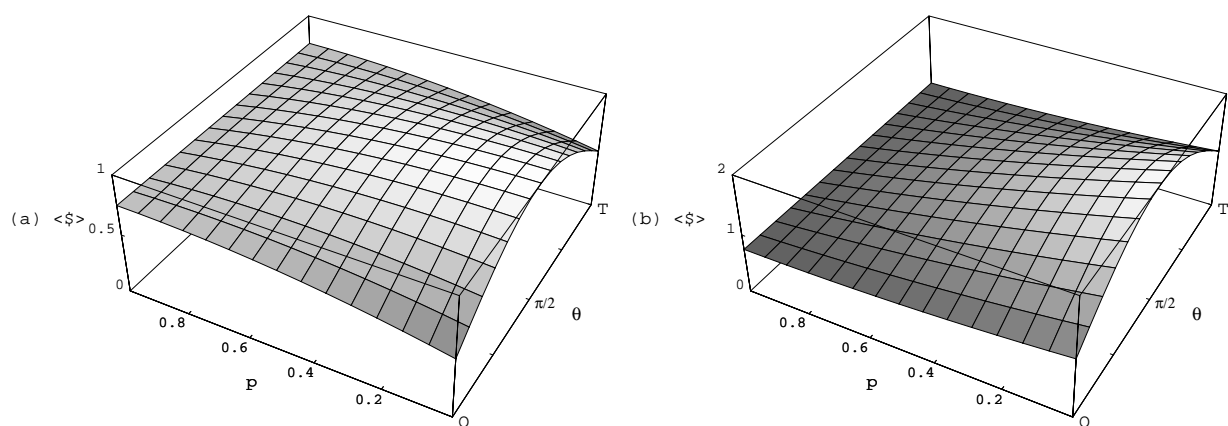
Figure 4. Payoffs for (a) Alice and (b) Bob in quantum battle of the sexes as a function of decoherence probability p and Alice's strategy θ, when Bob plays the optimum quantum strategy and Alice is restricted to a classical mixed strategy.
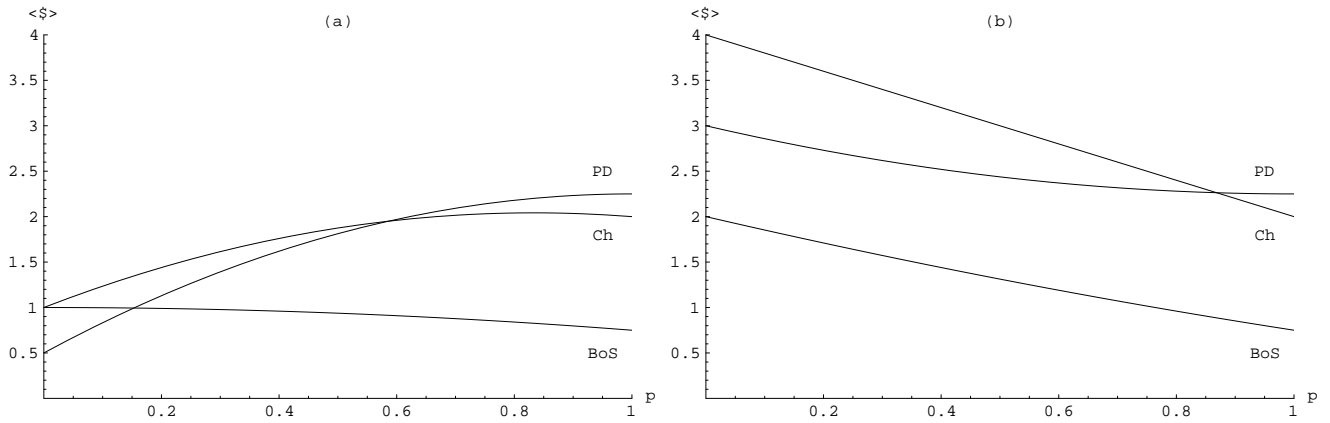
Figure 5. Payoffs as a function of decoherence probability p for (a) Alice and (b) Bob for the quantum games prisoners' dilemma (PD), chicken (Ch) and battle of the sexes (BoS). Bob plays the optimum quantum strategy and Alice her best classical counter strategy.

is still present for all levels of decoherence up to the maximum. At maximum decoherence ($p = 1$) the game result is randomized and the expectation of the payoffs are simply the average over the four possible results. The results presented in figures 2, 3 and 4 are comparable to the results for different levels of entanglement.[20] They are also consistent with the results of Chen and co-workers[19] who show that with increasing decoherence the payoffs to both players tend to the average of the four payoffs, as is the case in the present work.

4. DECOHERENCE IN OTHER QUANTUM GAMES

A simple effect of decoherence can be seen in Meyer's quantum penny-flip[3] between P, who is restricted to classical strategies, and Q, who has access to quantum operations. In the classical game, P places a coin heads up in a box. First Q, then P, then Q again, have the option of (secretly) flipping the coin or leaving it unaltered, after which the state of the coin is revealed. If the coin shows heads Q is victorious. Since the players moves are carried out in secret they do not know the intermediate states of the coin and thus the classical game is balanced.

In the quantum version, the coin is replaced by a qubit prepared in the $|0\rangle$ ("heads") state. Having access to quantum operations, Q applies the Hadamard operator to produce the superposition $(|0\rangle + |1\rangle)/\sqrt{2}$. This state is invariant under the transformation $|0\rangle \leftrightarrow |1\rangle$ so P's action have no effect. On his second move Q again applies the Hadamard operator to return the qubit to $|0\rangle$. Thus Q wins with certainty against any classical strategy by P.

Decoherence can be added to this model by applying a measurement with probability p after Q's first move. Applying the same operation after P's move has the same effect since his move is either the identity or a bit-flip. If the initial state of the coin is represented by the density matrix $\rho_0 = |0\rangle\langle0|$ the final state can be calculated by

$$
\begin{aligned}
\rho_f &= \hat{H}\hat{P}\hat{D}\hat{H}\rho_0\hat{H}^\dagger\hat{D}^\dagger\hat{P}^\dagger\hat{H}^\dagger \\
&= \frac{1}{4}\begin{pmatrix} 4-2p & 0 \\ 0 & 2p \end{pmatrix},
\end{aligned}
\tag{16}
$$

where $\hat{H}$ is the Hadamard operator, $\hat{P}$ is P's move ($\hat{I}$ or σ_x), and $\hat{D} = \sqrt{1-p}\,\hat{I} + \sqrt{p}(|0\rangle\langle0| + |1\rangle\langle1|)$ is a measurement in the computational basis with probability p. Again, the final state is independent of P's move. The expectation of Q winning decreases linearly from one to $\frac{1}{2}$ as p goes from zero to one. Maximum decoherence produces a fair game.
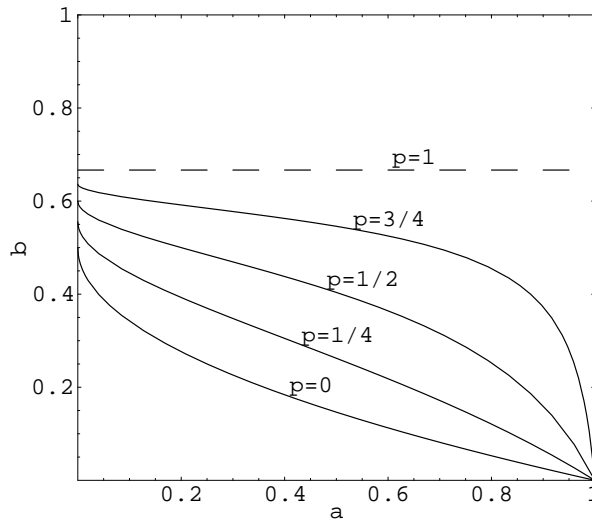
Figure 6. In a one round quantum truel with $c = 0$ and with decoherence, the boundaries for different values of the decoherence probability p below which Alice maximizes her expected payoff by doing nothing and above which by targeting Charles. There is a smooth transition from the fully quantum case ($p = 0$) to the classical one ($p = 1$). From Flitney and Abbott.[13]

As an example of the effect of decoherence on another quantum game consider a game analogous to a three player duel, or truel, between Alice, Bob and Charles.[13] The classical version can be described as follows. Each player has a bit, starting in the one state. The players move in sequence in alphabetic order. A move consists of either doing nothing or attempting to flip an opponent's bit with a known probability of failure of a, b or c, for Alice, Bob and Charles, respectively. A player can do nothing if their bit is zero. The payoffs at the completion of the game are 1/(number of bits in the one state) to a player whose bit is one, or zero otherwise. (The connection with a truel is made by considering one to correspond to "alive" and zero to "dead". A move is an attempt to shoot an opponent.) In some situation the optimal strategy is counter-intuitive. It may be beneficial for a player to do nothing rather than attempt to flip an opponent's bit from one to zero, since if they are successful they become the target for the third player.

The game is quantized by replacing the players' bits by qubits and by replacing the flip operation by an $SU(2)$ operator of the form of Eq. (8) operating on the chosen qubit. Maintaining coherence throughout the game removes the dynamic aspect since the players can get no information on the success of previous moves. Decoherence can be added to the quantum game by giving a probability p of a measurement being made after each move, and in the case of a measurement, allowing the players to choose their strategy depending on the result of previous rounds, which are now known. Figure 6 shows the regions of the parameter space (a, b) corresponding to Alice's preferred strategy in a one round truel when $c = 0$ (i.e., when Charles is always successful). The boundary between Alice maximizing her expected payoff by doing nothing and by targeting Charles depends on the decoherence probability p. We see a smooth transition from the quantum case to the classical one as p goes from zero to one. Note that the boundary in parameter space changes from linear in the classical case to convex in the quantum case. This is of interest since convexity is being intensely studied as the basis for Parrondo's paradox.[21]

5. CONCLUSION

A method of introducing decoherence into quantum games has been presented. One measure of the "quantumness" of a quantum game subject to decoherence is the advantage a quantum player has over a player restricted to classical strategies. As expected, increasing the amount of decoherence degrades the advantage of the quantum player. However, in the model considered, this advantage does not entirely disappear until the

decoherence is a maximum. When this occurs in 2×2 quantum games, the results are equally distributed over all the possible outcomes and the players' strategies are not relevant. The classical game is not reproduced. The loss of advantage to the quantum player is very similar to that which occurs when the level of entanglement between the players' qubits is reduced.

In the example of a one-round quantum truel, increasing the level of decoherence altered the regions of parameter space corresponding to different preferred strategies smoothly towards the classical regions. In this quantum game, maximum decoherence produces a situation identical to the classical game.

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