Information Entropy and Parrondo's Discrete-Time Ratchet

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Abstract. Parrondo's paradox involves two losing games of chance. The paradox is that a winning situation is produced when the two games are played in a random sequence in accordance with fixed probabilities. In this paper we investigate the relationship between these parameters and the entropy rates of the games.

INTRODUCTION

The apparent paradox that two losing games A and B can produce a winning outcome when played in a random sequence in accordance with fixed probabilities was devised by Parrondo as a pedagogical illustration of the Brownian ratchet [1].

It has recently come to the attention of the authors that a similar phenomenon was studied by Pinsky and Scheutzow [2]. They showed that a positive-recurrent diffusion process can be constructed by switching between a finite number of diffusion processes according to a Markov chain in such a way that the random diffusion, that is produced by following the sample path of the Markov chain, is almost surely transient. Similarly, a transient diffusion process can be constructed by switching between a finite number of diffusion processes in such a way that the random diffusion is almost surely positive-recurrent.

In this paper, we show how an analysis of Parrondo's games can be linked to information theory. A discrete-time Markov chain analysis is used to find the

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parameter space where the paradox of Parrondo's games exists. We investigate how this relates to the entropy rates of the games.

PARRONDO'S GAMES

Game A, which is described by (1) below, is straightforward and is given by the outcome of tossing a biased coin or taking a step in a random walk with drift.

$$\text{Game A:}\quad \begin{aligned} &\mathbf{P}[\text{winning}] = p \\ &\mathbf{P}[\text{losing}] = 1 - p. \end{aligned} \tag{1}$$

Game B is more complicated and can be described by a game in which one can either win or lose one unit of capital. If the present capital is a multiple of M, then the chance of winning is p_1, otherwise it is p_2. That is, we have (2) below.

$$\text{Game B:}\quad \begin{aligned} &\mathbf{P}[\text{winning}|\text{capital mod } M = 0] = p_1 \\ &\mathbf{P}[\text{losing}|\text{capital mod } M = 0] = 1 - p_1 \\ &\mathbf{P}[\text{winning}|\text{capital mod } M \neq 0] = p_2 \\ &\mathbf{P}[\text{losing}|\text{capital mod } M \neq 0] = 1 - p_2. \end{aligned} \tag{2}$$

A convenient parameterization is given in terms of a biasing parameter ε via the transformation $p = p' - \varepsilon$, $p_1 = p_1' - \varepsilon$ and $p_2 = p_2' - \varepsilon$.

SIMULATION RESULTS

Using Parrondo's original parameter values, $p' = 1/2$, $p_1' = 1/10$, $p_2' = 3/4$, $M = 3$ and $\varepsilon = 0.005$ [1], we simulated games A and B individually 100 times each and averaged the outcomes over many trials. Based upon this evidence, we concluded that both are losing, that is, after playing 100 games, on average we finish with less capital than we started with. We then simulated the situation in which we play two games of A, then two of B, two of A, and so on. The result was that, on average, we now win. Furthermore, we also win on average under the scenario in which we switch randomly between games A and B. Figure 1 shows the average progress when playing games A and B individually, as well as switching periodically or randomly. We refer to the game that is produced by randomly switching between games A and B as game C.

ANALYSIS

By regarding the current capital as the state of a discrete-time Markov chain, it can be shown that the paradox exists if the conditions

$$\frac{1-p}{p} > 1, \quad \frac{(1-p_1)(1-p_2)^{M-1}}{p_1 p_2^{M-1}} > 1 \quad \text{and} \quad \frac{(1-q_1)(1-q_2)^{M-1}}{q_1 q_2^{M-1}} < 1 \tag{3}$$

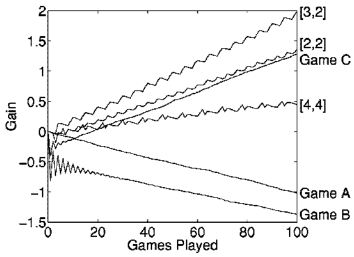

FIGURE 1. The progress of playing games A and B. Simulations were performed by playing game A a times, game B b times, and so on until a total of 100 games were played. This is represented by $[a, b]$. The results were averaged from 50 000 trials using Parrondo's original numbers with $\varepsilon = 0.005$.

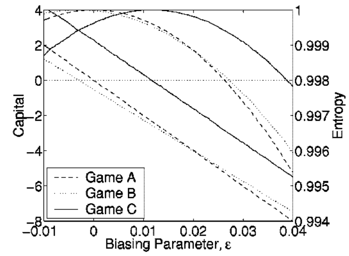

FIGURE 2. The relation between the fairness of the games and the entropy. The bottom three curves are the final capital after the 100th game has been played and the top three lines are the entropy rates. Simulations were performed by playing 100 games and averaging over 250 000 trials.

are met for games A, B and C respectively (see [3]). Here $q_1 = \gamma p + (1-\gamma)p_1$ and $q_2 = \gamma p + (1-\gamma)p_2$ are the winning probabilities of game C when the capital is (respectively is not) a multiple of M, γ is the probability of playing game A, and $1-\gamma$ is the probability of playing game B at each time point of game C. If the inequalities become equalities all the games are fair. For game A this occurs only when $p = 1/2$. For each value of p_1 in game B there is a p_2 for which fairness occurs. For $p_1 = 1/2$, that value is $p_2 = 1/2$.

ENTROPY

Now let us think of the sequences of wins and losses of the games in terms of information theory. Denote by X_j the random variable which represents the outcome at time point j when playing any of the games A, B or C. If the game wins at the jth time point, then $X_j = 1$, if it loses then $X_j = 0$.

For game A, the sequence $X_1, X_2, \ldots$ is an ergodic stationary sequence. In the case of games B and C, the sequence $X_1, X_2, \ldots$ is not stationary but we can regard it to be so after an initial period. Hence, by the Shannon-MacMillan-Breiman Theorem (see, for example, Durrett [4], page 314),

$$\lim_{n\to\infty} -\frac{1}{n}\log p(X_0, \ldots, X_{n-1}) = H \tag{4}$$

almost surely, where

$$H \equiv \lim_{n\to\infty} E\left[-\log p(X_n | X_{n-1}, \ldots, X_0)\right] \tag{5}$$

is the *entropy rate* of the sequence and $p(x_0, \ldots, x_{n-1}) \equiv P(X_0 = x_0, \ldots, X_{n-1} = x_{n-1})$ is the probability measure of the sequence.

In the current situation it is convenient to take logarithms with respect to base 2. For the case of game A, the right hand side of equation (5) reduces to

$$H^A = H(p) \equiv -p \log p - (1-p)\log(1-p) \qquad 0 < p < 1. \tag{6}$$

The random variables X_j for games B and C are correlated and so the expressions for their entropy rates are more complicated. To find the entropy rates of game B, we need to calculate the equilibrium probabilities π_i^B that the capital is congruent to i modulo M for $i = \{1, \ldots, M\}$. For $K =$ B or C the right hand side of equation (5) then becomes, using the appropriate probabilities,

$$H^K = [1 - \pi_M^K]H(p_2) + \pi_M^K H(p_1), \tag{7}$$

For both games B and C, we can calculate the entropy rates as if the random variables X_j were independent. For $K =$ B or C this results in

$$H_I^K = H([1 - \pi_M^K]p_2 + \pi_M^K p_1). \tag{8}$$

The argument of H on the right hand side of (8) is simply the stationary probability of producing a 1 at a given time point of game B. Since $H(p)$ is concave, it is easy to see that $H_I^B \geq H^B$ and $H_I^C \geq H^C$.

The entropy rate is a measure of how ordered a sequence is, the higher the entropy rate, the less order there is. Thus H^A is maximized when $p = 1/2$. Given this, we might expect the entropy rate of a game to be related to the fairness of the game. A fair game, one that has a net gain of zero, is likely to have have an equal number of wins and losses, hence 0s and 1s in the chain. It might be reasonable to expect that this is the most unordered state and has a maximal value for its entropy rate.

From the Shannon-MacMillan-Breiman Theorem, it follows that, by taking n large enough, we can get an estimate of the entropy rate of a stochastic process which generates a sequence from a sample path of that sequence via the left hand side of equation (4). We did this for the sample paths generated by the simulations reported above, using expressions (6) and (8). The average values of these estimates are plotted along with with the average gain in capital for the various games in Figure 2. Note that the entropy rates found in Figure 2 are calculated, *not* taking correlations into account. The effects of taking correlation into account will be dealt with in following section.

ENTROPY AND PARAMETER SPACE

In this section we explore the relations between the entropy rates generated from Parrondo's games and how they relate to each other and the parameter space. As we noted above, the successive values of X_j in games B and C are correlated, as can be seen from a simple case. If $p_1 = 0.9$ and $p_2 = 0.1$, then most of the time the capital will oscillate between two adjacent values. That is, given that we are in

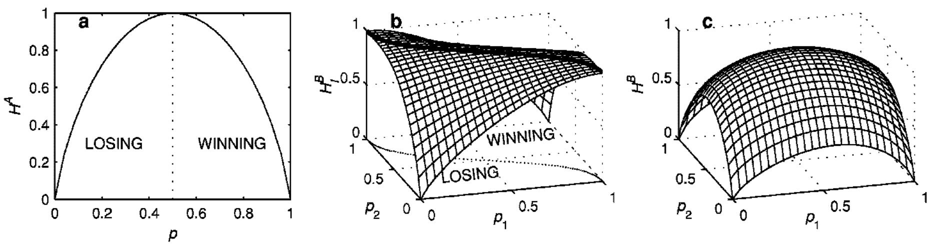

FIGURE 3. Entropy rates and parameter spaces: (a) game A; (b) uncorrelated entropy rate H^B, which agrees with simulations; (c) correlated entropy rate H^B.

one state, we can predict with high probability which state we will be in next and which digit will be generated next. Hence there is high correlation, even though there are approximately the same numbers of 0s and 1s.

Game A is simple, winning or losing depends only on p, and its entropy rate can be calculated directly from (6) (see Figure 3a). The maximum of the entropy rate curve occurs at $p = 1/2$: to the left we lose, to the right we win.

The condition for winning or losing game B depends on the parameters p_1 and p_2 (for a given value of M). The uncorrelated entropy rates H_I^B are shown in Figure 3b, while Figure 3c shows the correlated entropy rates H^B. The fairness of the game is actually reflected by H_I^B. When $H_I^B = 1$, the game is fair, shown by the thick line along the ridge in Figure 3b. This agrees with (3).

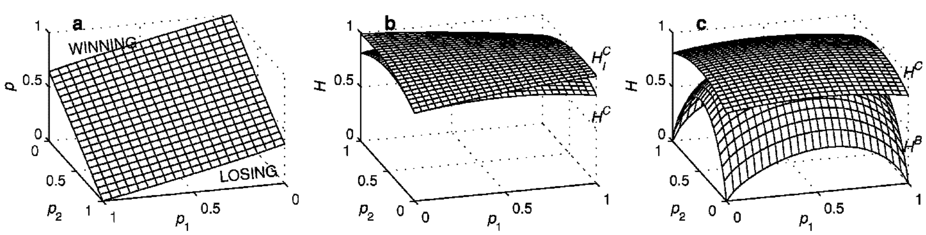

FIGURE 4. Entropy rates and the parameter space for game C: (a) this surface divides the winning and losing volumes; (b) comparing the correlated (H^C) and uncorrelated (H_I^C) entropy rates; (c) comparing the entropy rates H^B and H^C when $p = 1/2$.

Even though $H_I^B = 1$ occurs for *all* fair games, we can have $H^B < 1$ for a fair game (see [5]). From Figure 3c, we can conclude that the usual entropy rate of a game allowing for correlations is not related in any simple way to fairness.

For game C, the parameter space is dependent on the three parameters p, p_1 and p_2 (for given γ and M). The plane separating winning and losing games is described by (3), shown in Figure 4a. By plotting the three parameter spaces together we find an enclosed three-dimensional region within which games A and B both lose but game C wins (see [6]). That is, Parrondo's games are paradoxical in that region.

The entropy rates for game C are set naturally in four dimensions, which make them difficult to visualize. However the same traits can be carried from games A and B to game C. As before $H_I^C = 1$ occurs for all the fair games and we can verify that $H_I^C \geq H^C$ as shown in Figure 4b when $p = 1/2$.

Comparing the entropy rates from games B and C with (considered as sources), when $p = 1/2$, $H^C \geq H^B$ as shown in Figure 4c. If $p \neq 1/2$ then for some values of p_1 and p_2 $H^C < H^B$. This makes sense if we consider game C to be game B plus another source. If this other source is completely random ($p = 1/2$), then we are only adding disorder to game B, and cannot decrease the entropy rate.

CONCLUSION

We calculated the entropy rates of the games. It was revealed that the uncorrelated entropy rates are closely related to the parameter space. In particular, the games are fair when the uncorrelated entropy rates are 1. In fact, it is easy to see that measuring the uncorrelated entropy rate is just another way of counting the proportion of zeros and ones, and that the uncorrelated entropy rate is maximized when zeros and ones occur in equal proportion. The entropy rates allowing for correlations do not have a simple relation with the fairness of the games.

One way to think of this is as a new paradox in terms of uncorrelated entropy rates: with $\varepsilon = 0$, games A and B separately create sequences with maximum uncorrelated entropy rate. However the mixing of A and B creates a sequence with a smaller uncorrelated entropy rate.

This paradox does, however, have a very easy solution: there is no reason to think that mixing games with maximal uncorrelated entropy rate should produce another game with maximal uncorrelated entropy rate in the presence of correlation.

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