AIP Conference Proceedings

Stable processes in econometric time series: Are prices made out of noise?

Andrew Allison and Derek Abbott

Citation: AIP Conf. Proc. **511**, 221 (2000); doi: 10.1063/1.59978
View online: http://dx.doi.org/10.1063/1.59978
View Table of Contents: http://proceedings.aip.org/dbt/dbt.jsp?KEY=APCPCS&Volume=511&Issue=1
Published by the American Institute of Physics.

Stable processes in econometric time series: Are prices made out of noise?

Andrew Allison and Derek Abbott

Department of Electrical and Electronic Engineering
University of Adelaide
Australia, SA, 5005.

Abstract. If the rules that regulate a market allow speculation then prices in that market become volatile. It is possible to model the fluctuations in these prices as random variables. The empirical distributions of these random variables are unusual; they are distinctly non-Gaussian. They belong to a class of distributions called Lévy stable distributions and have paradoxical properties. The processes that generate these unusual distributions must also have special properties. One possible hypothesis is that markets are self organised critical systems. The discussion provides a brief history of these ideas and identifies some of the unsolved problems in the application of the theory of the Lévy process to economics.

INTRODUCTION

The dominant model for price determination in economic theory relies on the concept of equilibrium between supply and demand. This theory of general equilibrium was first stated, in mathematical form by Leon Walras in 1896 [1]. His economic theory was strongly influenced by contemporary theories in physics and engineering, such as statics. It resembles classical thermodynamics in that it attempts to define the equilibrium states of a system rather than describe the evolution of a system with time.

The basic structure of the general equilibrium theory has been continually refined and embellished, culminating in the work of John Hicks [2], Kenneth Arrow [3], and Gerard Debreu [3,4]. If we consider the theory as an example of pure mathematics then it is certainly rigorous. Unfortunately, the practical application has been limited. The classical theory predicts that prices should move to the equilibrium value and remain there with only very small variations. Any deviation from the equilibrium value should be punished by the "invisible hand" of the market. The equilibrium price should only change in response to fundamental structural changes in the underlying economy. Unfortunately, the typical empirical observation is that

CP511, *Unsolved Problems of Noise and Fluctuations*, edited by D. Abbott and L. B. Kish

prices fluctuate much more than the fundamental economy does. Many prices seem to wander about and do not show any tendency to stabilise at any particular fixed value.

It is clear that a new synthesis is required. The new theory must reduce asymptotically to the classical theory for the special restricted cases where the classical theory does appear to work. It must also account for the volatile fluctuations that are actually observed in most of the historical data.

THE CHOICE OF PARADIGM

The first step towards the creation of a new model is often the collection and analysis of data. Unfortunately, this never completely value free [5]. Possibly the most neutral way to present the data is as a simple graph, like Figure 1. In

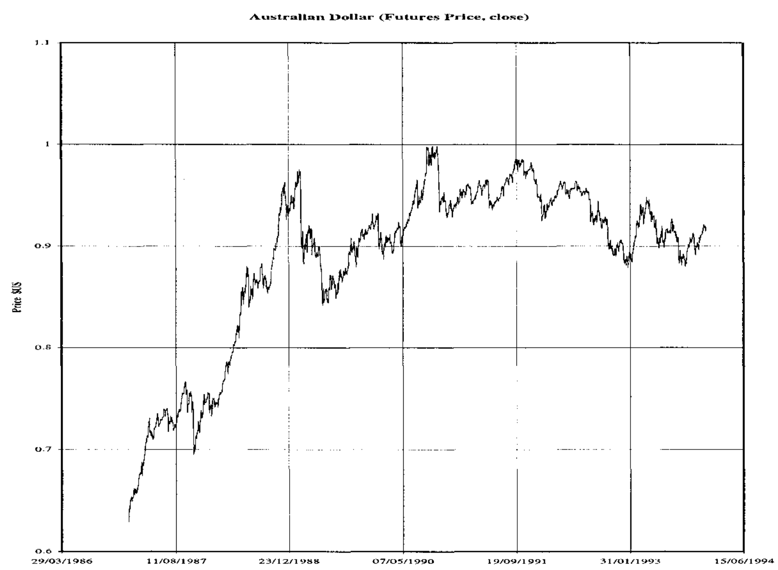

FIGURE 1. The price of Australian Dollar Futures contracts.

this example, the financial instruments being traded are Australian Dollar futures contracts. The prices are the daily closing prices and are quoted US dollars. The source of data was the Tick Data Inc. data base. The values of the prices have been adjusted to avoid the 'jumps' which occur at contract boundaries [6]. The dates are based on the Gregorian calendar. We can denote the price, as a function of time, using the symbol: $X(t)$. (A full discussion of the rules governing futures contracts is beyond our present scope but may be found in the literature [7].)

The vast majority of futures contracts are closed out prior to delivery [1] [7]. This indicates that the primary motive for most participants, is speculation. The problem with the simple graph is that it does not represent prices as they are considered by speculators, who are interested in short term relative changes in the price.

A most expedient, and yet exact, way to display relative changes is to use continuously compounded returns or 'log returns' [8]. We can define the return over a period, starting at time t and of duration d, as follows: $r(t,d) = \ln(X(t+d)) - \ln(X(t))$. The natural unit of a continuously compounded return is a Neper. In finance, a return is always quoted with respect to a time interval. The symbol, d, serves this purpose. In the special case as $d \to 0$, continuously compounded returns can be compared with simple percentage rates provided that they are multiplied by 100%.

Continuously compounded returns have a useful additive property. If we consider the time interval $[t, t+d_1+d_2]$ to be composed of two smaller intervals, $[t, t+d_1]$ and $[t+d_1, t+d_1+d_2]$ then we can write $r(t,(d_1+d_2)) = r((t+d_1), d_2) + r(t, d_1)$. This can readily be extended to N equally spaced time intervals:

$$r(t, Nd) = \sum_{k=0}^{(N-1)} r((t+kd), d) \ . \tag{1}$$

There are other reasons for using continuously compounded returns. Mandelbrot [9,10] has noted that, for a wide range of different financial instruments, volatility appears to be proportional to price.

In financial markets, it is rather unusual for trading to occur on every day of the week. This introduces an additional problem because the weekend break is longer than the overnight break between other days of the week. The usual solution to this problem is only to consider trading days and to regard all 'breaks' as equivalent. In practice, the difference does not appear to be very important, although this could be an area for further investigation.

Figure 2 displays the same information as Figure 1 but uses continuously compounded returns and elapsed trading days. A casual study of Figure 2 suggests that $r(t,1)$ is like a random variable. At first glance, it would appear to be white noise. A closer study seems to reveal some structure. The volatility does not seem to be uniform in time [2]. Large price fluctuations seem to be clustered together [3].

The most popular theories about the nature of price variation are as follows:

---

1) On 5th May 1993, the Guardian newspaper noted that the total volume of exchange in foreign currency markets was 20 times greater than the trade that it was supposed to finance.

2) In econometrics this phenomenon is called 'heteroskedasticity' [11].

3) The autocorrelation function of the square of the price variation provides more formal evidence for the existence of heteroskedasticity [8,12,13].

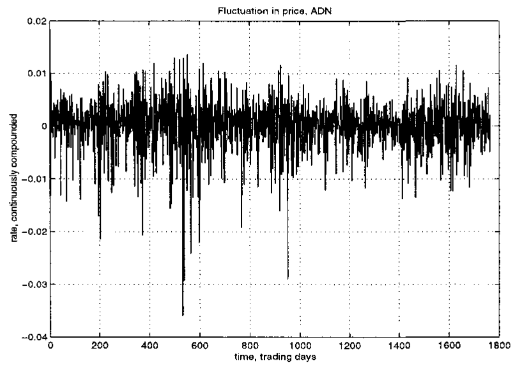

FIGURE 2. Continuously compounded daily rates of change

1. The process is non-stationary and is governed by a number of hidden variables. There may be some noise obscuring the underlying process. The presence of deterministic chaos may limit our ability to predict far into the future but some short term prediction is possible. This is the paradigm of technical trading. The technical trader tries to recognize a pattern that will repeat and act on that information before the pattern has completed [6].

2. Each variation, $r(t,1)$, can be modelled as a random variable which is independent of t and of any previous values. The distribution of $r(t,1)$ is Gaussian. The process is considered to be stationary, or at least quasi-stationary. No better prediction is possible. In the short term, all variations are Independently and Identically Distributed, IID. In the longer term the parameters may vary. Long term variations are assumed to be Independent but Not Identically Distributed, INID. This paradigm was first proposed by Bachelier [14]. It is regularly used in the financial industry to estimate the correct trade off between risk and expected return and forms the basis of the Capital Asset Pricing Model, CAPM [8]. A Gaussian model is also used in the derivation of the celebrated Black-Scholes equation, which is used to estimate the arbitrage price for derivatives [15].

3. Each variation, $r(t,1)$, can be modelled as a random variable which is independent of t and of any previous values. The distribution of $r(t,1)$ is non-Gaussian. The names 'Lévy stable' or 'α-stable' are used to describe these distributions. They are described as 'stable' because they are invariant under convolution and because they are the stable limits of other distributions, under convolution, within a domain of attraction [16]. The Gaussian distribution is a special limiting case of the Lévy stable distributions. In one sense, this paradigm is a generalisation of Paradigm 2. It was first proposed by B. B. Mandelbrot [9].

In some ways the most attractive point of view is Paradigm 1. For many people, the idea of predicting prices seems easier than working for a living. It would seem to be theoretically possible to model the mental states of every participant in the market using some sort of formalism, such as the theory of games. If enough data could be collected and the simulation could be executed faster than real time then prediction would be possible. One major problem is the complexity of the solution. As the number of independent variables increases then the combinatorial complexity increases so rapidly that it becomes completely intractable. This difficulty was anticipated by John Von Neumann who always hoped for a statistical simplification. No general statistical simplification has yet been found for games with very many players.

Another problem with Paradigm 1 is that price variations, such as Figure 2, do not exhibit any significant autocorrelation [12,8]. This means that any attempt at *linear* filtering is doomed to fail. Moving averages and Weiner filtering will not work, and since the Fourier transform is also a linear operation no advantage can be gained by working in the frequency domain. The square of the price variations do exhibit significant autocorrelation [12,8]. This may place limits on the use of a completely random model, although the result may be an artifact of poor or non-existent convergence [4] .

The discovery of deterministic chaos by Edward Lorenz has other implications for markets. It presents a practical problem that, even if simplification were possible and even if we could collect enough data to satisfy the initial conditions of our model, the accuracy of our predictions could quickly degrade due to the instability of the system. Even if prediction is possible, we can probably only predict a short time ahead and only with limited accuracy.

It is possible that some of the apparent 'noise' in price fluctuations is due to deterministic chaos and is not truly random. Attempts have been made to use the ideas of chaos and non-linear dynamics to predict price variations [19–21]. Prices do seem to have stable correlation dimensions. They are fractals. It is possible to calculate Lyapunov exponents for stock indices, which indicates that information takes a finite time to dissipate. Information is not immediately lost so price variations are not *pure* martingales. Unfortunately, the departure from randomness appears to be so small that it has not been possible to construct a successful trading scheme [21].

The final problem with Paradigm 1 is the Efficient Market Hypothesis, EMH, which states that the market already 'knows' everything that *can* be known about the future values of assets [22].

From a philosophical point of view price variations may not be truly random but they are effectively random in practice. They could be compared with pseudo-random numbers generated by a well designed computer program.

Paradigms 2 and 3 are essentially martingale models. The conditional probabilities are considered to be independent of past events. This implies that the long-horizon return, $r(t, Nd)$, will simply be a sum of N independent random vari-

[4]) It is possible that the covariance is not even defined [18].

ables [5]. The theory of additive processes can be invoked and it is possible that certain probability limit theorems will apply as $N \to \infty$. We can think of long-horizon returns as sums of random variables, represented by the short horizon returns. In other words, we are simply summing or integrating noise and the result is a random walk. More formally, this is called a Weiner process [23] in the case of a Gaussian distribution, or a Lévy flight in the case of a Lévy stable distribution.

The choice between these paradigms presumably depends on performance [6] and cost [7]. The final choice would depend on the particular application. There some theoretical reasons for preferring a Gaussian model, which are apparently contradicted by the Lévy stable model. In order to make an informed choice, we must consider the theoretical arguments in favour of the Gaussian distribution.

THE CENTRAL LIMIT THEOREM

There are many situations where a variable is a sum of a very large number of independent random factors. Each individual factor only has a very small effect on the process. We might consider the distribution of a sum,

$$S = \sum_{k=1}^{n} X_k \; . \tag{2}$$

The problem is to determine the distribution of S where very little is known about the distributions of X_k. The Central Limit Theorem states that, under a wide range of conditions, the distribution of S can be found and will be Gaussian [24]. One important premise assumed in the proof is that moments of order 2, and infinitesimally greater than 2, must converge [8]. Paul Lévy [26] realised that, even when these moments do *not* converge, the sum, S, may still converge to a stable limiting distribution. Some authors, notably W. Feller [16], developed the theory further using the concept of infinitely divisible distributions. In 1963 B. B. Mandelbrot published evidence that Lévy stable distributions occurred in econometric time series [9,10].

LÉVY STABLE DISTRIBUTIONS

All good scientific theories must have some aspect that can be tested. An important prediction of the theory of stable distributions is that the spread of the distribution should scale according to a power law. We can not use the standard

---

5) See Equation (1).

6) Performance is the ability of the model to accurately represent the phenomena.

7) Cost is the mathematical and computational difficulty associated with using the model.

8) Another assumption often made in practice is that convergence has *completely* occurred. Even if the moments eventually converge, they may do so very slowly [25].

deviation to measure spread since that does not converge for the Lévy stable distributions. Instead we use the symbol $\sigma(d)$ to denote one of the inter-hexile, 1/6 to 5/6 or 2/6 to 4/6, spreads [9] for the distribution of $r(t,d)$. The theory predicts that $\sigma(d)^\alpha = \sigma(1)^\alpha d$, where α is the stable index of the distribution [10]. The values of spread were estimated for various values of d and the results were plotted on log-log axes and are shown in Figure 3. The index [10] of the distribution was estimated to

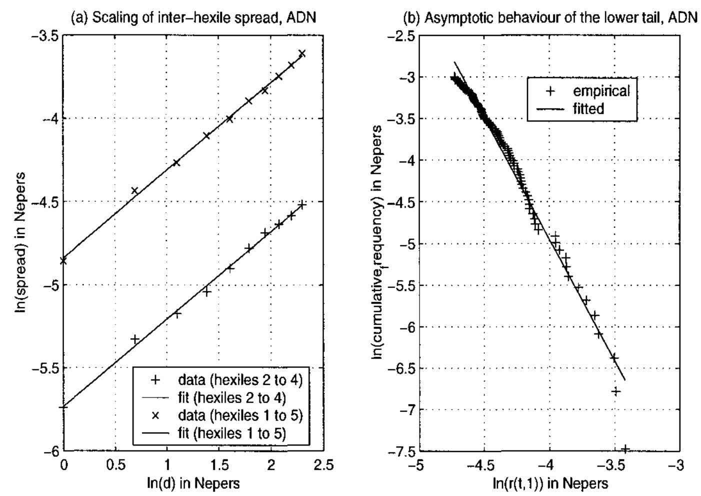

FIGURE 3. Scaling behaviour of Australian Dollar futures contracts, ADN.

be, $\alpha = 1.90$. Both inter-hexile spreads scale in the same way with time and differ significantly from the Gaussian value of 2.0. Given that $\alpha < 2.0$, we could predict that the formula for the variance would diverge. It is effectively infinite! This serves to remind us that there are a number of technical problems associated with work on the Lévy stable distributions. These include: (1) There is no general closed form expression for the probability density functions although there are series expansions and many special cases are known [16]. (2) Many of the moments diverge. The variance certainly diverges [16,10]. Measures that depend on the calculation of moments, such as kurtosis, may be unreliable. Analysis of variance becomes meaningless. (3) Even the mean can diverge, for $\alpha \leq 1.0$. Consider the case of the Cauchy distribution: $P(x) = \frac{1}{\pi(1+x^2)}$ [10]. (4) Expected values can be misleading even when they do converge This can lead to a number of paradoxes [10,27]. (5) Averaging can be useless! Consider the Cauchy distribution. The distribution of the average is the same as the original distribution [27]. This has implications for the 'moving average' methods used in technical trading. The standard formulae for

[9] The actual choice of percentiles should not matter. The scaling index should always be the same.

[10] The value of $\sigma(1)$ was estimated to be 0.0032 Neper over one day for hexiles 4 & 2 and 0.0079 Neper over one day for hexiles 1 & 5.

covariance and correlation may not converge [18]. (6) The principle of least squares is no longer relevant [10]. (7) No completely general method exists for estimation of all of the parameters [32].

These technical difficulties make the Lévy stable model less attractive and there is a tendency in the financial community to use other fat tailed distributions [8].

It has been proven that if we begin the limiting process with distributions, X_k, that are asmyptotically similar to a power law (Paretian) then it is possible to get a Lévy stable distribution in the limiting case. The necessary and sufficient conditions have been described by Mandelbrot [9,10], Fama [31], and Feller [16]. This is an important result because it provides a credible mechanism. The additive process described in Equation (1) gives rise to a limiting process that results in a Lévy stable distribution, as long as we begin with a distribution that is Paretian.

The next important step in understanding must be a theory that explains the origin of the Paretian distributions. There is considerable evidence that power laws are common in nature [30,27]. There are different mechanisms involved and it would not be prudent to claim that all phenomena with the same statistics must have similar mechanisms.

Power laws are certainly evident in the asymptotic behaviour of the Australian Dollar data. The probability density function can be inferred from the empirical equation for the asymptotic behaviour of the cumulative histogram. The asymptotic expression for the probability density function is $P(r) = P_0|r|^{-\alpha}$. Where $P_0 = 1.66 \times 10^{-7}$ and $\alpha = 2.93$. This numerical value of α is consistent with other values quoted in the literature. Gopikrishnan and Amaral [28,29] have studied the asymptotic behaviour of the distribution of price variations in great detail, using very large data bases. The fact that α is greater than 2 means that the distribution must eventually converge to a Gaussian if we could consider a sufficiently large time horizon. We may speculate that it only appears to be Lévy stable in the short term. We may also assume that the higher moments may be large and difficult to estimate but they are finite.

One of the more common mechanisms for generating power laws in physics, is the behaviour of systems when they are in a critical state. This occurs when matter is under the right conditions, of pressure and temperature, for a phase change to occur. There can be a very large change to the way in which the system is arranged in response to a very small change in energy. The most important model of phase transition is the Ising model [34,35]. The system is considered to be composed of many independent parts which only interact locally. The cumulative effect can give rise to very large scale events. This phenomenon is called 'long range dependence'. One important consequence of his theory is that the gross behaviour of the system does not depend on the microscopic details of the local interactions. This is called the principle of 'universality'. Computer simulations using the Ising model, and experiments reveal that changes have an energy spectrum which is Paretian [36].

SELF ORGANISED CRITICALITY

Most materials, like water, will only remain in a critical state if they are kept under special conditions so it would be difficult to see how the behaviour of critical systems could be a very common or important mechanism unless there was some means of maintaining or organising the critical state.

In 1987 Per Bak, and colleagues, [37] claimed that under certain conditions, a system can maintain *itself* in the critical state. These conditions have not been stated with the same precision as those of the Central Limit Theorem, but they include: (1) The system must be composed of a large number of elements that interact locally with each other. (2) The elements must have states. For example, a water molecule is either bound to an adjacent molecule, (in the solid state) or not bound (in a fluid state). (3)The elements can easily change state in response to nearby events. (4) There must be a constant flow of energy, from outside, through the system. (5) The flow of energy prevents the system from moving into the equilibrium condition of classical thermodynamics. (6) The system always has a distribution of states which is nearly stable or 'meta-stable'. (7) The system evolves as the distribution of states is rearranged. and it moves from one meta-stable state to the next.

Any rearrangement of the distributions of the states is perceived as an event, or a fluctuation, of a certain size or 'energy'. The histogram of these 'energies' reveals a power law. Computer simulations have been performed on several different systems of this type [39,36,40]. Some authors have claimed that many natural phenomena, as varied as earthquakes and the evolution of life, can be modelled using Self Organised Criticality, SOC. Simulations indicate that self organised systems naturally evolve into a critical state and then remain there, even if they are perturbed. These systems also have an interesting emergent property that they create fractal geometries. Other authors remain skeptical, claiming that the artificial systems that exhibit SOC do not obey physically realistic conservation laws and that the organisation in these systems is really concealed in the very special rules that have been chosen. Some authors have already applied the concept of SOC to economics [38].

CONCLUSIONS AND OPEN QUESTIONS

From what has been stated, and from a study of the literature, it should be clear that there is no widely accepted paradigm, or world view, that unifies the study of econometric time series and price variation. There are several competing paradigms which are very hard to compare objectively because each paradigm selects different problems as being significant [5]. The ultimate judgement must be made by the community of scientists who work in the area.

It is probably best to classify the unsolved problems in terms of the paradigms in which they have meaning.

1. Deterministic models that attempt to predict the future value of prices have, so far, proven to be computationally intractable. In spite of these difficulties, it seems hard to give up the quest for quick profit and there are some interesting theoretical questions to be examined. Why *does* volatility [11] seem to vary? Why do prices have stable fractal [12] correlation dimensions? Must we give up all hope of ever being able to predict the future of economic processes? The final, almost existential, question is: are prices *really* made out of noise? It seems absurd that anything so central to all our hopes and fears, as "value," should be so impermanent. Perhaps it is best to remain agnostic on this issue.

2. It is possible to build a model for the variations in prices using random variables. Gaussian models are easy to apply but not particularly accurate. There are still many arguments in their favour. Several authors have argued that very long horizon returns do converge slowly to a Gaussian distribution. The issue could be one of rate of convergence. The Gaussian paradigm has difficulty accounting for the way in which the distributions scale. There is also the issue of the apparent autocorrelation in the square of price variations. The main hope in this area rests on the Autoregressive Conditionally Heteroskedasticity, ARCH, and Generalised Autoregressive Conditionally Heteroskedastic, GARCH, models. These models rely on Gaussian random variables whose parameters are themselves regressions, or linear estimates, based on previous variations. The parameters vary within a similar time scale to the data that they model! Mandelbrot has criticised this principle quite severely on theoretical grounds [10]. Mantegna and Stanley have found that the empirical performance of GARCH is inferior to a Lévy stable model with exponentially tapered asymptotes [33].

3. Models based on Lévy stable distributions are empirically more accurate than the Gaussian models and there are possible mechanisms to explain how they could occur. The idea of Self Organised Criticality, SOC, should be investigated further.

   Unfortunately the Lévy models are difficult to apply and interpret although this is a technical rather than philosophical problem. There are a number of other interesting technical questions about this paradigm. What do the parameters of the distributions , such as the index, α, tell us about the structure of a market? Is there advantage to be gained from the fact that the Black-Scholes equation and the CAPM are based on theories that are not accurate?

   Although the Lévy model is empirically accurate, it is not perfect and it must compete with other non-Gaussian theories for our attention. Mittnik and Rachev [18] have proposed the Weibull distribution. They have asserted that it is more accurate than the pure Lévy model. The Weibull distribution is one of a family of extreme value distributions. We might postulate some

[11] Many technical traders believe that volatility holds the key to profitable investment.
[12] Some traders believe that there is deterministic chaos operating in markets.

sort of "auction" process. There is no a priori argument against alternative non-Gaussian distributions such as the Weibull. The final decision must be empirical [13].

It is hoped that this paper will, in a small way, encourage a new community of scientists to work towards a unifying paradigm which will explain the important features of price variation.

REFERENCES

1. Walras, L, *Etudes d'economie sociale*, Lausanne : F. Rouge, 1896
2. Hicks, J. R., *Value and capital* 1950
3. Arrow, K & Debreu, G, "Existence of an equilibrium for a competitive economy," Econometrica **22** (3), 1954 pp.265 to 290.
4. Debreu, G *Theory of value: an axiomatic analysis of economic equilibrium*, Yale University Press, 1959.
5. Kuhn, T. S., *The Structure of Scientific Revolutions*, The University of Chicago Press, 1970,
6. York, P., *The principles of profitable investment*, These course notes describe many of the standard techniques of technical trading.
7. McGrath M. & Viney C. M., *Financial Institutions, Instruments and Markets*, McGraw-Hill Australia, 1997
8. Campbell, J. Y. & Lo, A. W. & MacKinlay, A. C., *The Econometrics of Financial Markets*, Princeton University Press, 1997,
9. Mandelbrot, B. B., "The variation of certain speculative prices," The Journal of Business (Chicago),1963, **36**, 394-419.
10. Mandelbrot, B. B., *The fractal geometry of nature*, W.H. Freeman, 1983.
11. Gujarati, D. N., *Basic econometrics*, McGraw-Hill, 1995.
12. Cont, R. and Potters, M and Bouchaud, J. P., "Scaling in stock market data: stable laws and beyond," Physics WWW database, cond-mat/9705087 v3 9 May 1997.
13. Cont, R., " Scaling and correlation in financial data," Physics WWW database, cond-mat/9705075 v3 28 May 1997.
14. Bachelier, L. *Théorie de la spéculation*, Thesis for the Doctorate in Mathematical Sciences. (defended March 29,1900) Translation available in *The random character of stock market prices*, Ed. P.H.Cootner Cambridge MA: MIT Press, 1964
15. Baxter, M. *An introduction to derivative pricing*, Cambridge University Press, 1996.
16. Feller, W., *An introduction to probability theory*, John Wiley & Sons, Inc. 1966.
17. Bertoin, J. *Lévy processes*, Cambridge tracts in mathematics Cambridge University Press, 1996.
18. Taqqu, M. S., *Stable processes and related topics* , A selection of papers from the Mathematical Sciences Institute Workshop. Boston : Birkhauser, 1991.

[13)] The authors have tried to model price variations using Weibull and Fisher-Tippett distributions. The results were not very convincing.

19. Peters, E. E., "Fractal structure in capital markets," Financial Analysts Journal July/August 1989.
20. Peters, E. E., *Chaos and Order in the Capital Markets*, Wiley, 1990.
21. Casti, J. *Complexification*, Abacus, 1994.
22. Fama, E. F., "Efficient capital markets: a review of theory and empirical Work," Journal of Finance **25**, 383 (1970)
23. Papoulis, A., *Probability Random Variables, and Stochastic Processes*, McGraw Hill, 1991
24. Adams, W. J., *The life and times of the central limit theorem* Kaedmon Pub. Co., 1974.
25. Press, W. H. & Teukolsky, S. A. & Vetterling, W. T. & Flannery, B. P., *Numerical recipes in C*, Cambridge University Press, 2nd. ed. (1992).
26. Lévy, P., *Calcul des probabilités*, Paris: Gauthier Villars.
27. Schroeder, M., *Fractals, Chaos, Power Laws*, W.H Freeman and Company, (1991).
28. Gopikrishnan, P., Meyer, M., Amaral, L. A. N. & Stanley, H. E., "Inverse cubic law for the distribution of stock price variations," The European Physical Journal **3**, 139–140 (1998).
29. Gopikrishnan, P., Plerou, V., Amaral, L. A. N., Meyer, M., & Stanley, H. E., "Analysis of the distribution of returns for single companies," Proceedings of Applications of Physics in Financial Analysis, Paper: S1, Dublin, Ireland, (1999).
30. Stanley, H. E. & Ostrowsky, N., On growth and Form: Fractal and Non-Fractal Patterns in Physics, Proceedings 1985 Chargèse NATO ASI Institute, Martinus Nijhoff, (1986).
31. Fama, E. F., "Mandelbrot and the stable Paretian hypothesis," The Journal of Business (Chicago),1963, **36**, 394-419.
32. Zolotarev, V. M., *One-dimensional stable distributions* , Providence, R.I : American Mathematical Society, 1986, Translations of mathematical monographs.
33. Mantegna, R. N. & Stanley, H. E., "Scaling behaviour in the dynamics of an economic index," Nature **376**, 46–49 (1995).
34. Stanley, H., *Phase Transitions and Critical Phenomena* Clarendon Press, Oxford (1971).
35. Stauffer, D. & Stanley, H. E., *From Newton to Mandelbrot*, Springer, (1989).
36. Bak, P. & Chen, K., "Self Organized Criticality.," Sci. Am. **264** (Jan. 1991) 46–53
37. Bak, P. & Tang, C. & Wiesenfield, K., "Self organized criticality. an explanation of $1/f$ noise.," Physical Review Letters **59** (1987) 381
38. Bak, P. & Chen, K. & Scheinkman, J. A & Woodford, M. "Aggregate fluctuations from independent shocks: self-organized criticality in a model of production and inventory dynamics," Ricerche Economiche **47** (1993) 3.
39. Bak, P. & Tang, C. & Wiesenfield, K., "Self organized criticality.," Physical Review A **38** (1988) 364
40. Bak, P., *How nature works: The science of self organized criticality*, Oxford University Press, (1997).
41. Engle, R., "Autoregressive conditional heteroscedasticity with estimates of the variance of United Kingdom inflation," Econometrica, vol **50**, no 1, (1982), pp. 987-1007.