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Part I

Markov games

Chapter 1

State-Space Visualisation and Fractal Properties of Parrondo's Games

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1.1 Introduction

In Parrondo's games, the apparently paradoxical situation occurs where individually losing games combine to win [1, 2]. The basic formulation and definitions of Parrondo's games are described in Harmer et alii [3, 4, 5, 6]. These games have recently gained considerable attention as they are physically motivated and have been related to physical systems such as the Brownian ratchet [4], lattice gas automata [7] and spin systems [8]. Various authors have pointed out interest in these games for areas as diverse as biogenesis [9], political models [8], small-world networks [10], economics [8] and population genetics [11].

In this chapter, we will first introduce the relevant properties of Markov transition operators and then introduce some terminology and visualisation techniques from the theory of dynamical systems. We will then use these tools, later in the chapter, to define and investigate some interesting properties of Parrondo's games.

We must first discuss and introduce the mathematical machinery, terms and notation that we will use. The key concepts are :

state : This contains all of the information that we need to specify what is happening in the system at any given time.

time-varying probability vector : This is a time-varying probability distribution which specifies the probabilities that the system will be in certain states and any given time.

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transition matrix : This is a Markov operator which which determines the way in which the time varying probability vector will evolve over time.

These concepts are defined and discussed at length in many of the standard text books on stochastic processes [12, 13, 14, 15].

Time-homogeneous sequences of regular Markov transition operators have unique stable limiting state-probabilities. The state-space representations of the associated time-varying probability vectors converge to unique points. If the sequence of Markov transition operators is not homogeneous in time then the sequence time-varying probability vectors generated by the products of these different operators need not converge to a single point, in the original state space. It is possible to construct quite simple examples to show that this is the case.

If the sequences are periodic then it is possible to incorporate the finite memory of these systems into a new definition of "state." The new inhomogeneous systems can be re-defined as strictly homogeneous Markov processes. These new Markov processes, with new states, will generally have unique limiting probability vectors.

If we allow the sequence to become indefinitely long then the amount of memory required grows without bound. It is still possible, in principle, to define these indefinitely long periodic sequences as homogeneous Markov process although the definition, and encoding, of the states would require some care. We can consider any one indefinite sequence of operators as being one of many possible indefinite sequences of operators. If we do this then most of the possible sequences will appear to be "random." We can learn something about the general case by studying indefinitely long random sequences.

If the sequence of operators is chosen at random then the time varying probability vector, as defined in the original state-space, does not generally converge to a single unique value. Simulations show that the time-varying probability vector assumes a distribution in the original state-space which is self-similar, or "fractal," in appearance. The existence of fractal geometry is established, with rigor, for some particular Markov games. We establish a transcendental equation which allows the calculation of the Hausdorff dimensions of these fractal objects.

If state-transitions of the time-inhomogeneous Markov chains are associated with rewards then it is possible to show that even simple, "two-state," Markov chains can generate a Parrondo effect, as long as we are free to choose the reward matrix. Homogeneous sequences of the individual games generate a net loss over time. Inhomogeneous mixtures of two games can generate a net gain.

We show that the expected rates of return, or moments of the reward process, for the time-inhomogeneous games are identical to the expected rates of return from a homogeneous sequence of a time-averaged game. This is a logical consequence of the Law of Total Probability and the definition of expected value.

Two different views of the time-inhomogeneous process emerge, depending of the viewpoint that one takes:

- If you have access to the history of the time-varying probability vector and you have a memory to store this information and you choose to represent this data in state-space then you will see distributions with fractal geometry. This is more or less the view that a large casino might have if they were to visualise the average states of their many customers.

- If you do not have access to the time-varying probability vector or you have no memory in which to store this information then all that you can see is a sequence of rewards from a stochastic process. The internal details of this process are hidden from you. You have no way of knowing precisely how this process was constructed from an inhomogeneous sequence of Markov operators. There is no experiment that you can perform to distinguish between the time-inhomogeneous process and the time-averaged process. The time-averaged process is a homogeneous sequence of a single operator. We can calculate a single unique limiting value for the probability vector. This is more or less the view that a single, mathematically inclined, casino patron might have if they were playing against some elaborate poker machine. The internal workings of the machine would be hidden from the customer but it would be possible to perform some analysis of the outcomes and form an estimate of the parameters for the time-averaged model.

We show that the time inhomogeneous process is consistent in the sense that the "casino" and the "customer" will always agree on the expected winnings or losses of the customer. In more technical terms, the time-average, which the customer sees, is the same as the ensemble-average over state-space, which the casino can calculate.

1.2 Time-Homogeneous Markov Chains and Notation

Finite discrete-time Markov chains can be represented in terms of matrices of conditional transition probabilities. These matrices are called Markov transition operators. We denote these by capital letters in brackets, eg : $[A]$ where $A_{i,j} = \Pr\{K_{t+1} = j | K_t = i\}$ and $K \in \mathcal{Z}$ is some measure of displacement or the "state" of the system. The Markov property requires that $A_{i,j}$ cannot be a function of K but it can be a function of time, t. In Parrondo's original games, K, represents the amount of capital that a player has. There is a one-to-one mapping between Markov games and the Markov transition operators for these games. We will refer to the games and the transition operators interchangeably.

The probability that the system will be in any one state at a given instant of time can be represented by a distribution called the time-varying probability vector. We represent this probability mass function, at time t, using a row vector, $\mathbf{V_t}$. We can represent the evolution of the Markov chain in time using a simple Matrix equation,

$$\mathbf{V_{t+1}} = \mathbf{V_t} \cdot [A] \ . \tag{1.1}$$

This can be viewed as a multi-dimensional finite difference equation. The initial value problem can be solved using generating function, or Z transform, methods. Sequences of identical Markov transition operators, where $[A]$ does not vary, are said to be time-homogeneous. A Markov transition operator is said to be regular if some positive power of that operator has all positive elements. Time-homogeneous sequences of regular Markov transition operators always have stable limiting probability vectors, $\lim_{t\to\infty}(\mathbf{V_t}) = \mathbf{\Pi}$. The time varying probability vector reliably converges to a single point [12, 13, 14, 15].

We can think of the space which contains the time-varying probability vectors, and the stable limiting probability vector, as a vector space which has a strong analogy to the state-space which is used in the theory of control. We shall refer to this space as "state-space," $[0,1]^N$, and we will refer to the time-varying probability vector as a state vector. This terminology is used in the engineering literature [15]. We emphasise that the "*state-vector',*" $\mathbf{V_t} \in \Re^N$ is distinct from the "*state*" of the system, $K \in \mathcal{Z}$, used in Markov chain terminology. As a simple example, we can consider the regular Markov transition operator

$$[A] = \begin{bmatrix} \frac{13}{16} & \frac{3}{16} \\ \frac{1}{16} & \frac{15}{16} \end{bmatrix} \tag{1.2}$$

using the initial condition

$$\mathbf{V_t} = [V_0, V_1] \ = \left[\frac{3}{4}, \frac{1}{4}\right] \text{ when } t = 0. \tag{1.3}$$

The components of $\mathbf{V_t}$ are V_0 and V_1 and these can be considered to be the dimensions of the Cartesian space which we call "state-space". This space has a clear analogy with the phase-space of Poincare and the state-space used in the theory of control. It also has some analogy with the "γ" or gaseous phase-space of Gibbs and the phase-space used in Lagrangian dynamics although we must be careful not to press these analogies too far since the state-spaces of physics and of Markov chains use different transition operators which obey different conservation laws.

A fundamental question in the study of dynamical systems is to classify how they behave as $t \to \infty$ and all transient effects have decayed. The evolution of the state vector of a discrete-time Markov chain generally traces out a sequence of points or "trajectory" in the state-space. The natural technique would be to draw a graph of this trajectory. As an example of this, we can consider the trajectory of the time homogeneous Markov chain, described by Equations 1.2 and 1.3, which is shown in Figure 1.1.

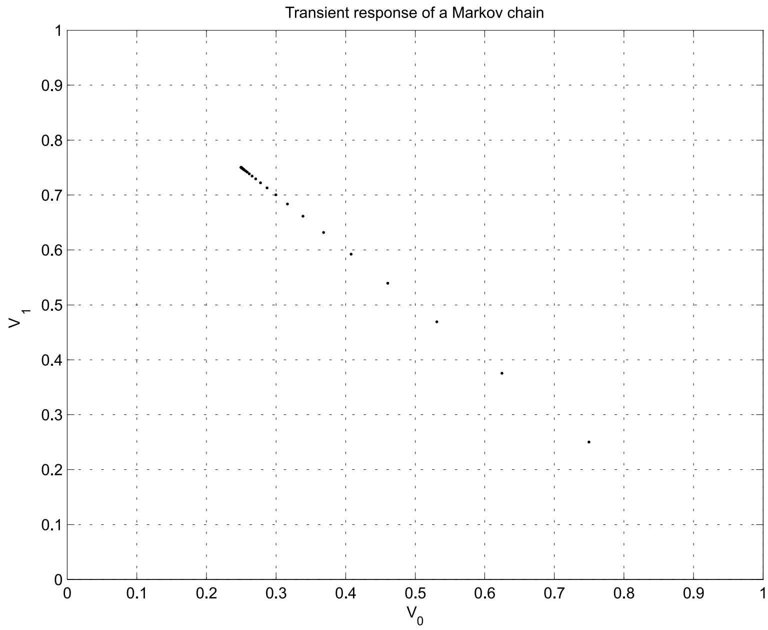

FIGURE 1.1. State-space trajectory of a Markov chain

The state vector, $\mathbf{V_t}$, always satisfies the constraint, $V_0 + V_1 = 1$. This follows from the law of total probability. The state-vector is always constrained to lie within an $N-1$ dimensional subspace of the N dimensional state-space. The dynamics of the system all occur within this sub-space. This is clearly visible in Figure 1.1. We can think of the set

$$M = \{[V_0, V_1] \mid (0 \leq V_0 \leq 1) \wedge (0 \leq V_1 \leq 1) \wedge (V_0 + V_1 = 1)\} \ , \qquad (1.4)$$

as a state manifold for the dynamical system defined by Equations 1.2 and 1.3. The state manifold has a dimension which is smaller than the embedding state-space. This is a result of the fact that there is a conservation law (the law of total probability) which constrains the dynamics of the system. For this example, the sequence converges to a stable fixed point at $\mathbf{\Pi} = \left[\frac{1}{4}, \frac{3}{4}\right]$. It can be shown that sequences of this type always converge to single stable fixed points as long as the Markov transition operators are regular and time-homogeneous [12, 13, 14, 15]. The convergent points are the appropriate state-space representation of the stable limiting probabilities for the Markov chain.

1.3 Time-Inhomogeneous Markov Chains

The existence, uniqueness and dynamical stability of the fixed point are important parts of the theory of Markov chains but we must be careful not to apply these theorems to systems where the basic premises are not satisfied. If the Markov transition operators are not homogeneous in time then there may no longer a single fixed point in state-space. The state vector can perpetually move through two or more points without ever converging to any single stable value. To demonstrate this important point, we present a simple example, using two regular Markov transition operators :

$$[S] = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{3}{4} & \frac{1}{4} \end{bmatrix} \tag{1.5}$$

and

$$[T] = \begin{bmatrix} \frac{1}{4} & \frac{3}{4} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix} . \tag{1.6}$$

The rows of these matrices are all identical. This indicates that the outcome of each game is completely independent of the initial state. The limiting stable probabilities for these regular Markov transition operators are $\mathbf{\Pi_S} = \left[\frac{3}{4}, \frac{1}{4}\right]$ and $\mathbf{\Pi_T} = \left[\frac{1}{4}, \frac{3}{4}\right]$ respectively. The time-varying probability vector immediately moves to the stable limiting value after even a single play of each game.

$$[Q] \cdot [S] = [S] \tag{1.7}$$

and

$$[Q] \cdot [T] = [T] \tag{1.8}$$

for any conformable stochastic matrix $[Q]$. This leads to some interesting corollaries:

$$[T] \cdot [S] = [S] \tag{1.9}$$

and

$$[S] \cdot [T] = [T] \tag{1.10}$$

If we play an indefinite alternating sequence of these games, $\{STST\cdots\}$, then there are two simple ways in which we can associatively group the terms:

$$\mathbf{V_{2N}} = \mathbf{V_0}\left([S]\,[T]\right)\left([S]\,[T]\right)\cdots\left([S]\,[T]\right) \tag{1.11}$$

$$= \mathbf{V_0}\,[T] \tag{1.12}$$

$$\Rightarrow \mathbf{\Pi} = \mathbf{\Pi_T} \tag{1.13}$$

and

$$\mathbf{V_{2N+1}} = (\mathbf{V_0}[S])([T][S])([T][S])\cdots([T][S]) \quad (1.14)$$
$$= \mathbf{V_0}[S] \quad (1.15)$$
$$\Rightarrow \mathbf{\Pi} = \mathbf{\Pi_S} \ . \quad (1.16)$$

If we *assume* that there is a unique probability limit then we must conclude that $\mathbf{\Pi_S} = \mathbf{\Pi_T}$ and hence $\frac{1}{4} = \frac{3}{4}$ which is a contradiction. We can invoke the principle of excluded middle (reductio ad absurdum) to conclude that the assumption of a single limiting stable value for $\lim_{t\to\infty}(\mathbf{V_t})$ is false. In the asymptotic limit as $t \to \infty$, the state vector alternately assumes one of the *two* values $\mathbf{\Pi_S}$ or $\mathbf{\Pi_T}$. We refer to the set of all recurring state vectors of this type, $\{\mathbf{\Pi_S}, \mathbf{\Pi_T}\}$, as the *attractor* of the system. In more general terms an attractor is a set of points in the state-space which is invariant under the system dynamics in the asymptotic limit as $t \to \infty$.

1.3.1 Reduction of the periodic case to a Time-Homogeneous Markov Chain

In the last section, we considered a short sequence of length 2. This can be generalised to an arbitrary length, $N \in \mathcal{Z}$. It is possible to associatively group the operators into sub-sequences of length N. As with the sequences of length two, the choice of time origin is not unique. We are free to make an arbitrary choice of time origin with the initial condition at $t = 0$. We can think of the operators as having an offset of $n \in \mathcal{Z}$, where $0 \leq n \leq N-1$ within the sub-sequence. We can also calculate a new equivalent operator to represent the entire sequence, $A_{\text{eq}} = \prod_{n=0}^{N-1} A_n$. We can then calculate the steady-state probabilities associated with this operator, $\mathbf{\Pi}_{\text{eq}} = \mathbf{\Pi}_{\text{eq}} \cdot A_{\text{eq}}$. We can refer the asymptotic trajectory of the time varying probability vector to this fixed point, $\mathbf{V_{(t \pmod N)}} = \mathbf{\Pi}_{\text{eq}} \cdot \Pi_{n=0}^{(t \pmod N))-1} A_n$. In the periodic case, there is generally not a single fixed point in the original state-space but the time varying probability vector settles into a stable limit cycle of length N. If we aggregate time, modulo N then we can redefine what we mean by "state" and we can define a new state-space in which the time-varying vector does converge to a single point.

If we allow the length of the period, N, to become indefinitely long $N \to \infty$ then our new definition of "state" becomes infinitely complicated. We would have to contemplate indefinitely large offsets, $n \to \infty$, within the infinitely long cycle. If we wish to avoid the many paradoxes that infinity can conceal then we really should consider the case with "infinite" period as being qualitatively different from the case with finite period, N.

1.3.2 Random Selection of Markov transition operators

1.4 Two simple Markov Games that Generate a Simple Fractal in State-Space

We proceed to construct a simple system in which operators are selected at random and we will use the standard theories regarding probability and expected values to derive some useful results. If we modify the system specified by Equations 1.5 and 1.6 :

$$[S] = \begin{bmatrix} \frac{5}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \tag{1.17}$$

and

$$[T] = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{6} & \frac{5}{6} \end{bmatrix} \tag{1.18}$$

and select the sequence of transition operators at random then the attractor becomes an infinite set. If we were to play a homogeneous sequence of either of these games then they would have the same stable limiting probabilities as before, $\mathbf{\Pi_S}$ and $\mathbf{\Pi_T}$, and the dynamics would be similar to those shown in Figure 1.1. In contrast, if we play an indefinite *random* sequence of the new games S and T, $\{STSSTSTTSTT\cdots\}$, then there are no longer any stable limiting probabilities and the attractor has a fractal or "self-similar" appearance which is shown in Figure 1.2.

1.4.1 The Cantor Middle-Third Fractal

These games have been constructed in such a way that they generate the Cantor middle-third fractal.

It should be noted that the Cantor Middle-Third fractal is an uncountable set and so a, countably infinite, random sequence of operators will ever generate enough points to cover the entire set. The solution to this problem is to consider the uncountably infinite set generated by all possible infinite, random sequences of operators. We can construct a probability measure on the resulting set and then we can calculate probabilities and expected values. It is also reasonable to talk about the probability density function of the time-varying probability vector in the state-space.

In order to stimulate intuition, we can simulate the process and generate a histogram, showing the distribution of the time varying probability vector. The result is shown in Figure 1.3. For the x axis in this figure, we *could* have chosen the first element of the time varying probability vector, V_0 but this would not have been the easiest way to analyse the dynamics. It is better if we choose another parameterization. If we examine the eigenvectors of the matrices in Equations 1.17 and 1.18 then we find that

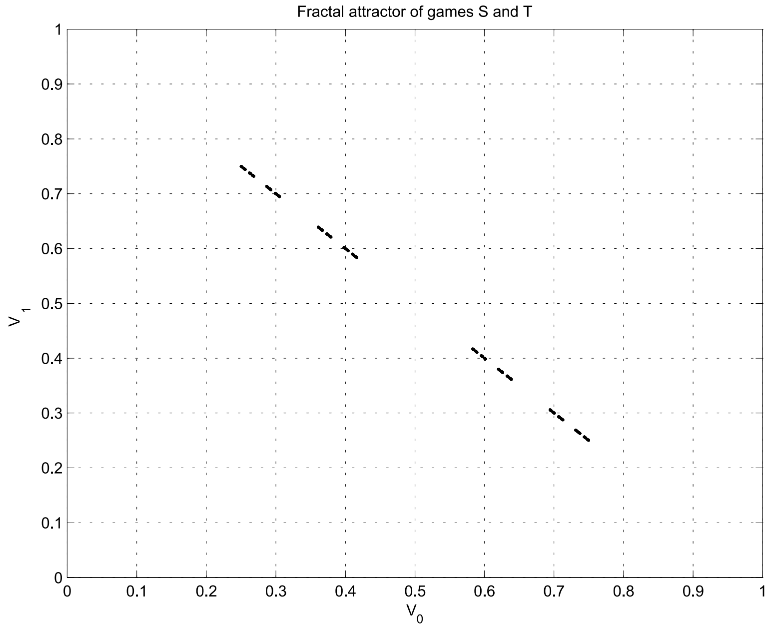

FIGURE 1.2. A fractal attractor generated by games S and T

a better re-parameterization is:

$$x = V_0 - V_1 \tag{1.19}$$

and

$$y = V_0 + V_1 \ . \tag{1.20}$$

Of course, we always have $y = 1$ and x is a new variable in the range $-1 \leq x \leq +1$. The Cantor fractal lies in the unit interval $-\frac{1}{2} \leq x \leq \frac{1}{2}$ which is the x interval shown in Figure 1.3. The transformation for matrix $[S]$, in Equation 1.17 reduces to:

$$\left(+\frac{1}{2} - x_{t+1}\right) = \frac{1}{3} \cdot \left(+\frac{1}{2} - x_t\right) \tag{1.21}$$

and the transformation for matrix $[T]$, in Equation 1.18 reduces to:

$$\left(-\frac{1}{2} - x_{t+1}\right) = \frac{1}{3} \cdot \left(-\frac{1}{2} - x_t\right) \ . \tag{1.22}$$

The transformation S has a fixed point at $x = +\frac{1}{2}$ and the transformation T has a fixed point at $x = -\frac{1}{2}$. If we choose these transformations as random then the recurrent values of x lie in the interval between the fixed points, $-\frac{1}{2} \leq x \leq \frac{1}{2}$. This is precisely the iterated function system for the Cantor Middle-Third Fractal. These are described in Barnsley [16].

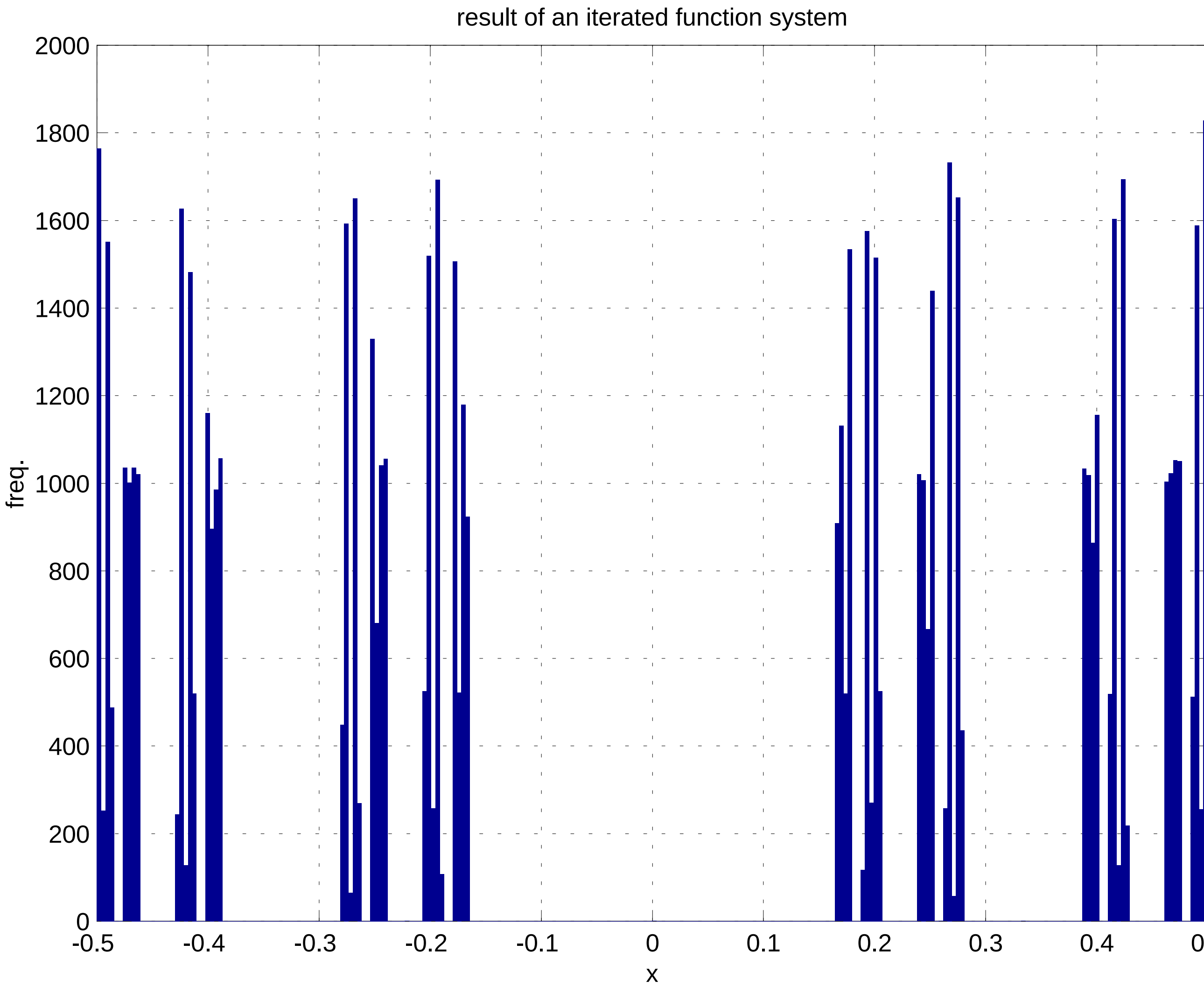

FIGURE 1.3. A histogram of the distribution of $\mathbf{V_t}$ in state-space

The most elementary analysis that we can perform is to calculate the dimension of this set. If we assume conservation of measure then every time we perform a transformation, we reduce the diameter by a factor of $\frac{1}{3}$ but the transformed object is geometrically half of the original object so we can write

$$\frac{1}{2} = (\frac{1}{3})^D \tag{1.23}$$

where D is the fractional dimension. This is the law of conservation of measure for this particular system. We can solve this equation for D to get

$D = \frac{\log(2)}{\log(3)} \approx 0.630929\cdots$.

We can invert the rules described in Equations 1.21 and 1.22 giving:

$$x_t = 3x_{t+1} - 1 \tag{1.24}$$

and

$$x_t = 3x_{t+1} + 1\ . \tag{1.25}$$

If we consider these equations, together with the law of conservation of total probability then we get a self-similarity rule for the PDF (or Probability Density Function), $p(x)$, of the time varying probability vector, $\mathbf{V_t}$:

$$\frac{3}{2}p\left(3x-1\right)+\frac{3}{2}p\left(3x+1\right)=p\left(x\right)\ . \tag{1.26}$$

This PDF, $p(x)$ is the density function towards which the histogram in Figure 1.3 would converge if we could collect enough samples. The self-similarity rule for the PDF gives rise to a recursion rule for the moment generating function, $\Phi\left(\Omega\right)=E\left(e^{j\Omega x}\right)$:

$$\Phi\left(\Omega\right)=\Phi\left(\frac{\Omega}{3}\right)\cdot\cos\left(\frac{\Omega}{3}\right)\ . \tag{1.27}$$

We can evaluate the derivatives at $\Omega = 0$ and calculate as many of the moments as we wish. We can calculate the mean, μ, and the variance σ^2 :

$$\mu = 0 \tag{1.28}$$

$$\sigma^2 = \frac{1}{8} \tag{1.29}$$

These algebraic results are consistent with results from numerical simulations.

1.4.2 Iterated Function Systems

The cause of the fractal geometry is best understood if we realise that Markov transition operators perform affine transformations on the state-space. An indefinite sequence of different Markov transition operators is equivalent to an indefinite sequence of different affine transformations which is called an "Iterated Function System". We refer the reader to the work of Michael Barnsley [16] and the theory of Iterated Function Systems to show that fractal geometry is quite a general property of a system of randomly selected affine transformations.

1.5 An Equivalent Representation of the Random Selection of Markov Transition Operators

Consider two mutually exclusive events, $A \cap B = \emptyset$, embedded within some probability space $(\Omega, \mathcal{F}, P)$. Consider any third event $C \subseteq A \cup B$. These

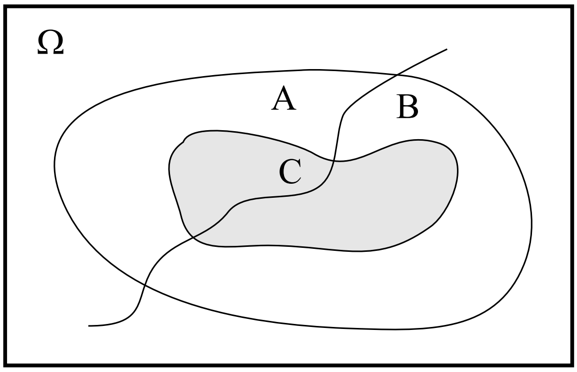

FIGURE 1.4. Set Relationships and Change of Probability

events are represented in Figure 1.4. The law of total probability asserts that

$$\Pr(C) = \Pr(C|A) \cdot \Pr(A) + \Pr(C|B) \cdot \Pr(B) \ . \tag{1.30}$$

We can now make the following particular identifications:

$$C \equiv \{X \in \Omega \mid K_{t+1} = i \ \wedge \ K_t = j\} \tag{1.31}$$
$$A \equiv \{\text{played game } A\} \tag{1.32}$$
$$B \equiv \{\text{played game } B\} \ . \tag{1.33}$$

If we select games A and B at random with probabilities of γ and $(1-\gamma)$ respectively then we can write $\Pr(A) = \gamma$ and $\Pr(B) = (1-\gamma)$. By definition, the Markov matrices for games A and B contain conditional probabilities for state transitions :

$$A_{i,j} = \Pr\{(K_{t+1} = j \mid K_t = i) \ \wedge \ \text{played game } A\} \tag{1.34}$$
$$B_{i,j} = \Pr\{(K_{t+1} = j \mid K_t = i) \ \wedge \ \text{played game } B\} \ . \tag{1.35}$$

Note that in this case $C = A \cup B$. We can define a new operator corresponding to the events $C_{i,j}$:

$$C_{i,j} = \Pr\{K_{t+1} = j \mid K_t = i\} \tag{1.36}$$

and Equation 1.30 reduces to

$$C_{i,j} = A_{i,j} \cdot \gamma + B_{i,j} \cdot (1-\gamma) \ . \tag{1.37}$$

The conditional probabilities of state transitions of the inhomogeneous Markov process generated by games A and B are the same as the conditional probabilities of a new equivalent game called "Game C." The transition matrix for Game C is a linear convex combination of the matrices for the original basis games, A and B. Even if we have complete access to the state of the system then there is no function that we can perform on the state, or state transitions, which could allow us to distinguish between a homogeneous sequence of Games C and an inhomogeneous *random* sequence of Games A and B. We refer to game C as the time-average model. This is analogous to the state-space averaged model found in the theory of control [17].

1.6 The Phenomenon of Parrondo's Games

1.6.1 Markov Chains with Rewards

Suppose that we apply a reward matrix to the process:

$$R_{i,j} = \text{reward if } (K_{t+1} = j) \mid (K_t = i) \ . \tag{1.38}$$

There is a specific reward associated with each specific state transition. We can think of $R_{i,j}$ as the reward that we earn when a transition occurs from state i to state j. The state transitions, rewards and probabilities of transition, for "Game A" are shown in Figure 1.5. The state transition dia-

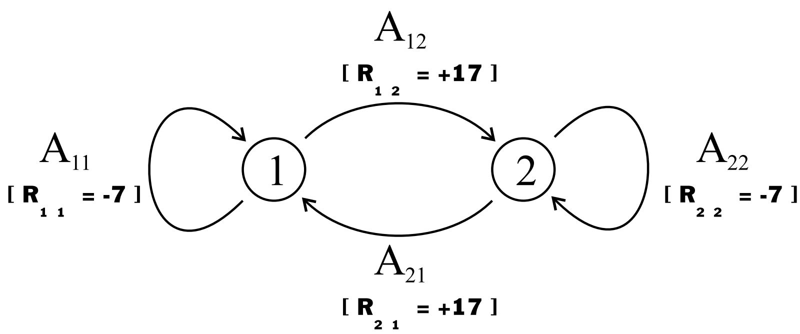

FIGURE 1.5. State Transition Diagram for "Game A" with rewards.

grams for "Game B" and the time averaged "Game C" would have identical topology and have identical reward structure, although the probabilities of transition between states would be different. Systems of this type have been analysed by Howard [18] although we use different, matrix, notation to perform the necessary multiplications and summations.

The expected reward from each transition of the time-averaged homogeneous process is :

$$Y_{i,j} = E\left[R_{i,j} \cdot C_{i,j}\right] \ . \tag{1.39}$$

If we wish to calculate the mean expected reward then we must sum over all recurrent states in proportion to their probability of occurrence. This will be a function of the transition matrix, C, and the relevant steady state probability vector, $\mathbf{\Pi_C}$:

$$Y\left(C\right) = \mathbf{\Pi_C} \cdot \left(\left[R\right] \circ \left[C\right]\right) \cdot \mathbf{U}^T \tag{1.40}$$

where "$\circ$" represents the Hadamard, or element by element, product and $\mathbf{U}^T$ is a unit column vector of dimension N. Post-multiplication by $\mathbf{U}^T$ has the effect of performing the necessary summation. We recall that $\mathbf{\Pi_C}$ represents the steady state probability vector for matrix C. The function $Y\left(C\right)$ represents the expected asymptotic return, in units of "reward," per unit time when the the games are played.

If we include the definition of C in Equation 1.37 in Equation 1.39 then we can write :

$$\begin{aligned} Y_{i,j} &= E\left[R_{i,j} \cdot \left(\gamma A_{i,j} + \left(1-\gamma\right) B_{i,j}\right)\right] && (1.41) \\ &= \gamma E\left[R_{i,j} \cdot A_{i,j}\right] + \left(1-\gamma\right) E\left[R_{i,j} \cdot B_{i,j}\right] \ . && (1.42) \end{aligned}$$

We can also define :

$$Y(A) = \mathbf{\Pi_A} \left(\left[R\right] \circ \left[A\right]\right) \mathbf{U}^T \tag{1.43}$$

and

$$Y(B) = \mathbf{\Pi_B} \left(\left[R\right] \circ \left[B\right]\right) \mathbf{U}^T \tag{1.44}$$

and we might *falsely* conclude that

$$Y(C) = \gamma Y(A) + \left(1-\gamma\right) Y(B) \ . \tag{1.45}$$

This would be equivalent to saying that :

$$Y(C) = \gamma \left(\mathbf{\Pi_A} \left(\left[R\right] \circ \left[A\right]\right) \mathbf{U}^T\right) + \left(1-\gamma\right) \left(\mathbf{\Pi_B} \left(\left[R\right] \circ \left[B\right]\right) \mathbf{U}^T\right) \ . \tag{1.46}$$

but these equations 1.45 and 1.46 are in **error** because Equation 1.42 must be summed over all of the recurrent states of the *mixed* inhomogeneous games but in the **false** Equation 1.46, the first term is summed with respect to the recurrent states of Game "A" and the second term is summed with respect to the recurrent states of game "B." This is an error. The dependency on state makes the reward process non-linear. The correct expression for $Y(C)$ would be :

$$Y(C) = \gamma \left(\mathbf{\Pi_C} \left(\left[R\right] \circ \left[A\right]\right) \mathbf{U}^T\right) + \left(1-\gamma\right) \left(\mathbf{\Pi_C} \left(\left[R\right] \circ \left[B\right]\right) \mathbf{U}^T\right) \ . \tag{1.47}$$

The difference between the intuitively appealing but **false** Equations 1.45 and 1.46 and the correct Equation 1.47 is the cause of "Parrondo's paradox."

1.6.2 Parrondo's Paradox Defined

The essence of the problem is that when we say that "Game A is losing" or "Game B is losing" we perform summation with respect to the steady state probability vectors for Games "A" and "B" respectively. When we say that "a random sequence of games A and B is winning," we perform the summation with respect to the steady state probability vector for the time-averaged game, Game "C."

We can say that the "paradox" exists whenever we can find two games A and B and a reward matrix R such that :

$$Y\left(\gamma A + (1-\gamma)\, B\right) \neq \gamma Y(A) + (1-\gamma)\, Y(B)\ . \tag{1.48}$$

The "paradox" is equivalent to saying that the reward process is not a linear function of the Markov transition operators.

1.6.3 A simple "Two-State" Example of Parrondo's Games

We can show that Parrondo's paradox does exist by constructing a simple example. We can define

$$[A] = \begin{bmatrix} \frac{5}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \tag{1.49}$$

and

$$[B] = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{6} & \frac{5}{6} \end{bmatrix} \tag{1.50}$$

The steady state probability vectors are: $\mathbf{\Pi_A} = \left[\frac{3}{4}, \frac{1}{4}\right]$ and $\mathbf{\Pi_B} = \left[\frac{1}{4}, \frac{3}{4}\right]$. These games are the same as games "S" and "T" defined earlier but we analyse them using the theory of Markov chains with rewards. We can define a reward matrix

$$[R] = \begin{bmatrix} -7 & +17 \\ +17 & -7 \end{bmatrix} \tag{1.51}$$

and we can apply Equations, 1.43, 1.44 and 1.47 to get :

$$Y(A) = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \end{bmatrix} \left(\begin{bmatrix} -7 & +17 \\ +17 & -7 \end{bmatrix} \circ \begin{bmatrix} \frac{5}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \right) \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \ -1 \tag{1.52}$$

and

$$Y(B) = \begin{bmatrix} \frac{1}{4} & \frac{3}{4} \end{bmatrix} \left(\begin{bmatrix} -7 & +17 \\ +17 & -7 \end{bmatrix} \circ \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{6} & \frac{5}{6} \end{bmatrix} \right) \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \ -1 \tag{1.53}$$

and, for the time-average we get :

$$Y\left(C\right) = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix} \left(\begin{bmatrix} -7 & +17 \\ +17 & -7 \end{bmatrix} \circ \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} \right) \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \ +1\ . \tag{1.54}$$

Games "A" and "B" are losing and the mixed time-average game, game, $C = \frac{1}{2}(A + B)$, is winning. Equation 1.48 is satisfied and so we have Parrondo's "paradox" for the two-state games "A" and "B" as defined in Equations 1.49 and 1.50. We can simulate the dynamics of this two-state version of Parrondo's games. Some typical sample paths are shown in Figure 1.6. The results from the simulations are consistent with the algebraic results.

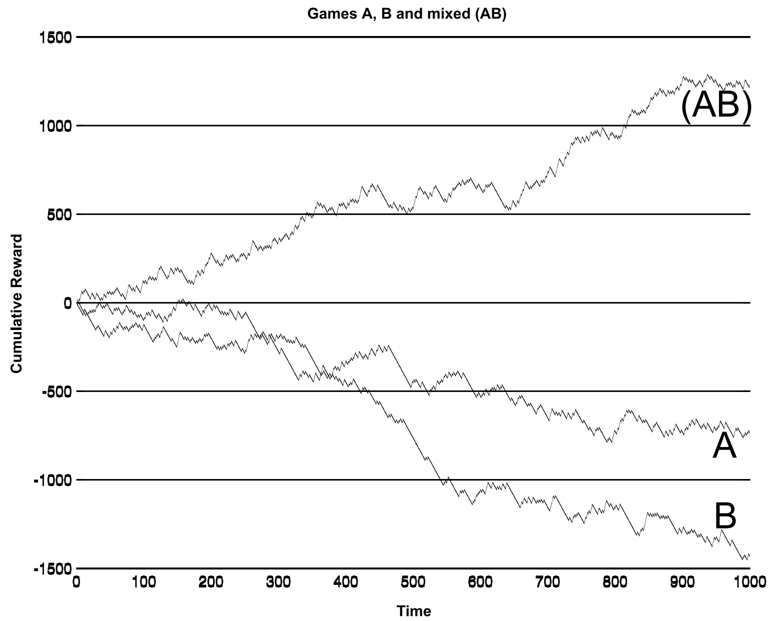

FIGURE 1.6. Simulation of a Two-State version of Parrondo's games

If we refer back to Figure 1.5 then an intuitive explanation for this phenomenon is possible. The negative or "punishing" rewards are associated with transitions that do not change state. The good positive rewards are associated with the changes of state. If we play a homogeneous sequence of Games "A' or "B" then there are relatively few changes of state and the resulting weighted sum of all the rewards is negative. If we play the mixed game then the rewarding changes of state are much more frequent and the resulting weighted sum of rewards is positive.

1.7 Consistency between State-Space and Time averages

In order for the "fractal view" of the process, in state-space, to be consistent with the time average view of the process we require :

$$E\left[\mathbf{V_t}\right] = \mathbf{\Pi_C} \tag{1.55}$$

The value of $E[\mathbf{V_t}]$ follows from the argument in Section 1.4.1. We can use the mean as defined in Equation 1.29 to state that

$$\begin{aligned} E\left[\mathbf{V_t}\right] &= \left[\frac{1}{2} + \frac{1}{2}E\left[x\right], \frac{1}{2} - \frac{1}{2}E\left[x\right]\right] && (1.56) \\ &= \left[\frac{1}{2} + \frac{1}{2}\mu, \frac{1}{2} - \frac{1}{2}\mu\right] && (1.57) \\ &= \left[\frac{1}{2}, \frac{1}{2}\right] && (1.58) \end{aligned}$$

The value of $\mathbf{\Pi_C}$ follows from the arguments in Section 1.6.3. Specifically we require $\mathbf{\Pi_C} = \mathbf{\Pi_C} \cdot C$ which gives:

$$\mathbf{\Pi_C} = \left[\frac{1}{2}, \frac{1}{2}\right] \tag{1.59}$$

which is consistent with Equation 1.58. Which proves this special case. To prove the more general case we need to have some notation for an entire fractal set, like the one shown in Figure 1.2. We use $\{F\}$ to denote the attractor generated by two operators A and B. We can write :

$$E\left[\{F\}\right] = \gamma E\left[\{F\}\right] A + (1-\gamma) E\left[\{F\}\right] B \ . \tag{1.60}$$

This follows from conservation of measure under the affine transformations A and B. We note that *everything* in these equations is linear and so we can write

$$\begin{aligned} E\left[\{F\}\right] &= E\left[\{F\}\right](\gamma A + (1-\gamma) B) && (1.61) \\ &= E\left[\{F\}\right] \cdot C && (1.62) \end{aligned}$$

which is the defining property of $\mathbf{\Pi_C}$ which implies that

$$E\left[\{F\}\right] = \mathbf{\Pi_C} \ . \tag{1.63}$$

The two ways of viewing the situation are consistent which means that we can use the time averaged game to calculate expected values of returns from Parrondo's games.

1.8 Parrondo's original games

1.8.1 Original Definition of Parrondo's Games

In their original form, Parrondo's games spanned infinite domains, of all integers or all non-negative integers [3]. If our interest is to examine the asymptotic behaviour of the games as $t \to \infty$ and to study asymptotic rates of return or moments then it is possible to reduce these games by aggregating states of the Markov chain modulo three. We can do this without losing any information about the rate of return from the games. After reduction, the Markov transition operators take the form :

$$[A] = \begin{bmatrix} 0 & a_0 & (1-a_0) \\ (1-a_1) & 0 & a_1 \\ a_2 & (1-a_2) & 0 \end{bmatrix} . \tag{1.64}$$

where a_0, a_1 and a_2 are the conditional probabilities of winning, given the current state modulo three. This form of the games has been published by Pearce [6].

1.8.2 Optimised form of Parrondo's Games

Simulations reveal that *periodic* inhomogeneous sequences of Parrondo's games have the strongest Parrondo effect. Further investigation by the authors, using Genetic Algorithms, suggest that the most powerful form of the games is a set of three games that are played in a strict periodic sequence $\{G_0, G_1, G_2, G_0, G_1, G_2, \cdots\}$. The transition probabilities are as follows :
Game G_0 **:** $[a_0, a_1, a_2] = [\mu, (1-\mu), (1-\mu)]$
Game G_1 **:** $[a_0, a_1, a_2] = [(1-\mu), \mu, (1-\mu)]$
Game G_2 **:** $[a_0, a_1, a_2] = [(1-\mu), (1-\mu), \mu]$
where μ is a small probability, $0 < \mu < 1$. We can think of μ as being a very small, ideally "microscopic", positive number. The rate of return form any **pure sequence** of these games is approximately

$$Y \approx \frac{1}{2} \cdot \mu \tag{1.65}$$

which is close to zero and yet the return from the **cyclic combination** of these games is approximately

$$Y \approx 1 - 3 \cdot \mu \tag{1.66}$$

which is close to a certain win. We can engineer a situation where we can deliver an almost certain win every time using games that, on their own, would deliver almost no benefit at all! These games clearly work better as a team than on their own. Just as team players may pass the ball in a game of soccer, the games $\{G_0, G_1, G_2\}$ carefully pass the state vector from one trial to the next as this sequence of Parrondo's games unfolds.

1.8.3 An Exquisite Fractal Object

It is possible to de-rate these games by increasing μ. In the limit as $\mu \to \frac{1}{2}$ the Parrondo effect vanishes and the attractor collapses to a single point in state-space. Just before this limit the attractor takes the form of the very small and exquisite fractal shown in Figure 1.7. This fractal is embedded

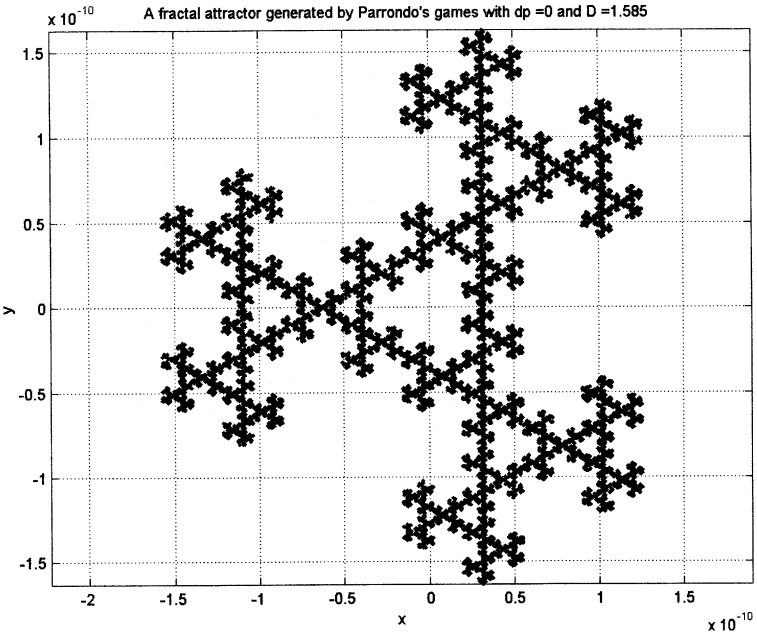

FIGURE 1.7. A 2D projection of a fractal attractor generated by the "last gasp" of Parrondo's games

in a two dimensional sub-space of the three dimensional state-space of the games $\{G_0, G_1, G_2\}$. The two dimensional sub-space has been projected onto the page in order to make it easier to view. The projection preserves dot product, length and angle measure. The coordinates "x" and "y" are linear combinations of the the components of the original state vector, $\mathbf{V_t} = [V_0, V_1, V_2]$. The orientation of the image is such that the original "V_2" axis is projected onto the new "y" axis. (The direction of "up" is preserved.) The negative numbers on the axes represent negative offsets rather than negative probabilities. This is the same concept that is used when we write down a probability $(1-p)$. If p is a valid probability then so is $(1-p)$. The number $-p$ is an offset that just happens to be negative.

The dimension of this fractal is $D \approx \frac{\log(9)}{\log(4)} \approx 1.585$. We define the amount of Parrondo effect, Δp, as the difference in rate of return, Y, between the mixed sequence of games $\{G_0, G_1, G_2\}$ and the best performance from any pure sequence of a single game. For this limiting case, $\Delta p \approx 0$. There are are some interesting qualitative relationships between the Hausforff dimension and the amount of Parrondo effect which deserve further investigation to see if it is possible to state a general quantitative law.

1.9 Summary

In this paper we have analysed Parrondo's games in terms of the theory of Markov chains with rewards. We have illustrated the concepts constructively, using a very simple two-state version of Parrondo's games and we have shown how this gives rise to fractal geometry in the state-space. We have arrived at a simple method for calculating the expected value of the asymptotic rate of reward from these games and we have shown that this can be calculated in terms of an equivalent time-averaged game. We have used graphic representations of trajectories and attractors in state-space to motivate some of the arguments.

The use of state-space concepts opens up new lines of enquiry. Simulation and visualisation encourage intuition and help us to grasp the essential features of a new system. This would be much more difficult if we were to use a purely formal algebraic approach at the start. We do not propose visualisation as a *replacement* for rigorous analysis. We see it as a guide to help us to decide which problems are worthy of more detailed attention and which problems might later yield to a more formal approach. We believe that state-space visualisation will be as useful for the study of the dynamics of Markov chains as it has already been for the study of other dynamical systems.

Finally, we conclude that Parrondo's games are not really "paradoxical" in the true sense. The anomaly arises because the reward process is a non-linear function of the Markov transition operators and our "common sense" tells us the reward process "ought" to be linear. When we combine the games by selecting them at random, we perform a linear convex combination of the operators but the expected asymptotic value of the rewards from this combined process is not a linear combination of the rewards from the original games.

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