# Geometric intuition

Richard Feynman once made a statement to the effect that the history of mathematics is largely the history of improvements in notation — the progressive invention of ever more efficient means for describing logical relationships and making them easier to grasp and manipulate. The Romans were stymied in their efforts to advance mathematics by the clumsiness of Roman numerals for arithmetic calculations. After Euclid, geometry stagnated for nearly 2,000 years until Descartes invented a new notation with his coordinates, which made it easy to represent points and lines in space algebraically.

Feynman himself, of course, introduced into physics a profound change in notation with his space–time diagrams for quantum field theory. Previously, writing out the terms in an infinite series for a probability amplitude involved a laborious algebraic procedure, which Feynman replaced with simple pictures and explicit rules to translate them into mathematical expressions. This was an advance in housekeeping, if you will, but also among the most important advances in twentieth-century mathematical physics.

However, one of the most important and elegant advances in mathematical notation has perhaps not yet achieved the wide recognition it deserves. In 1873, the English mathematician and philosopher William Clifford invented a deceptively simple algebraic system unifying Cartesian coordinates with complex numbers, and offering a compact representation of lines, areas and volumes, as well as rotations, in 3-space. In more advanced physics, Clifford's algebra — he called it 'geometric algebra' — is now well recognized as the natural algebra for describing physics in 3-space, but it hasn't yet caught on in engineering, or even in standard treatments of electricity and magnetism or fluid dynamics, where vector analysis with its ugly cross product still holds sway.

Clifford's geometric algebra begins with the three coordinate vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ inherited from Descartes for the three independent directions in space. These satisfy the usual rules of orthonormality, $\mathbf{e}_i \bullet \mathbf{e}_j = \delta_{ij}$; they are mutually perpendicular and of unit length. Clifford then introduced another kind of multiplication between vectors, denoted as $\mathbf{e}_i\mathbf{e}_j$. His key point was to assume that this kind of multiplication would be anti-commutative for $i$ not equal

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to $j$, that is, $\mathbf{e}_i\mathbf{e}_j = -\mathbf{e}_j\mathbf{e}_i$. Another way to put it is that multiplication between parallel vectors is commutative, whereas it is anti-commutative for orthogonal vectors.

These rules are enough to define the algebra, and it's then easy to work out various implications. For example, $(\mathbf{e}_1\mathbf{e}_2)^2 = (\mathbf{e}_2\mathbf{e}_3)^2 = (\mathbf{e}_3\mathbf{e}_1)^2 = -1$. Something like $\mathbf{e}_1\mathbf{e}_2$ is called a bi-vector, but isn't a vector at all; rather it is a novel thing in its own right. Similarly, $\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ also isn't a vector, or a bi-vector, but a tri-vector, another totally new thing, the square of which also comes to −1. Within this algebra, the most general object is a multi-vector — the sum of a scalar, vector, bi-vector and tri-vector. In a sense, this is an advance over Descartes in that it provides a way to combine lines, areas and volumes within one formalism.

The resulting algebra has remarkable richness within it. The bi-vectors $\mathbf{e}_1\mathbf{e}_2$, $\mathbf{e}_2\mathbf{e}_3$, $\mathbf{e}_3\mathbf{e}_1$, for example, can be thought of as oriented areas. They are linked to rotations respectively about the $\mathbf{e}_3$, $\mathbf{e}_1$, $\mathbf{e}_2$ axes, and act identically to the basis elements of William Rowan's quaternions, which he introduced in 1843 in an attempt to generalize complex numbers to three dimensions. The tri-vector — for simplicity, we can denote it as $\check{I}$ — acts analogously to $i = \sqrt{-1}$; it's square is −1 and it commutes with all the basis vectors. Using this shorthand, the bi-vectors and tri-vectors together satisfy the Pauli algebra $\mathbf{e}_i\mathbf{e}_j = \check{I}\varepsilon_{ijk}\mathbf{e}_k$ central to the description of rotations in three dimensions (here $k$ is summed over, and $\varepsilon_{123} = 1$ changes sign for any permutation of indices, and vanishes if any two are equal).

For example, the rotation of any vector about the $\mathbf{e}_3$ axis is generated by multiplying the vector from the left by $\mathbf{e}_2\mathbf{e}_1$; this bi-vector is a 'rotor' that acts as an operator generating a rotation through the arc defined as $\mathbf{e}_1$ sweeps through to $\mathbf{e}_2$. For any two unit vectors $\mathbf{e}_a$ and $\mathbf{e}_b$, $\mathbf{e}_b\mathbf{e}_a$ generates a similar rotation in the plane defined by the two vectors. Of course, these rotations satisfy a non-commutative algebra as must be true if they are to represent the consequences of rotations in 3-space faithfully.

It is also completely natural not only to add or subtract multi-vectors, but to multiply or divide them — something not possible with ordinary vectors. The result is always another multi-vector. In the particular case of a multi-vector that is an ordinary vector $\mathbf{V}$, the inverse turns out to be $\mathbf{V}/v^2$, where $v^2$ is the squared magnitude of $\mathbf{V}$. It's a vector in the same direction but of reciprocal magnitude.

Write out the components for the product of two vectors $\mathbf{U}$ and $\mathbf{V}$, and you find the result $\mathbf{UV} = \mathbf{U}\bullet\mathbf{V} + \check{I}\mathbf{U} \times \mathbf{V}$, with $\bullet$ and $\times$ being the usual dot and cross product of vector analysis. Hence, geometric algebra blends both operations in a natural way.

For nearly 40 years, physicist David Hestenes of Arizona State University has waged a one-man crusade to advertise Clifford's geometric algebra and to lift it up to what he sees as its rightful place in physics. It hasn't worked yet. The standard techniques of vector analysis as originally introduced by Gibbs remain dominant instead, which is too bad.

Maxwell's equations in vector notation are often cited as a prime example of the beauty of physics, but the elegance is only enhanced in geometric algebra. It's natural to combine the electric and magnetic fields into one field quantity: $\mathrm{F} = \boldsymbol{E} + \check{I}c\boldsymbol{B}$. The full equations then take the simple form $\nabla\mathrm{F} = \mathbf{J}$ where $\nabla$ is the four-gradient $1/c\,\partial_t + \partial_r$ and $\mathbf{J}$ the four-current $1/\varepsilon_0\,\rho - c\mu_0\mathbf{J}$. By combining the dot and cross products, Maxwell's four equations collapse into one (of course, this can also be achieved in tensor notation).

The improvement is even more startling for the Dirac equation, which actually takes the form of a simple generalization of Maxwell's equations in which the field $F$ becomes a full multi-vector. This and other examples are explored in more detail in a short review of geometric algebra (J. M. Chapell *et al.*, http://arxiv.org/abs/1101.3619; 2011), and Hestenes has created a wide variety of introductory materials (http://geocalc.clas.asu.edu).

One day, perhaps, Clifford's geometric algebra will be taught routinely to students in place of vector analysis. It would probably eliminate a great deal of confusion, and improve the geometric intuition of many practising scientists. ❐

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Corrected online: 28 October 2011

## Correction

An incorrect version of the Thesis 'Geometric intuition' went to press. In the final equation quoted in the article, the term $1/c$ should instead have been $1/\varepsilon_0$. The text has been rectified for the HTML and PDF versions.