A NEW CIRCUIT THEORY PARADOX IN THE NOISE ANALYSIS OF A 2-STAGE RC LADDER

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A new paradox is described where the correlation between the thermal noise voltages in a 2-stage RC ladder behaves unexpectedly at limiting component values. A future resolution to this unsolved problem may possibly uncover a limitation in the circuit theory formalism for handling noise. There may be be a connection between this problem and Penfield's motor paradox proposed in 1966.

1 Introduction

For the first time, we present an unsolved paradox in a simple two stage RC ladder (Fig. 1). In the Appendix, we solve the relevant complex integral for the circuit showing that correlation between the two capacitor noise voltages is zero, ie. $\langle v_1 v_2 \rangle = 0$. However, we also show that if $R_2 \to 0$ or $R_1 \to \infty$ then $\langle v_1 v_2 \rangle$ suddenly becomes non zero! For low R_2 we obtain some correlation, whereas for large R_1 we get anticorrelation. In practice, zero correlation will not be observed as we impose a limited measurement bandwidth. Zero correlation is only obtained when we consider the total frequency band. Nevertheless, the predicted crossover from correlation to anticorrelation, as resistor values change, is a surprise result.

This dilemma is unsolved and may highlight a limitation in the circuit theory formalism for noise. If the capacitors are replaced by inductors, it may be that the problem has similar roots to Penfield's motor paradox[1,2].

It would be interesting to recalculate the correlation terms if we replace kT with the one dimensional form of Planck's law[3], to impose the quantum limit to bandwidth. Closed form solutions of the resulting integrals appear to be exceedingly difficult and could probably be expressed in terms of the χ function (the derivative of $\ln \Gamma(z)$). However, as $R_1 \to \infty$, the integration anomalies occur near $f = 0$, so the quantum form would not affect the result in this particular case.

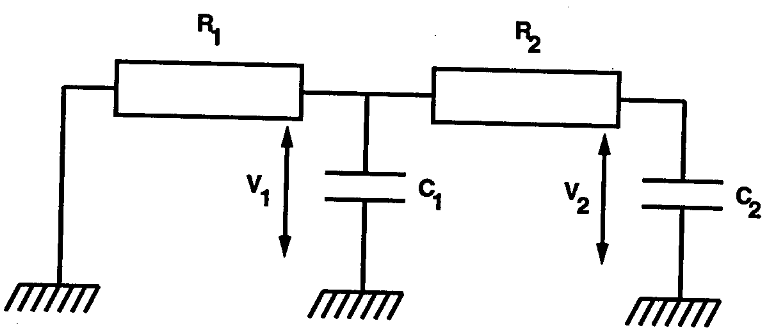

Figure 1: Two stage RC ladder – a source of an unsolved paradox.

2 Discussion

It may be that there are anomalies introduced by letting $R_1 \to \infty$ or $R_2 \to 0$ *before* we do the integration. It would therefore be instructive to take limits after the integration, for comparison. However, we do not get the opportunity to do this because $\langle v_1 v_2 \rangle = 0$ and thus there are no variables in this expression to manipulate. To overcome this problem, we write a new variable $\langle v_{ij} \rangle$ which is the voltage noise across capacitor C_i due to resistor R_j. So we can separate the noise contributions from the two resistors as

$$\langle v_1 \rangle = \langle v_{11} \rangle + \langle v_{12} \rangle \tag{1}$$

and

$$\langle v_2 \rangle = \langle v_{21} \rangle + \langle v_{22} \rangle \tag{2}$$

hence the correlation can be written as

$$\langle v_1 v_2 \rangle = \langle v_{11} v_{21} \rangle + \langle v_{12} v_{22} \rangle. \tag{3}$$

It can then be easily shown that

$$\langle v_{11} v_{21} \rangle = \frac{kT}{C_1 + C_2 + \frac{R_2 C_2}{R_1}} \tag{4}$$

and

$$\langle v_{12} v_{22} \rangle = -\frac{kT}{C_1 + C_2 + \frac{R_2 C_2}{R_1}}. \tag{5}$$

Therefore the sum of these two terms, which is the total correlation, is zero as before. Now if we let $R_1 \to \infty$ or $R_2 \to 0$, the total correlation is still zero. This contradicts our initial non-zero results, when the limits were taken *before* the integration. This basically summarizes the dilemma. The differences *can be* mathematically explained: for instance if $R_1 \to \infty$, before integration, the positive part of the curve becomes a delta function and is no longer captured by the integral. Hence we effectively integrate under the negative portion of the curve and the correlation becomes negative. The unsolved problem is the physical interpretation, in terms of noise, of the ordering of the limits.

3 Conclusion

We have outlined an unsolved problem regarding thermal noise correlation in a 2-stage RC ladder. A solution may improve understanding of the circuit theory formalism. If the capacitors are replaced by inductors, there may be some similarities with Penfield's paradox posed in 1966, though this requires further investigation.

Acknowledgments

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Appendix

General Complex Integral for Capacitor Problem

We need to solve the integral of the general form:

$$I = \frac{1}{2\pi j}\int_{-j\infty}^{+j\infty} \frac{a_0 + a_1 s + a_2 s^2}{(b_0 + b_1 s + b_2 s^2)(b_0 - b_1 s + b_2 s^2)} ds \tag{6}$$

Let $b_0 + b_1 s + b_2 s^2 = b_2(s - s_1)(s - s_2)$, so by factorizing the denominator and taking a contour integral we have,

$$I = \frac{1}{2\pi j}\oint_C \frac{a_0 + a_1 s + a_2 s^2}{b_2^2(s - s_1)(s - s_2)(s + s_1)(s + s_2)} ds. \tag{7}$$

Taking the sum of the residues,

$$I = \frac{a_2 s_1 s_2(s_1 - s_2) - a_0(s_1 - s_2)}{2 s_1 s_2 (s_1 - s_2)(s_1 + s_2) b_2^2} \tag{8}$$

and using $s_1 s_2 = b_0/b_2$ with $s_1 + s_2 = -b_1/b_2$, finally gives

$$I = \frac{a_0 b_2 - a_2 b_0}{2 b_0 b_1 b_2}. \tag{9}$$

This result is quite fascinating as the a_1 term has totally dropped out giving a purely real result. This can be explained because $a_1 s$ is an *odd* function of s.

Another curious matter is that because of the conjugation on the denominator of the original integral, the Cauchy-Riemann equations are *not* satisfied. However, the method of residuals happens to nevertheless work because the integral is essentially real given that the a_1 term drops out. As a precaution, we integrated the real part of the integral the long hand way and found that it did indeed reduce to the same result provided by the method of residues. Due to the great length of this procedure, this was accomplished using the MAPLE math editor software.

The fact that the method of residues is found to work on a non-analytic integral, is apparently not discussed in the complex analysis literature. There maybe be scope for future work to formally define a class of such integrals.

Noise Analysis of 2-Stage RC Ladder

From nodal analysis of the circuit we find that,

$$v_1 = \frac{e_1(1 + sC_2R_2) - e_2 sC_2R_1}{1 + s(C_1R_1 + C_2R_2 + C_2R_1) + s^2 C_1C_2R_1R_2} \tag{10}$$

and

$$v_2 = \frac{e_1 + e_2(1 + sC_1R_1)}{1 + s(C_1R_1 + C_2R_2 + C_2R_1) + s^2 C_1C_2R_1R_2}. \tag{11}$$

Using $e_1 = 2kTR_1$ and $e_2 = 2kTR_2$ and multiplying by complex conjugates, gives us the spectral densities,

$$S_{11} = 2kT\frac{R_1|1 + sC_2R_2|^2 + R_2|sC_2R_1|^2}{|1 + s(C_1R_1 + C_2R_2 + C_2R_1) + s^2 C_1C_2R_1R_2|^2} \tag{12}$$

$$S_{22} = 2kT\frac{R_1 + R_2|1 + sC_1R_1|^2}{|1 + s(C_1R_1 + C_2R_2 + C_2R_1) + s^2 C_1C_2R_1R_2|^2} \tag{13}$$

$$S_{12} = 2kT\frac{R_1(1 + s^2 C_1C_2R_1R_2)}{|1 + s(C_1R_1 + C_2R_2 + C_2R_1) + s^2 C_1C_2R_1R_2|^2}. \tag{14}$$

These spectral densities are now integrated using the general solution given in the last section. This yields the noise voltages in Volts squared per Hertz, and the integrals simply reduce to

$$\langle v_1^2 \rangle = \frac{kT}{C_1}, \quad \langle v_2^2 \rangle = \frac{kT}{C_2}, \quad \langle v_1 v_2 \rangle = 0$$

but if $R_2 \rightarrow 0$,

$$S_{11} = S_{22} = S_{21} = \frac{2kTR_1}{|1 + s(C_1 R_1 + C_2 R_1)|^2}$$

therefore,

$$\langle v_1^2 \rangle = \langle v_2^2 \rangle = \langle v_1 v_2 \rangle = \frac{kT}{(C_1 + C_2)}.$$

If $R_1 \rightarrow \infty$,

$$S_{11} = \frac{2kTR_2C_2^2}{|C_1 + C_2 + sC_1C_2R_2|^2}, \quad S_{22} = \frac{2kTR_2C_1^2}{|C_1 + C_2 + sC_1C_2R_2|^2},$$

$$S_{12} = \frac{-2kTR_2C_1C_2}{|C_1 + C_2 + sC_1C_2R_2|^2},$$

therefore,

$$\langle v_1^2 \rangle = \frac{kTC_2}{C_1(C_1 + C_2)}, \quad \langle v_2^2 \rangle = \frac{kTC_1}{C_2(C_1 + C_2)}, \quad \langle v_1 v_2 \rangle = -\frac{kT}{(C_1 + C_2)}.$$

References

1. P. Penfield, *Proc. IEEE,* **54,** 9 (1966).
2. D. Abbott *et al, IEEE Trans. Edu.* **39**, 1 (1996).
3. H. Nyquist, *Phys. Rev.* **32** (1928).