33

Parrondo's Capital and History Dependent Games[1]

Gregory P. Harmer, Derek Abbott and Juan M. R. Parrondo

ABSTRACT It has been shown that it is possible to construct two games that when played individually lose, but alternating randomly or deterministically between them can win. This apparent paradox has been dubbed "Parrondo's paradox." The original games are capital dependent, which means that the winning and losing probabilities depend on how much capital the player currently has. Recently, new games have been devised, that are not capital dependent, but history dependent. We present some analytical results using discrete-time Markov-chain theory, which is accompanied by computer simulations of the games.

33.1 Introduction

It has recently been shown [1, 2] that a discrete-time version of the flashing ratchet [3, 4, 5] can be interpreted as simple gambling games. There exists two losing games that can be combined to form a game with a winning expectation, much in the same way as a flashing ratchet can be made to move Brownian particles uphill with the use of mechanisms that individually let the particles move downhill. More information regarding this analogy can be found in [6].

However, this original incarnation of the games has the probabilities depend on the value of the current capital of the player, that is, the games are capital dependent. Though this is useful in certain applications [7], a version of the games that does not depend on capital is more natural. This led to a construction of the games where the probabilities depend on the results of the previous two games, referred to as history dependent games [8].

In this chapter, we analyse the games using simple discrete-time Markov chain theory and show analytical results from numerical simulations of the games. We also offer an explanation of the games in terms of their equilibrium distributions.

[1]Manuscript received date: April 12, 2001

33.2 Parrondo's Capital Dependent Games

In this section we section we construct the capital dependent games and explain how the concept of fairness applies to these games. Certain results of playing the games are also shown. The results have been found analytically, that is, what would be expected if we averaged over almost an infinite number of games.

33.2.1 Construction of the Games

Game A is straight forward and can be thought of as tossing a weighted coin that has probability p of winning. Game B is a little more complex and can be generally described by the following statement. If the present capital is a multiple of M then the chance of winning is p_1, if it is not a multiple of M the chance of winning is p_2. Thus, the respective losing probabilities are $1-p_1$ and $1-p_2$.

The two games can be represented diagrammatically using branching elements, shown in Fig. 33.1. The notation (x, y) at the top of the branch gives the probability or condition for taking left and right branch respectively.

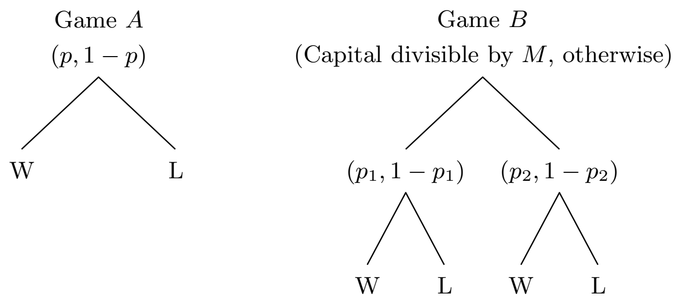

FIGURE 33.1. Construction of the capital dependent games. The games could be formed using three biased coins.

If we require to control the three probabilities p, p_1 and p_2 via a single variable, a biasing parameter ϵ can be used to represent a subset of the probability space with the transformation

$$\begin{aligned} p &= 1/2-\epsilon, \\ p_1 &= 1/10-\epsilon \quad \text{and} \\ p_2 &= 3/4-\epsilon. \end{aligned} \tag{33.1}$$

This parameterisation along with $M = 3$ gives Parrondo's original numbers for the games [1].

33.2.1.1 The Randomised Game

Dealing with the randomised game is not as difficult as it first appears. Let us define a mixing parameter γ that gives the probability of playing game A, which is assumed to be a 1/2 unless otherwise stated. When the capital is a multiple of M, the probability of winning is

$$q_1 = \gamma p - (1 - \gamma)p_1. \tag{33.2}$$

This is the chance of playing game A multiplied by the chance of winning it and correspondingly the chance of playing game B multiplied by the chance of winning. Alternatively, when the capital is not a multiple of M, the probability of winning is

$$q_2 = \gamma p - (1 - \gamma)p_2. \tag{33.3}$$

The respective losing probabilities are $1-q_1$ and $1-q_2$. Using these probabilities we can treat the randomised game exactly the same as game B, except replace each p_i with a q_i.

33.2.1.2 Fairness

An issue that needs to be clarified is the question of how to define whether the games are losing, fair or winning. To classify a game as either winning or losing is trivial, but when it comes to deciding if it is fair, the issue can become controversial. The reason is the behaviour of game B differs from game A as we are likely to win or lose a small amount depending on the value of the capital that we start with. If the starting capital is a multiple of M then it is likely we will lose a little, if not, it is likely we will gain a little.

A brief discussion of fairness follows. A more detailed mathematical formulation of fairness relating to Parrondo's games is given by [9]. Consider a gambler repeatably playing a game and after the nth game has capital $X(n)$, or X_n for short. Classically, as defined by [10], a fair game is one where given all the past results, the expectation of the next result is the same as the present result for any given game. That is, the game has to be a martingale where the expected value of capital after playing a game is the same as the present value.

The difficulty with game B is when X_0 is a multiple of M, $E[X_1|X_0] > X_0$ and correspondingly when X_0 is not a multiple of M, $E[X_1|X_0] < X_0$. This makes it troublesome to classify game B as either winning, losing or fair [9]. Suffice to say it is argued in [9] that fairness can be defined in terms of drift rates. Thus, if the capital tends to drift toward infinity then it classifies as winning ($\epsilon < 0$) or if it drifts towards negative infinity it is losing ($\epsilon > 0$). If there is no drift, then the game is fair ($\epsilon = 0$).

Therefore, using the above criterion, both games A and B are fair when ϵ set to zero in (33.1). This is true of game A because the probabilities of moving up and down in capital are equal for all n. It is also true of game B even though the value of starting capital influences the probability of going up and down for small values of n because as $n \to \infty$, there is no change in capital. The transient response actually decays to almost nothing very quickly, after about 20 games. The drift rates that determine fairness can be easily verified by considering a detained balance [11] of the corresponding system.

Although there is some concern over whether game B is technically fair, it is not that important in the context of the apparently paradoxical nature of the games as they definitely lose when $\epsilon > 0$. This is satisfactory since the only prerequisite we have in later sections are games A and B lose when $\epsilon > 0$.

33.2.2 Playing the Games Analytically

As has been implied in the introduction, the mode of analysis for the games is via discrete-time Markov chains (DTMCs). Each value of capital is represented by a state, and the transition probabilities are determined by the rules of the games. Since in every game we must either incrementally win or lose, i.e. go up or down the chain by one state, the DTMC is referred to as *skip-free*.

The transition probabilities p_{ij} form the entries of the transition matrix $\mathbb{P}$, which defines the DTMC. Since the matrix represents a skip-free DTMC, $\mathbb{P}$ is tridiagonal with the main diagonal all zeros and all the columns sum to unity. Since the DTMC that represents the games is doubly-infinite, the dimensions of $\mathbb{P}$ also extend to $\pm\infty$. However, in practice the dimensions only need to extend to twice the number of games that are being played.

The transition matrix modelling game B is given by

$$\mathbb{P}_B = \begin{bmatrix} 0 & 1-p_2 & & & & \\ p_1 & 0 & \ddots & & & \\ & p_2 & \ddots & 1-p_1 & & \\ & & \ddots & 0 & 1-p_2 & \\ & & & p_1 & 0 & \ddots \\ & & & & p_2 & \ddots \\ & & & & & \ddots \end{bmatrix}. \tag{33.4}$$

This matrix shows the state dependency that is exhibited with the probabilities p_1 and $1 - p_1$ leaving the state that are divisible by M.

Since game A is a specific case of game B where $p_1 = p_2 = p$, $\mathbb{P}_A$ can be easily found from $\mathbb{P}_B$. Recalling from (33.2) and (33.3), anything derived for game B equally holds true for the randomised game, thus $\mathbb{P}_R$ can be determined. This is sufficient for all the analysis since the combination of two DTMCs simply forms another DTMC that obeys Markov chain theory.

From the transition matrices representing the games, the equilibrium probabilities (or stationary distribution) $\boldsymbol{\pi} = [\ldots, \pi_{-1}, \pi_0, \pi_1, \ldots]^T$ can be found. This contains the probabilities of finding the capital in each of the states. The expected outcome when playing a game can then be found by applying $\mathbb{P}$ to $\boldsymbol{\pi}$. Hence, the posterior distribution after playing n games is given by

$$\boldsymbol{\pi}_n = \mathbb{P}^n \boldsymbol{\pi}_0,$$

where the $\boldsymbol{\pi}_0$ is the starting capital. As $n \to \infty$ this gives the stationary distribution. To initially start (i.e. $n = 0$) with zero capital we would have $\boldsymbol{\pi}_0 = [0, \ldots, 0, 1, 0, \ldots, 0]^T$. By using the appropriate transition matrix the individual or randomly mixed games can be played.

To play a deterministic mix of games, the appropriate $\mathbb{P}$ must be substituted. Thus, we could have

$$\boldsymbol{\pi}_n^{[a,b]} = \mathbb{P}_X^n \boldsymbol{\pi}_0,$$

where the notation $[a, b]$ represents playing game A a times, game B b times and so on, thus

$$\mathbb{P}_X = \begin{cases} \mathbb{P}_A & \text{if } n \bmod (a+b) < a \\ \mathbb{P}_B & \text{otherwise.} \end{cases}$$

The deterministically mixed games can be implemented using a single transition matrix by grouping the periodic sequence. For example, $\mathbb{P}_{2,2} = \mathbb{P}_B^2 \mathbb{P}_A^2$ represents the equivalent transition matrix of playing $AABB$. Applying $\mathbb{P}_{2,2}$ is then equivalent to playing four consecutive games. Due to the multiple paths the capital can take within those four games, the algebra becomes tedious – a symbolic programming language is most advantageous.

Using the stationary distribution we can determine some statistical properties of the games, namely the mean μ, and standard deviation σ. We define a capital vector $\mathbf{x} = [-n, \ldots, -1, 0, 1, \ldots, n]$ so that the values correspond to the stationary probabilities in $\boldsymbol{\pi}$, thus the 0 in $\mathbf{x}$ should be aligned with the 1 in $\boldsymbol{\pi}_0$. The mean is then given by

$$\mu_n \equiv E[X_n] = \mathbf{x}\boldsymbol{\pi}_n \tag{33.5}$$

and the standard deviation by

$$\sigma_n = \sqrt{(\mathbf{x} - \mu_n)^2 \boldsymbol{\pi}_n}, \tag{33.6}$$

where the squared vector term is an element-wise operation.

Several characteristics of the games are plotted in Fig. 33.2. The probability density functions (PDF) $p(x, n)$ of the games, which are equivalent to the stationary probabilities $\boldsymbol{\pi}$ are shown in Fig. 33.2a. However, since the capital must increase or decrease after each game, it leaves every second state with a zero stationary probability. To correct for this misleading characteristic a centered mean is taken, denoted by a hat,

$$\hat{p}(\mathbf{x}, n) = \frac{p(\mathbf{x}, n-1) + 2p(\mathbf{x}, n) + p(\mathbf{x}, n+1)}{4}, \tag{33.7}$$

which is the quantity plotted in Fig. 33.2a.

To better observe the ratchet potential that is exhibited by game B, a higher value of M is preferable, $M = 7$ with $p_1 = 0.075$ and $p_2 = 0.6032$ for example. This clearly shows the Brownian ratchet mechanism that the games were based from [1].

In Fig. 33.2b the expected outcome of the games using (33.5) is plotted against the number of games played. This shows clearly the paradoxical result of the games – two losing games can combine to form a game with a winning expectation. One should note however, this is an apparent paradox, even

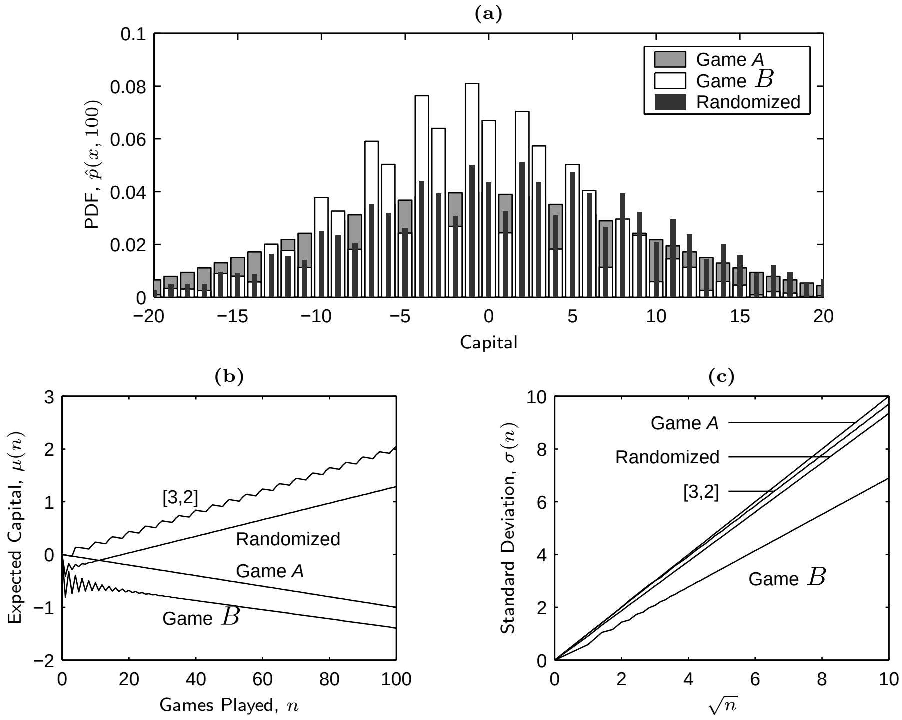

FIGURE 33.2. Characteristics for the capital dependent games using (33.1). (a) The probability density function of the games using the centered mean of (33.7) with $\epsilon = 0$. (b) The expected outcome when playing the games individually and mixing with $\epsilon = 0.05$. The notation $[3, 2]$ for example, refers to playing the sequence $AAABB\ldots$. (c) The standard deviations of the games, which are proportional to $\sqrt{n}$.

though it has a counter-intuitive result that even experienced mathematicians find surprising a proof is available that explains the situation.

Figure 33.2c plots the standard deviations using (33.6) against $\sqrt{n}$ for the same games in Fig. 33.2b. This shows the behaviour of the games does not diverge rapidly, but in fact the standard deviation of the games are all proportional to $\sqrt{n}$ and less than that of game A's.

33.2.3 Analysis using Equilibrium Distributions

When analysing the games, it is sufficient to only consider whether the capital $X(n)$ is in a state relative to the modulus rule. Thus we can define a cyclic DTMC by

$$Y(n) \equiv X(n) \bmod M, \tag{33.8}$$

where $Y(t)$ has the states $\{0, \ldots, M-1\}$. If we win at the highest state $M-1$ we go back to state 0, and vice versa from state 0 to $M-1$. Thus, given an initial distribution of capital among the states and as $n \to \infty$ the probability of the capital being in any of the states reaches an equilibrium, $\boldsymbol{\pi}_n \to \boldsymbol{\pi} = [\pi_0, \ldots, \pi_{M-1}]$. From this equilibrium distribution, many properties of the games can be found analytically. The transition matrix associated with $Y(t)$ is

$$\mathbb{P}_B = \begin{bmatrix} 0 & 1-p_2 & & & p_2 \\ p_1 & 0 & \ddots & & \\ & p_2 & \ddots & 1-p_2 & \\ & & \ddots & 0 & 1-p_2 \\ 1-p_1 & & & p_2 & 0 \end{bmatrix}, \tag{33.9}$$

which will used to represent game B (or the randomised game by replacing each p with a q). This is restricted to $M \times M$ in size and the two extra entries (c.f. (33.4)) provide the cyclic nature of the chain.

From the transition matrix, there are many ways to find the stationary distribution, see [12, 13] for example. Using $M = 3$ to simplify the algebra, the stationary distribution is

$$\boldsymbol{\pi}^B = \frac{1}{D} \begin{bmatrix} 1 - p_2 + p_2^2 \\ 1 - p_2 + p_1 p_2 \\ 1 - p_1 + p_1 p_2 \end{bmatrix}, \tag{33.10}$$

where $D = 3 - p_1 - 2p_2 + 2p_1p_2 + p_2^2$ is the normalisation constant. If we let $p_1 = p_2 = p$ to represent game A, then the stationary distribution simplifies to $\boldsymbol{\pi}^A = (1/3)[1, 1, 1]^T$ as expected for a three state chain. Using the probabilities of (33.1) with $\epsilon = 0$, the stationary distribution for game B turns out to be

$$\boldsymbol{\pi}^B = (1/13)[5, 2, 6]^T. \tag{33.11}$$

33.2.3.1 Capital Dependent Games Constraints

It would be desirable, given a set of parameters, if constraints could be found to determine if Parrondo's paradox exists. An intuitive approach is finding the probability of winning using the stationary distribution, which is given by

$$p_{\text{win}} = \sum_{i=0}^{M-1} \pi_i p_i, \tag{33.12}$$

where p_i is the winning probability in state π_i. The games are winning, losing or fair when p_{win} is greater than, less than or equal to a half, which implies that $\langle X(n) \rangle$ is a decreasing, increasing or constant function with respect to n respectively.

For game A to lose, from (33.12) we get $p < 1/2$, or alternatively

$$\frac{1-p}{p} > 1. \tag{33.13}$$

The probability of winning game B by expanding (33.12) is

$$p_{\text{win}} = \pi_0 p_1 + (1 - \pi_0) p_2, \tag{33.14}$$

recalling that $\sum \pi_i = 1$. Subjecting $p_{\text{win}} < 1/2$ and using the stationary probabilities $\boldsymbol{\pi}^{B'}$ of (33.10) yields

$$\frac{(1-p_1)(1-p_2)^2}{p_1 p_2^2} > 1, \tag{33.15}$$

for $M = 3$. This is the condition that needs to be satisfied for game B to be losing.

For the randomised game we use the expression for game B except replacing each p_i with a q_i and conditioning the game to win by setting $p_{\text{win}} > 1/2$ leads to

$$\frac{(1-q_1)(1-q_2)^2}{q_1 q_2^2} < 1. \tag{33.16}$$

This is the condition for the randomised game to win. Therefore, in order for Parrondo's paradox to be exhibited we require probabilities and parameters to satisfy (33.13), (33.15) (i.e. to make game A and B lose) and (33.16) (i.e. make the randomised game win). This happens to be the case for $p = 5/11$, $p_1 = 1/121$, $p_2 = 10/11$ and $\gamma = 1/2$.

This type of analysis becomes tedious as M becomes larger due to the necessity of finding the equilibrium distribution. An alternative analysis, which can be solved for the general modulo M game, considers the conditions for recurrence of the corresponding DTMC and is given in [6]. The conditions that need to be satisfied for the generalised games are

$$\frac{1-p}{p} > 1, \tag{33.17}$$

$$\frac{(1-p_1)(1-p_2)^{M-1}}{p_1 p_2^{M-1}} > 1 \quad \text{and} \tag{33.18}$$

$$\frac{(1-q_1)(1-q_2)^{M-1}}{q_1 q_2^{M-1}} < 1. \tag{33.19}$$

Using this type of analysis it is possible to find other properties such as rate of return, range of ϵ where the paradox occurs and the probability space for example.

33.2.4 Explanation in Terms of Distributions

When investigating game B *prima facie*, it can be mistakenly interpreted as a winning game, thus invalidating the paradoxical result. This is due to taking the wrong line of analysis by considering the games statistically. This approach assumes the capital spends an equal amount of time in all states. When $M = 3$ it would be mistakenly assumed the capital is in each of the three states a third of the time. Then using the probabilities in (33.1) with $\epsilon = 0$ so the games are fair, the winning probability is calculated as

$$p_{\text{win}} = \frac{1}{3} \cdot \frac{1}{10} + \frac{1}{3} \cdot \frac{3}{4} + \frac{1}{3} \cdot \frac{3}{4} = \frac{16}{30},$$

which is greater than a half. This implies that the game B is winning, which is incorrect – it is actually fair.

As we have seen, the correct analysis is via DTMCs. Using the correct distribution probabilities from (33.11) the probability of winning is

$$p_{\text{win}} = \frac{5}{13} \cdot \frac{1}{10} + \frac{2}{13} \cdot \frac{3}{4} + \frac{6}{13} \cdot \frac{3}{4} = \frac{1}{2},$$

which correctly dictates that the game is fair. Subtracting a small amount ϵ from each of the probabilities makes $p_{\text{win}} < 1/2$ and the game is losing.

We notice that the construction of the game keeps the stationary distribution $\boldsymbol{\pi}^B$ locked at these values and manages to weight the probabilities so game B is losing. We can think of game B as consisting of two coins, a bad (C_1) and good (C_2) coin biased to win according to p_1 and p_2 respectively. Then we use coin C_1, 5/13 of the time and C_2 for the remaining time. If we can somehow 'flatten' the distribution of the game it can be made to win. This is achieved by mixing game B with something completely random like game A. This has the effect of playing the better coin C_2 more often than C_1, and hence produce a winning game.

This can be related to several observations in Fig. 33.2. The distributions of game B have a very definite shape whereas that of game A is smooth. It is this well defined shape of game B that allows it to lose using both good and bad coins. When mixing the two games evenly together the new PDF loses some of its shape. This is enough to allow the new game to be more evenly distributed, as seen from the skinny bars in Fig. 33.2a, to produce a winning game. It is this breaking up of the PDF of game B that leads to the paradoxical result. Note, in Fig. 33.2a the PDFs for games A and B have drifted to the left and that for the randomised game to the right.

The consequences of breaking the distribution are seen in the standard deviations in Fig. 33.2c. Since game A essentially represents free diffusion it has the largest standard deviation, whereas the standard deviation of game B is the smallest due to the capital being caught by the rules of the game. It should be of no then surprise then to find the standard deviation of the games formed by mixing A and B to lie between the standard deviation of the individual games.

33.3 The History Dependent Games

It has been shown that two losing capital dependent games can win, but are there any other types of games that have this characteristic? Although state dependent games are applicable in certain areas (see [7] for examples), it may be desirable to have a version of the games independent of capital. The answer to the aforementioned question is in the affirmative, in the form of history dependent games. These were also devised by Parrondo [8], although other implementations are possible [9].

33.3.1 Construction and Results

Game A is the same as before and we introduce game B', the modified version of the original game B. The probabilities that we use for the new game depend on the results of the two previous games, hence there are four options. Game B' is shown by a branching process in Fig. 33.3 and could be played using four biased coins.

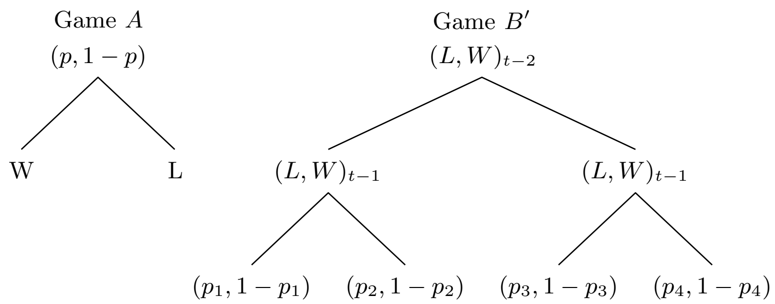

FIGURE 33.3. Construction of the history dependent games, game B' has four possible options {LL,LW,WL,WW}.

We can also parameterise the history dependent games as

$$\begin{aligned} p &= 1/2-\epsilon, \\ p_1 &= 9/10-\epsilon, \\ p_2 &= p_3 = 1/4-\epsilon \quad \text{and} \\ p_4 &= 7/10-\epsilon. \end{aligned} \tag{33.20}$$

This parameterisation gives Parrondo's original probabilities for the history dependent games [8], which behave in much the same way as the parameterisation of the capital dependent games in (33.1). That is, games are fair when $\epsilon = 0$, losing when $\epsilon > 0$ and winning when $\epsilon < 0$. The method of analysis closely follows that of the capital dependent games.

The same counter intuitive result occurs when playing games A and B', that is, when playing the games individually they are losing, but switching between them creates a winning expectation. The switching can be either stochastic or deterministic as shown by various games that are plotted in Fig. 33.4. Similarly, there are initial stating transients, the magnitude and shape depending on the initial conditions used, i.e. LL, LW, WL or WW. The sequences shown in Fig. 33.4 are averaged from each of the four starting conditions, thus eliminating much of the transient behaviour.

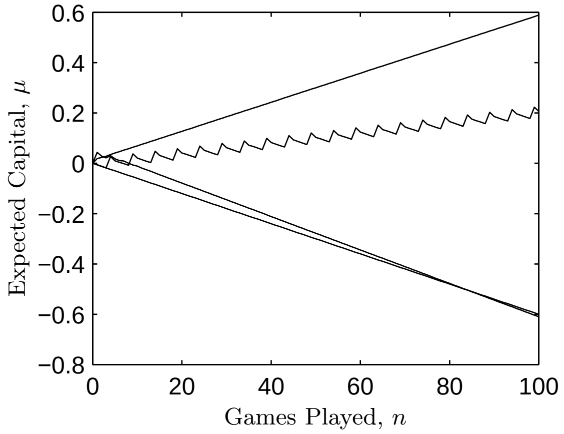

FIGURE 33.4. Games were played using the probabilities in (33.20) with $\epsilon = 0.003$, the results were averaged over each of the four starting conditions.

33.3.2 Analysis using DTMCs

When analysing the chain that is associated with game B' we notice the capital $X(n)$ is not a Markovian process [8]. However, there are two ways to overcome this limitation; to model the game as a quasi-birth-and-death (QBD) process or define a state space $Y'(t)$ similar in nature to $Y(n)$ for the capital dependent games.

With either method we require to record the past two events to determine what probability to use for the current game. Using a QBD process this is achieved by the use of phases, the second index in the state space E [14]. Details of the QBD formulation and the transition matrix directly representing game B' can be found in [9].

If we consider $Y(n)$ used to analyse game B, it only records where the capital is in each period in the periodic structure, not the absolute value of the capital. Similarly, we can define $Y'(n)$ as

$$Y'(n) = [X(n-1) - X(n-2), X(n) - X(n-1)], \tag{33.21}$$

which records the past events of the game. This gives four states as $[-1,-1]$, $[-1,+1]$, $[+1,-1]$ and $[+1,+1]$, where $+1$ represents winning and -1 losing. Using this representation we can perform the same types of analysis as for the capital dependent games. The corresponding transition matrix to $Y'(n)$ is

$$\mathbb{P}_{B'} = \begin{bmatrix} 1-p_1 & 0 & 1-p_3 & 0 \\ p_1 & 0 & p_3 & 0 \\ 0 & 1-p_2 & 0 & 1-p_4 \\ 0 & p_2 & 0 & p_4 \end{bmatrix}, \tag{33.22}$$

with the rows and columns representing the four states LL, LW, WL and WW labelling from the top left corner. This matrix is always 4×4 as only the results of the previous two games are recorded. The stationary probabilities can be calculated as

$$\boldsymbol{\pi}^{B'} = \frac{1}{D'} \begin{bmatrix} (1-p_3)(1-p_4) \\ (1-p_4)p_1 \\ (1-p_4)p_1 \\ p_1 p_2 \end{bmatrix}, \tag{33.23}$$

where the normalisation constant $D' = p_1p_2 + (1+2p_1-p_3)(1-p_4)$. Using the probabilities of (33.20) with $\epsilon = 0$ gives $\boldsymbol{\pi}^{B'} = (1/22)[5, 6, 6, 5]^T$.

When randomly mixing the games, the probabilities can be given by

$$q_i = \gamma p + (1-\gamma)p_i \tag{33.24}$$

for $i = 1, \ldots, 4$ and γ is the mixing parameter.

Thus, we can simply use the probability of winning to find constraints for the games paradox to exist. Using

$$p_{\text{win}} = \sum_{i=1}^{4} \pi_i p_i. \tag{33.25}$$

with the stationary probabilities of game B' in (33.23) yields

$$p_{\text{win}}^{B} = \frac{p_1(1+p_2-p_4)}{p_1p_2 + (1-p_4)(1+2p_1-p_3)}. \tag{33.26}$$

Subjecting this to the constraint $p_{\text{win}} > 1/2$ for a winning game or $p_{\text{win}} < 1/2$ for a losing game, we have the following conditions,

$$\frac{1-p}{p} > 1, \tag{33.27}$$

$$\frac{(1-p_4)(1-p_3)}{p_1p_2} > 1 \quad \text{and} \tag{33.28}$$

$$\frac{(1-q_4)(1-q_3)}{q_1q_2} < 1 \tag{33.29}$$

for game A and B' to lose and the randomised game to win.

The explanation of the games in terms of the equilibrium distribution is the same as that for the capital dependent games with the only difference being that for each value of capital there are four amounts to plot, LL, LW, WL and WW. If one plots the PDFs for history dependent games it is easy to see how the introduction of game A breaks up the equilibrium distribution of game B'.

33.4 Summary

We have described two versions of Parrondo's games and given simple DTMC analysis of them. The analytical results derived match closely with computer simulations. The analysis was performed via use of simple Markov-chain theory and the apparent paradox explained in terms of breaking the equilibrium distribution set by game B. Observing the similarity between the capital and history dependent games, one may assume that further investigation may reveal other settings where the games can be applied.

Acknowledgements

This work was supported by the Dirección General de Enseñanza Superior e Investigación Científica (Spain) Project No. PB97-0076-C02, GTECH (U.S.A.), the Sir Ross and Sir Keith Smith Fund (Australia), and the Australian Research Council (ARC).

References

[1] G. P. Harmer and D. Abbott. Parrondo's paradox. *Statistical Science*, 14(2):206–213, 1999.

[2] G. P. Harmer and D. Abbott. Parrondo's paradox: losing strategies cooperate to win. *Nature*, 402:864, 1999.

[3] A. Adjari and J. Prost. Drift induced by a periodic potential of low symmetry: pulsed dielectrophoresis. *C. R. Acadamy of Science Paris, Série II*, 315:1635–1639, 1992.

[4] R. D. Astumian and M. Bier. Fluctuation driven ratchets: Molecular motors. *Physical Review Letters*, 72(11):1766–1769, 1994.

[5] C. R. Doering. Randomly rattled ratchets. *Nuovo Cimento*, 17D(7–8):685–697, 1995.

[6] G. P. Harmer, D. Abbott, P. G. Taylor, and J. M. R. Parrondo. Parrondo's paradoxical games and the discrete Brownian ratchet. In D. Abbott and L. B. Kish, editors, *Second International Conference on Unsolved Problems of Noise and Fluctuations*, volume 511, pages 189–200, Adelaide, Australia, 2000. American Institute of Physics.

[7] G. P. Harmer, D. Abbott, P. G. Taylor, and J. M. R. Parrondo. Parrondo's games and Brownian ratchets. *Chaos* 11(3):705–714.

[8] J. M. R. Parrondo, G. P. Harmer, and D. Abbott. New paradoxical games based on Brownian ratchets. *Physical Review Letters*, 85(24):5226–5229, 2000.

[9] A. Costa, M. Fackrell, and P. G. Taylor. Two issues surrounding Parrondo's paradox. *Birkhäuser Annals of Dynamic Games*, This volume, 2004.

[10] J. L. Doob. *Stochastic Processes.* John Wiley & Sons, Inc., New York, 1953.

[11] L. Onsager. Reciprocal relations in irreversible processes I. *Physical Review*, 37:405–426, 1931.

[12] C. E. M. Pearce. Entropy, Markov information sources and Parrondo games. In D. Abbott and L. B. Kish, editors, *Second International Conference on Unsolved Problems of Noise and Fluctuations*, volume 511, pages 207–212, Adelaide, Australia, 2000. American Institute of Physics.

[13] R. Pyke. On random walks related to Parrondo's games. *Preprint math.PR/0206150*, 2001.

[14] M. F. Neuts. *Matrix-Geometric Solutions in Stochastic Models: An Algorithmic Approach.* The John Hopkins University Press, USA, 1981.