The Moving Plate Capacitor Paradox

B. R. Davis*, D. Abbott*and J. M. R. Parrondo† 1

*Department of Electrical and Electronic Engineering, The University of Adelaide, Adelaide SA 5005, Australia.
† Department Fisica Atomica, Nuclear y Molecular, Universidad Complutense de Madrid, 28040-Madrid, Spain.

Abstract. For the first time we describe an apparent paradox concerning a moving plate capacitor driven by thermal noise from a resistor. A demon restores the plates of the capacitor to their original position, only when the voltage across the capacitor is small – hence only small forces are present for the demon to work against. The demon has to work harder than this to avoid the situation of perpetual motion, but the question is how? We explore the concept of a moving plate capacitor, driven by noise, a step further by examining the case where the restoring force on the capacitor plates is provided by a simple spring, rather than some unknown demon. We display simulation results with interesting behavior, particularly where the capacitor plates collide with each other.

THE MOVING PLATE CAPACITOR DEMON

Consider a capacitor C with charge q and voltage v connected in parallel with a resistor R at a temperature T. The average force of attraction between the plates of a capacitor of area A and spacing x is given by $\langle f \rangle = -0.5 \langle v^2 \rangle \, dC/dx = \langle q^2 \rangle / 2\epsilon A = kT/2x$ since $\langle q^2 \rangle = kTC$, where ϵ is the permittivity of the material between the plates (in our case, air or a vacuum). The force f does work when the plates move together. Now suppose a demon is used to determine when the voltage across the capacitor is zero ($V = 0$) and at that instant the resistor is disconnected. There is no force between the plates, so they can be restored to their original position and then the resistor is reconnected. Clearly, so that we do not violate the laws of thermodynamics, the demon *must* do work, but the open question is where and how exactly is this work done? Can the work of Szilard, for instance, be used to explain this? Even if the voltage is not exactly zero, as long as it is small enough, the work done in restoring the plates to their initial position will be negligible, and

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the 'paradox' still remains. This problem only arises due to the rectification effect of force being proportional to the *square* of the voltage. This bears some similarity to Penfield's paradox [1] where the torque of a motor was related to the square of the current.

The dependence of the force on the square of either the voltage or the charge is an interesting situation since the fluctuations create a net force of attraction. The capacitor then acts as a rectifier of thermal fluctuations which is the aspect that we will focus on in this paper. Moreover, it is related to the old 'adiabatic piston problem' [2,3]. In that problem two gases with different densities and temperatures, but the same pressure, are separated by an adiabatic wall. The gases are in equilibrium from the point of view of classical thermodynamics, but it is known that the fluctuations push the piston to the cold side. The adiabatic piston problem is perhaps more involved than the present one, since the piston acts as a heat conductor. Nevertheless, the two problems are related, because in both cases a system is in equilibrium from the point of view of thermodynamics, but evolves due to fluctuations. The capacitor paradox is useful in the study of this problem because this effect is achieved without the use of adiabatic constraints. Note that when references are made to 'paradoxes' or 'demons', in this paper, these are merely *heuristic devices* used to highlight *incompleteness* in the present modelling of the system. The open question is to find the *simplest* description that completes the model, so that no apparent violations occur. This is of importance for increasing our understanding of the modelling of fluctuating systems.

CAPACITOR WITH A SPRING

In order to gain further appreciation of this problem, we decided to simulate the system shown in Figure 1. A spring of stiffness λ is connected to a moving plate of mass m, a resistor R is connected in parallel to provide noise, and i is the thermal noise due to R with power spectral density $S_{ii}(f) = 2kT/R$.

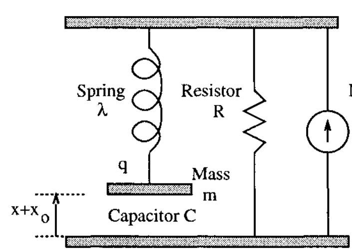

Figure 1. Capacitor, Mass and Spring System.

The relevant equations are:

$$\frac{dq}{dt} + \frac{q}{RC} = i, \qquad m\frac{d^2x}{dt^2} + \lambda x + f = 0, \qquad C = \frac{\epsilon A}{x + x_o}, \qquad f = \frac{q^2}{2\epsilon A} \tag{1}$$

where x_o is the distance the plates are apart when the the spring force is zero, and x is the deviation from this value.

A mechanical damper was not included because the mechanical system is damped by its coupling to the electrical system, and in addition to providing a force $f = d\,u$ proportional to the velocity u, it has associated with it a thermal noise force generator of power spectral density $S(f) = 2kTd$ which would obscure the interaction between the electrical and mechanical parts of the system.

In the simulation, a demon was not included, the object being solely to investigate the energy transfer between the electrical and mechanical systems. The parameters were arbitrarily chosen so that the RC circuit had a bandwidth of 2×10^5 rad/sec and the mechanical system was resonant at 10^5 rad/sec. For a system in thermal equilibrium with independent degrees and harmonic of freedom, the energy in each of the degrees of freedom would be expected to be $\frac{1}{2}kT$. For small perturbations of the capacitor plate, the mean square values of q, x and $u = dx/dt$ would be:

$$\hat{\sigma}_q^2 = kTC_o, \qquad C_o = \frac{\epsilon A}{x_o}, \qquad \hat{\sigma}_x^2 = \frac{kT}{\lambda}, \qquad \hat{\sigma}_u^2 = \frac{kT}{m} \tag{2}$$

The carets are used to designate that these may not be the true values as will be discussed later. The parameters were initially chosen with $\hat{\sigma}_x = 0.2\,x_o$ so that the capacitor plate perturbations were small. With arbitrary choices of $x_o = 0.1$ mm and $C_o = 50$ pF, the other parameters are then all determined. The values are $kT = 4.0 \times 10^{-21}$ J, $x_o = 0.1$ mm, $C_o = 50$ pF, $R = 1.0 \times 10^5\ \Omega$, $\lambda = 1.0 \times 10^{-11}$ N/m, $m = 1.0 \times 10^{-21}$ kg.

In the steady state, the average value of x will not be zero since the average force due to the noise voltage on the capacitor will cause the spring to extend slightly. Taking the ensemble average of the second equation in (1) and solving for the displacement $\bar{x}$ gives:

$$\bar{x} + \frac{kT}{2(\bar{x} + x_o)\lambda} \approx 0, \qquad \bar{x} \approx -2.04 \times 10^{-6} \tag{3}$$

where the approximation is $\bar{q^2} \approx kT\epsilon A/(\bar{x} + x_o)$. The approximation might be expected to be valid if $\bar{x}$ and $\hat{\sigma}_x$ are both $<< x_o$. The approximate equation has no solution if $x_o^2 < 2kT/\lambda = 2\hat{\sigma}_x^2$.

Figure 2 shows simulation results when the system had reached an apparent steady state. The variables plotted are normalised values $Q = q/\hat{\sigma}_q$, $X = x/\hat{\sigma}_x$ and $U = u/\hat{\sigma}_q$, where $\hat{\sigma}_q$, $\hat{\sigma}_x$ and $\hat{\sigma}_u$ are defined by (2).

The plates have an initial spacing $X_o = x_o/\hat{\sigma}_x = 5$, so $\bar{X}$ from (3) is -0.102. For the time interval shown, the mechanical variables X and U evolved only slowly due to the lack of mechanical damping, so the values are not representative of the average values. However the time average mean and variances calculated over the last 5 ms of a 10 ms simulation were $\langle Q \rangle = -0.076$, $\sigma_Q^2 = 0.968$, $\langle X \rangle = -0.098$, $\sigma_X^2 = 0.866$, $\langle U \rangle = -0.003$ and $\sigma_U^2 = 0.853$.

If the capacitor plates were fixed, the 95% confidence limits for σ_Q^2 would be approximately $0.88 \leq \sigma_Q^2 \leq 1.13$ and for small perturbations the result might be expected to be similar. The confidence limits for σ_X^2 and σ_U^2 are difficult to estimate, but are considerably wider because of the slow evolution of X and U.

With $\hat{\sigma}_x = 0.2\,x_o$, the plates of the capacitor remain separated for most of the time and for the parameters chosen no collisions between the plates were observed in the simulation. However if $\hat{\sigma}_x$ is increased, it was found that the plates collided regularly. In the simulation these collisions were assumed elastic so there was no energy loss. However it is not clear what should happen to q. If the plates are shorted together, the charge q might be expected to go to zero. On the other hand, if there is assumed to be an infinitesimally thin insulating sheet between the plates, then q would remain unaltered. Since the capacitor voltage is zero at this point, there does not seem to be any energy implications in setting $q = 0$, but clearly the system will evolve differently in time.

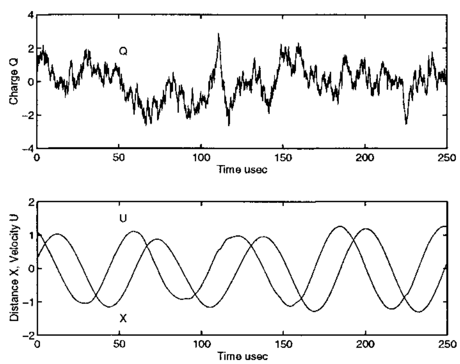

Figure 2. Small Perturbations.

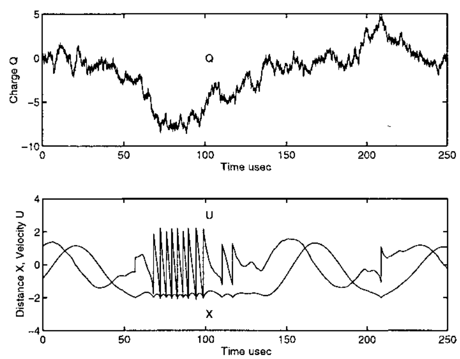

Figure 3. Large Perturbations.

Figure 3 shows simulation results for the same system as before, except that λ and m were reduced by a factor of 6.25, for which $\hat{\sigma}_x = 0.5\,x_o$. The charge q was not altered at the collision point. The simulation results show the plates colliding regularly when $X = -X_o$. In this case $X_o = 2.0$ and $\bar{X} = -0.293$ from (3).

In the time interval shown, multiple collisions occurred rapidly. When this occurs, Q can become large (due to the fact that the voltage across the capacitor is near zero, so no charge is lost through the resistor). The time average means and variances over an interval of 5.0 ms were $\langle Q \rangle = -0.258$, $\sigma_Q^2 = 3.013$, $\langle X \rangle = -0.284$, $\sigma_X^2 = 0.903$, $\langle U \rangle = -0.004$ and $\sigma_U^2 = 0.891$.

The main points to note are that $\langle X \rangle$ was close to its predicted value and the variances of X and U were close to unity, but the variance of Q was significantly greater than unity. When elastic collisions between the plates occur at times t_i determined by $x + x_o = 0$, the second equation in (1) becomes:

$$m\,\frac{du(t)}{dt} + \lambda\,x(t) + \frac{q^2(t)}{2\epsilon A} + 2\,m \sum_i u(t_i^-)\,\delta(t - t_i) \;=\; 0 \qquad (4)$$

where t_i^- is an infinitesimally small time before the time of collision t_i. The extra term accounts for the fact that during an elastic collision, the velocity $u(t) = dx/dt$ reverses in sign so that $u(t_i^+) = -u(t_i^-)$. Taking the time average of this equation over $0 \leq t \leq T$ then yields:

$$\langle q^2(t) \rangle \;=\; -2\epsilon A\lambda \langle x \rangle - \frac{4\epsilon Am}{T} \sum_i u(t_i^-) \tag{5}$$

Since $-x_o < \langle x \rangle < 0$ and and $u(t_i^-) < 0$, a positive value for $\langle q^2 \rangle$ is obtained. However, whether it remains bounded or not depends on the number of plate collisions in the interval $(0, T)$ in relation to T. From considerations discussed later, it seems that this number may grow faster than T.

If q is set to zero when the plates collide, the mechanical oscillations increase without limit. This seems to correspond to a Maxwell demon, although it is not clear how energy is being supplied to the system. Figure 4 shows a plot of the mean square value of the velocity U computed over consecutive intervals of length 250 μs for the original simulation when q was unaltered and for the case when q was set to zero.

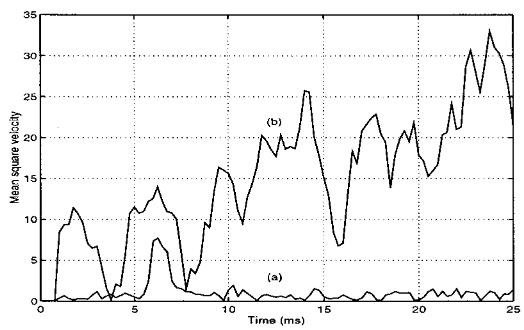

Figure 4. Mean Square Velocity (a) q not set to zero, (b) q set to zero.

DISCUSSION

Classical thermodynamics gives the stationary joint probability density function of q, x and u as

$$p(q, x, u) \;=\; \frac{1}{Z} \exp\left[-\frac{1}{kT}\left(\frac{q^2(x + x_o)}{2\epsilon A} + \frac{\lambda x^2}{2} + \frac{mu^2}{2}\right)\right] \tag{6}$$

where Z is a normalising constant.

Now (6) indicates that $\sigma_U^2 = 1$ and $\sigma_Q^2 = \infty$ regardless of the parameters, the latter being a consequence of collisions between the plates. Since in the first simulation no collisions were observed, the system had clearly not reached a steady state, and it would take an extremely long time to do so. The second simulation confirmed that with collisions between the plates, σ_Q^2 becomes larger as expected. The results from the third simulation were not consistent with (6), so it is concluded that setting $q = 0$ at a collision is not correct.

A summary of the results obtained for the variable X are shown below. Simulation 1 (Figure 2) corresponds to $X_o = 5$ (for which no collisions occurred), simulation 2 (Figure 3) corresponds to $X_o = 2$ with elastic collisions and leaving q unaltered, and simulation 3 corresponds to $X_o = 2$ with elastic collisions and setting $q = 0$. The theoretical values were obtained by numerical integration of the probability density function in (6). The theoretical mean values agree well with the approximate analysis presented earlier.

	Sim(1)	Theory(1)	Sim(2)	Sim(3)	Theory(2,3)
Mean of X	-0.098	-0.107	-0.284	1.376	-0.296
Variance of X	0.866	1.039	0.903	3.421	1.141

CONCLUSIONS AND OPEN QUESTIONS

We have shown that setting $q = 0$ cannot be correct, however:

- It seems that setting $q = 0$ when the plates collide drives the system out of equilibrium via the boundary conditions, rather than by the more usual situation where *explicit* terms added to the equations break the detailed balance. Can a prescription or a new generalised form of the fluctuation dissipation relation (FDR) be formulated that automatically predicts whether a given boundary condition yields equilibrium or not?
- Why does setting $q = 0$ seem to be a Maxwell demon? Theoretically there are no energy implications, or are there?
- Is it necessary to include the radiation pressure or Casimir force between the plates?
- While it is obvious how the electrical system couples energy into the mechanical system, it is not clear how the electrical system provides damping to the mechanical system.
- Does this problem have relevance to the 'adiabatic piston problem' [2,3], where the system is in equilibrium from the point of view of thermodynamics (average values) but not from the viewpoint of statistical mechanics (fluctuations)?

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