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A Semi-quantum Version of the Game of Life[1]

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ABSTRACT A version of John Conway's game of Life is presented where the normal binary values of the cells are replaced by oscillators which can represent a superposition of states. The original game of Life is reproduced in the classical limit, but in general additional properties not seen in the original game are present that display some of the effects of a quantum mechanical Life. In particular, interference effects are seen.

35.1 Introduction

John Conway's game of Life [10] is a well known two dimensional cellular automaton where cells are arranged in a square grid and have binary values generally known as dead or alive. The status of the cells change in a discrete fashion each "generation" depending upon the number of neighbouring cells that are alive, the general idea being that a cell dies if there is either overcrowding or isolation. There are many different rules that can be applied for birth or survival of a cell and a number of these give rise to interesting properties such as still lives (stable patterns), oscillators (patterns that periodically repeat), spaceships or gliders (fixed shapes that move across the Life universe), glider guns, and so on [3, 12, 11]. Conway's original rules are one of the few that are balanced between survival and extinction of the Life "organisms." In this version a dead (or empty) cell becomes alive if it has exactly three living neighbours, while an alive cell survives if and only if it has two or three living neighbours. Much literature on the game of Life and its implications exists. For a recent discussion on the possibilities of this and other cellular automata the interested reader is referred to reference [24]. The simplest still lives and oscillators are given in figure 35.1,

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while figure 35.2 shows a glider, the simplest and most common moving form. A large enough random collection of alive and dead cells will, after a period of time, usually decay into a collection of still lives and oscillators like those shown here while firing a number of gliders off towards the outer fringes of the Life universe.

The recent interest in quantum games [1, 2, 4, 6, 7, 16, 17, 18, 20, 19, 9] suggests the possibility of applying the idea of superposition of states in quantum mechanics to the game of Life. Unfortunately Conway's Life is irreversible while, in the absence of a measurement, quantum mechanics is reversible. In particular, operators that represent measurable quantities must be unitary. A full quantum Life would be problematic given the known difficulties of quantum cellular automata [21]. Recently, in an attempt to generalise von Neumann's universal constructor [22] to quantum mechanics, it was found that a quantum universal constructor capable to self-reproduction cannot exist with finite resource in a deterministic universe [23]. This could have important bearing in understanding life from a quantum theoretic viewpoint.

Interesting behaviour can still be obtained in a semi-quantum mechanical Life by representing the cells by classical sine-wave oscillators with a period equal to one generation, an amplitude between zero and one, and a variable phase. The amplitude of the oscillation represents the coefficient of the alive state so that the square of the amplitude gives the probability of finding the cell in the alive state when a measurement of the "health" of the cell is taken. If the initial state of the system contains at least one cell that is in a superposition of eigenstates the neighbouring cells will be influenced according to the coefficients of the respective eigenstates, propagating the superposition to the surrounding region.

If the coefficients of the superpositions are restricted to positive real numbers we do not expect to see qualitatively new phenomena. By allowing the coefficients to be complex, that is, by allowing phase differences between the oscillators, qualitatively new phenomena, for example interference effects, may arise. The interference effects we see are those due to an array of classical oscillators with phase shifts and are not fully quantum mechanical. Our cellular automaton should be distinguished from quantum cellular automata discussed in references [5, 8, 13, 15, 14].

35.2 A First Model

To represent the state of a cell we introduce the following notation: [4]

$$|\psi\rangle = a|\text{alive}\rangle + b|\text{dead}\rangle \; , \tag{35.1}$$

subject to the normalization condition

$$|a|^2 + |b|^2 = 1 \; . \tag{35.2}$$

[4] $|\ldots\rangle$ is the standard quantum mechanical notation to be read as "the state of ... "

$|a|^2$ and $|b|^2$ represent the probabilities of measuring the cell as alive or dead, respectively. If the values of a and b are restricted to non-negative real numbers we cannot get destructive interference. The model still differs from a classical probabilistic mixture since it is the amplitudes that are added and not the probabilities. In our model $|a|$ is the amplitude of the oscillator. Restricting a to non-negative real numbers corresponds to the oscillators all being in phase.

The birth, death and survival operators have the following effects

$$\begin{aligned} B|\psi\rangle &= |\text{alive}\rangle \\ D|\psi\rangle &= |\text{dead}\rangle \\ S|\psi\rangle &= |\psi\rangle \ . \end{aligned} \tag{35.3}$$

A cell can be represented by the vector

$$\begin{pmatrix} a \\ b \end{pmatrix} .$$

The B and D operators are not unitary. Indeed they can be represented in matrix form by

$$\begin{aligned} B &\propto \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} , \\ D &\propto \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} , \end{aligned} \tag{35.4}$$

where the proportionality constant is not relevant for our purposes. After applying B or D (or some mixture) the new state will require (re-) normalization so that the probabilities of being dead or alive still sum to unity.

A new generation is obtained by determining the number of living neighbours each cell has and then applying the appropriate operator to that cell. The number of living neighbours in our model is the amplitude of the superposition of the oscillators representing the surrounding eight cells. This process is carried out on all cells effectively simultaneously. When the cells are permitted to take a superposition of states, the number of living neighbours need not be an integer. Thus a mixture of the B, D and S operators may need to be applied. For consistency with standard Life the following conditions will be imposed upon the operators that produce the next generation:

- If there are an integer number of living neighbours the operator applied must be the same as that in standard Life.
- The operator that is applied to a cell must continuously change from one of the basic forms to another as the sum of the a coefficients from the neighbouring cells changes from one integer to another.
- The operators can only depend upon this sum and not on the individual coefficients.

If the sum of the a coefficients of the surrounding eight cells is

$$A = \sum_{i=1}^{8} a_i \tag{35.5}$$

then the following set of operators, depending upon the value of A, is the simplest that has the required properties

$$\begin{aligned} 0 \le A \le 1; \quad & G_0 = D\ , \\ 1 < A \le 2; \quad & G_1 = (\sqrt{2}+1)(2-A)D + (A-1)S\ , \\ 2 < A \le 3; \quad & G_2 = (\sqrt{2}+1)(3-A)S + (A-2)B\ , \\ 3 < A < 4; \quad & G_3 = (\sqrt{2}+1)(4-A)B + (A-3)D\ , \\ A \ge 4; \quad & G_4 = D\ . \end{aligned} \tag{35.6}$$

For integer values of A, the G operators are the same as the basic operators of standard Life, as required. For non-integer values in the range $(1, 4)$, the operators are a linear combination of the standard operators. The factors of $\sqrt{2}+1$ have been inserted to give more appropriate behaviour in the middle of each range. For example, consider the case where $A = 3 + 1/\sqrt{2}$, a value that may represent three neighbouring cells that are alive and one the has a probability of one half of being alive. The operator in this case is

$$G = \frac{1}{\sqrt{2}} B + \frac{1}{\sqrt{2}} D\ , \tag{35.7}$$

or in matrix form

$$M = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\ . \tag{35.8}$$

Applying this to either a cell in the alive, $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or dead, $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ states will produce the state

$$|\psi\rangle = \frac{1}{\sqrt{2}} |\text{alive}\rangle + \frac{1}{\sqrt{2}} |\text{dead}\rangle \tag{35.9}$$

which represents a cell with a 50% probability of being alive. That is, G is an equal combination of the birth and death operators, as we might have expected given the possibility that A represents an equal probability of three or four living neighbours. Of course the same value of A may have been obtained by other combinations of neighbours that do not lie half way between three and four living neighbours, but one of our requirements is that the operators can only depend on the sum of the a coefficients of the neighbouring cells and not on how the sum was obtained.

The new state of a cell is obtained by calculating A, applying the matrix G corresponding to the appropriate operator:

$$\begin{pmatrix} a' \\ b' \end{pmatrix} = G \begin{pmatrix} a \\ b \end{pmatrix}\ , \tag{35.10}$$

and then normalizing the resulting state so that $|a'|^2+|b'|^2 = 1$. It is this process of normalization that means that multiplying the matrix by a constant has no effect. Hence, for example, G_2 for $A = 3$ has the same effect as G_3 in the limit as $A \to 3$, despite differing by the constant factor $(\sqrt{2}+1)$.

35.3 Semi-quantum Life

To get qualitatively different behaviour from classical Life we need to introduce a phase associated with the coefficients, that is, a phase difference between the oscillators. We require the following features from this version of Life:

- It must smoothly approach the classical mixture of states as all the phases are taken to zero.
- Interference, that is the partial or complete cancellation between cells of different phases, must be possible.
- The overall phase of the Universe must not be measurable. That is, multiplying all cells by $e^{i\varphi}$ for some real φ will have no measurable consequences.
- The symmetry between $(B, |\text{alive}\rangle)$ and $(D, |\text{dead}\rangle)$ that is a feature of the original game of Life should be retained. That is, if the state of all cells is reversed ($|\text{alive}\rangle \longleftrightarrow |\text{dead}\rangle$) and the operation of the B and D operators is reversed the system will behave in the same manner.

In order to incorporate complex coefficients while keeping the above properties, the basic operators are modified in the following way:

$$\begin{aligned} B|\text{dead}\rangle &= e^{i\varphi}|\text{alive}\rangle \,, \\ B|\text{alive}\rangle &= |\text{alive}\rangle \,, \\ D|\text{alive}\rangle &= e^{i\varphi}|\text{dead}\rangle \,, \\ D|\text{dead}\rangle &= |\text{dead}\rangle \,, \\ S|\psi\rangle &= |\psi\rangle \,, \end{aligned} \tag{35.11}$$

where the superposition of the surrounding oscillators is

$$\alpha = \sum_{i=1}^{8} a_i = Ae^{i\varphi} \,, \tag{35.12}$$

A and φ being real positive numbers. That is, the birth and death operators are modified so that the new alive or dead state has the phase of the sum of the surrounding cells. The operation of the B and D operators on the state $\begin{pmatrix} a \\ b \end{pmatrix}$ can be written as

$$\begin{aligned} B\begin{pmatrix} a \\ b \end{pmatrix} &= \begin{pmatrix} a + |b|e^{i\varphi} \\ 0 \end{pmatrix} \,, \\ D\begin{pmatrix} a \\ b \end{pmatrix} &= \begin{pmatrix} 0 \\ |a|e^{i\varphi} + b \end{pmatrix} \,, \end{aligned} \tag{35.13}$$

with S leaving the cell unchanged. The modulus of the sum of the neighbouring cells, A, determines which operators apply, in the same way as before (see Eqn. (35.6)). The addition of the phase factors for the cells allows for interference effects since the coefficients of alive cells may not always reinforce in taking the sum, $\alpha = \sum a_i$. A cell with $a = -1$ still has a unit probability of being measured in the alive state but its effect on the sum will cancel that of a cell with $a = 1$. We are free to make the phase of the dead cell have some effect, but this does not fit the physical model presented in the introduction. Also, we wish to ensure that standard Life, in which empty cells have no effect, is a subset of our model. Hence we have chosen for the phase of the dead cells to have no effect. It is retained in order to maintain the alive $\longleftrightarrow$ dead symmetry.

A useful notation to represent semi-quantum Life is to use an arrow whose length represents the amplitude of the a coefficient and whose angle with the horizontal is a measure of the phase of a. That is, the arrow represents the phaser of the oscillator at the beginning of that generation. For example

$$\begin{aligned}
\longrightarrow &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \qquad (35.14)\\
\uparrow &= e^{i\pi/2} \begin{pmatrix} 1/2 \\ \sqrt{3}/2 \end{pmatrix} = \begin{pmatrix} i/2 \\ i\sqrt{3}/2 \end{pmatrix} , \\
\nearrow &= e^{i\pi/4} \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} (1+i)/2 \\ (1+i)/2 \end{pmatrix} ,
\end{aligned}$$

etc. Then α is the vector sum of the arrows. This notation includes no information about the b coefficient. The magnitude of this coefficient can be determined from a and the normalization condition. As noted previously, the phase of the b coefficient has no effect on the future progression of the game so it is not necessary to represent this.

35.4 Results and Discussion

The above rules have been implemented [5] in *Mathematica* [25]. All the structures of standard Life can be recreated by making the phase of all the alive cells equal. We are interested in whether there are new effects in our model or whether existing effects can be reproduced in simpler or more generalized structures.

The most important aspect of our model not present in standard Life is interference. Two live cells can work against each other as indicated in figure 35.3 that shows an elementary example in a block still life with one cell out of phase with its neighbours. In standard Life there are linear structures called wicks that die or "burn" at a constant rate. The simplest such structure is a diagonal line of live cells as indicated in figure 35.4a. In this, it is not possible to stabilize an end without introducing other effects. In our model a line of cells of alternating phase, that is of units of $\longrightarrow\longleftarrow$'s, is a generalization of this effect

[5] A version is available from the leading author.

(figures 35.4b and 35.4c) since it can be in any orientation and the ends can be stabilized easily. A line of alternating phase live cells can be used to create other structures such as the loop in figure 35.5a. This is a generalization of the boat still life (figure 35.5b) in the standard model that is of a fixed size and shape. The stability of the line of $\longrightarrow\longleftarrow$'s results from the fact that while each cell in the line has exactly two living neighbours, the cells above or below this line have a net of zero (or one at a corner) living neighbours, due to the cancelling effect of the opposite phases. No new births around the line will occur unlike the case where all the cells are in phase.

Oscillators (figure 35.1) and spaceships (figure 35.2) cannot be made simpler than the minimal examples shown for standard Life. Figure 35.6 shows a stable boundary that results from the appropriate adjustment of the phase differences between the cells. The angles have been chosen so that each cell in the line has between two and three living neighbours, while the empty cells above and below the line have either two or four living neighbours and so remain life-less. Such boundaries are known in standard Life but require a more complex structure.

In Conway's Life interesting effects can be obtained by colliding gliders. In our model we can obtain additional effects from colliding gliders and "anti-gliders," where all the cells have a phase difference of π with those of the original glider. For example, a head-on collision between a glider and an anti-glider as indicated in figure 35.7, causes annihilation, where as the same collision between two gliders leaves a block. However, there is no consistency with this effect since other glider-antiglider collisions produce alternative effects, sometimes being the same as those from the collision of two gliders.

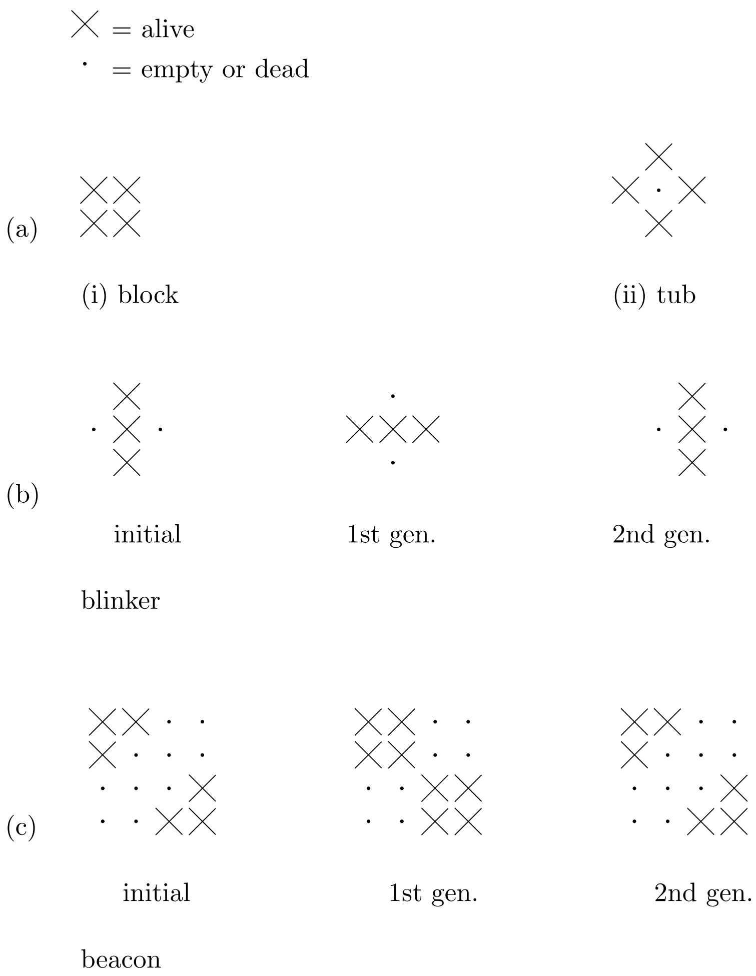

FIGURE 35.1. A small sample of the simplest structures in Conway's Life: (a) the simplest still-lives (stable patterns), the block and the tub, and the simplest oscillators (periodic patterns), (b) the blinker and (c) the beacon, both of period two. A number of blocks and blinkers will normally evolve from any moderate sized random collection of alive and dead cells.

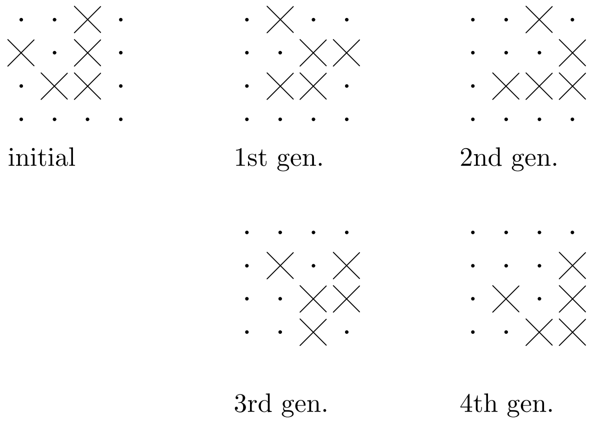

FIGURE 35.2. In Conway's Life, the simplest spaceship (a pattern that moves continuously through the Life universe), the glider. The figure shows how the glider moves one cell diagonally over a period of four generations.

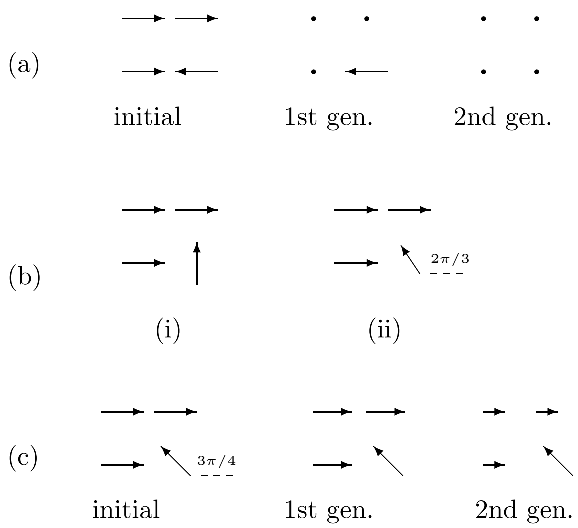

FIGURE 35.3. (a) A simple example of destructive interference in semi-quantum Life: a block with one cell out of phase by π dies in two generations. (b) Blocks where the phase difference of the fourth cell is insufficient to cause complete destructive interference; each cell maintains a net of at least two living neighbours and so the patterns are stable. In the second of these, the fourth cell is at a critical angle. Any greater phase difference causes instability resulting in eventual death as seen in (c), which dies in the fourth generation.

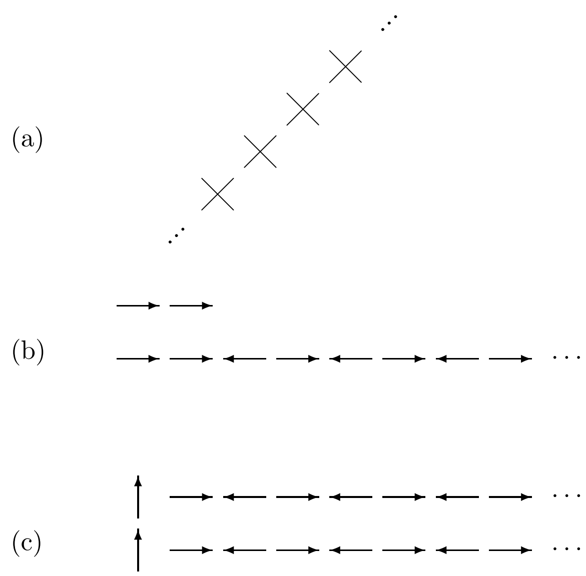

FIGURE 35.4. (a) A wick (an extended structure that dies, or "burns", at a constant rate) in standard Life that burns at the speed of light (one cell per generation), in this case from both ends. It is impossible to stabilize one end without giving rise to other effects. (b) In semi-quantum Life an analogous wick can be in any orientation. The block on the left-hand end stabilizes that end; a block on both ends would give a stable line; the absence of the block would give a wick that burns from both ends. (c) Another example of a light-speed wick in semi-quantum Life showing one method of stabilizing the left-hand end.

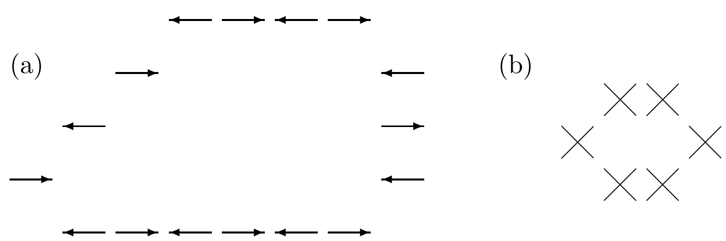

FIGURE 35.5. (a) An example of a stable loop made from cells of alternating phase. Above a certain minimum, such structures can be made of arbitrary size and shape. Compare this with (b), the boat still life in Conway's scheme, that cannot be extended without added complexity.

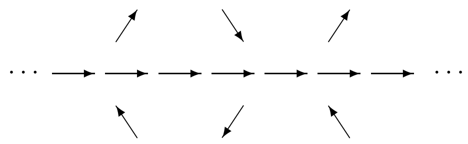

FIGURE 35.6. A boundary utilizing appropriate phase differences to produce stability. The upper cells are out of phase by $\pm\pi/3$ and the lower by $\pm 2\pi/3$ with the central line.

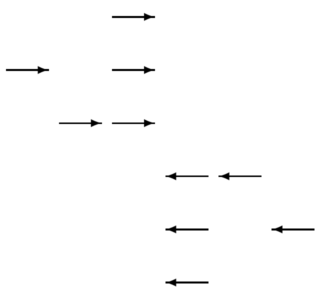

FIGURE 35.7. A head on collision between a glider and its phase reversed counter part, an anti-glider, produces annihilation in six generations.

35.5 Conclusion

John Conway's game of Life is a two dimensional cellular automaton where the new state of a cell is determined by the sum of the neighbouring states that are in one particular state generally referred to as "alive." In semi-quantum Life cells may be in a superposition of the alive and dead states with the coefficient of the alive state being represented by an oscillator. The equivalent of evaluating the number of living neighbours of a cell is to take the superposition of the oscillators of the surrounding states. The amplitude of this superposition will determine which operator(s) to apply to the central cell to determine its new state, while the phase gives the phase of any new state produced. Such a system is able to reproduce some of the aspects of quantum mechanics such as interference.

Obviously this paper just touches on some of the results that can be obtained with this new scheme but it can be seen that some new effects and structures occur and that some of the known effects in Conway's Life can occur in a simpler manner.

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