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Introduction to Quantum Games and a Quantum Parrondo Game[1]

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ABSTRACT In this paper, we provide an introduction to quantum game theory through discussion of ways of converting classical games into the quantum regime. We illustrate how a quantum-based approach can simulate all possible classical game histories in parallel, for the example of Parrondo's games.

34.1 Introduction

Game theory has been used to describe and model the world in a variety of ways. A large group of these models rely, at some level, on probability or stochastic modelling. Quantum mechanics is a theory based on probability. So it is natural to extend classical game theory into the quantum regime. It is has been suggested that classical game theory is, in fact, a limiting subset of quantum game theory.

Meyer introduced the idea of quantum game theory for two-person zero-sum games using deterministic and probabilistic strategies [6]. He proved that, in dynamic games[2], quantum strategies are indeed always at least as good as classical ones. Eisert *et al.* later published corresponding principles for static games[3] through examining the prisoner's dilemma in the quantum regime [2]. This work was later generalised by Benjamin and Hayden [1]. Attempts have even been made to produce a quantum Monty Hall game [4]. For a summary of quantum game theory, see Marinatto [5].

Section two describes briefly the classical Parrondo's paradox [3], in which we have the counter-intuitive phenomenon where two losing games combine to result in a winning game. For simplicity we have used Parrondo's original parameters - but note, however, that the games need not be restricted to this particular parameter set. The third section provides a brief introduction to quantum mechanics and general principles of quantum game theory. These have been simplified greatly but, we believe, provide sufficient explanation for those readers with no previous quantum mechanics background.

[1]Presented at the 9th International Symposium on Dynamic Games and Applications held in Adelaide, South Australia in December 18-21, 2000

[2]Dynamic games are games where players play sequential moves in turn, e.g. chess

[3]Static games are where players make simultaneous decisions, e.g. the game of paper-stone-scissors

Sections four and five describe one method of playing classical Parrondo games through a quantum computer. The results are classical, but the ability of quantum computers to simulate classical systems efficiently is demonstrated. Classical systems require 2^n bits to represent the parameter space of an n bit system, where quantum computers only require n qubits.

34.2 Classical Parrondo's Paradox

Parrondo's Paradox [3] is a counter-intuitive result in which two statistically "losing" games (Game A and Game B) combine to create a "winning" game. This is best demonstrated by tossing coins where the coins are biased one way or another (towards winning or losing). The original game [3] is a capital-dependent (CD) game requiring feedback loops. Parrondo *et al.* later published a capital-independent but history-dependent (HD) game requiring feed-forward loops [7].

In both CD and HD games, Game A involves tossing a single coin which is slightly biased towards losing. i.e. $p_{1,\text{win}} = 1 - p_{1,\text{lose}} = 0.5 - \epsilon$, where ϵ is a small number.

Game B is where CD and HD games differ. In the CD game, there are two coins, biased at $p_{2,\text{win}} = 0.1 - \epsilon$ and the other at $p_{3,\text{win}} = 0.75 - \epsilon$. We play either coin 2 or coin 3 depending on the amount of capital (money), C, that we have at the moment, hence a Capital Dependent (CD) game. If C is a multiple of 3, then we play coin 2, otherwise we play coin 3. This means that on average, coin 3 will be played more often than coin 2, but because coin 2 has such a poor probability of winning, it outweighs coin 3. This makes Game B a losing game overall.

On the other hand, 3 coins are used in Game B of the HD game. Depending on whether we won or lost in the previous game history, we choose one of the 3 coins to toss (see Table 34.1). The probabilities are given as $p_{2,\text{win}} = 0.9 - \epsilon, p_{3,\text{win}} = 0.25 - \epsilon, p_{4,\text{win}} = 0.7 - \epsilon$. As coin 3 is played much more frequently than the other coins, this is a losing game as well. It can be shown that the starting condition does not influence subsequent games, and so it is convenient to start the game with coin 2, and then coin 2 or 3 depending on the result of the first game.

TABLE 34.1. The choice of the next coin to play, Game_n**, depends on the results of the previous two games. This table shows which coin to play.**

$\text{Game}_{\text{n}-2}$	$\text{Game}_{\text{n}-1}$	Coin Played
Loss	Loss	2
Loss	Win	3
Win	Loss	3
Win	Win	4

It has been shown that for both the CD [3] and HD [7] game, by combining their respective losing Game A and B's, the combined game has a winning expectation overall.

34.3 Basic quantum mechanics and quantum game theory

The difference between quantum game theory and conventional classical game theory comes from the ability to place the bits into a superposition of states and the ability to entangle the bits. These bits are thus called qubits (quantum bits). A qubit has two distinct states. They may be arbitrarily labeled "Heads" and "Tails" if we are dealing with coin tosses, or more generally, Win/Lose. For the purpose of computation, we shall label them "1" and "0". These are orthogonal states in Hilbert space. Now, when a qubit is in a superposition, we can think of it as being both 0 and 1 at the same time. However, when we measure the qubit, the superposition will collapse into one of the two states with the probability defined by the nature of the superposition.

The standard notation for expressing these quantum states is the Dirac Bra-Ket notation. Each state is written as $|\psi\rangle$. So, the 0 state is $|0\rangle$, called the 0 ket. A ket is a complex vector in Hilbert space. Superpositions are expressed as vector sums of state kets. In the case of qubits, it is $a|0\rangle + b|1\rangle$, where a and b are, in general, complex probability amplitudes [4] of the respective kets. A measurement is a projection of this superposition onto the basis kets, with each of the magnitudes being the probability that we will find the qubit in a particular state. In other words, $|a|^2$ and $|b|^2$ are the probabilities that when we measure the qubit, we will find 0 and 1 respectively. From this, we can also conclude that $|a|^2 + |b|^2 = 1$.

One of the easiest ways to picture a qubit is by considering photon polarisations (Fig. 34.1). We can define vertical polarisation of the photon as $|0\rangle$ and horizontal polarisation as $|1\rangle$. Now imagine that we have a single photon of 45° polarisation - this is a superposition of $|0\rangle$ and $|1\rangle$. What happens when this photon arrives at a vertically polarised filter? This is a measurement of the photon, and thus the superposition will collapse. The photon will collapse into either a vertically polarised or a horizontally polarised state with 50/50 probability. This is an even superposition of $|0\rangle$ and $|1\rangle$. i.e. $\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$. Obviously if the photon is vertically polarised, it will pass through, otherwise, it will not. Now if the photon is 30° polarised, then we can see that this is $\frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle$. This means that we have a $|\frac{\sqrt{3}}{2}|^2 = \frac{3}{4}$ chance of detecting that the photon has passed through the filter.

Quantum mechanically, we place a qubit into a superposition by rotating the state ket in Hilbert space. Suppose we start with $|\psi_i\rangle = |0\rangle$. To create

[4] Quantum probability amplitudes differ to classical probability by obeying Feynman's rules rather than the classical Bayesian rules. In fact, a complex probability amplitude multiplied by its conjugate results in classical probability.

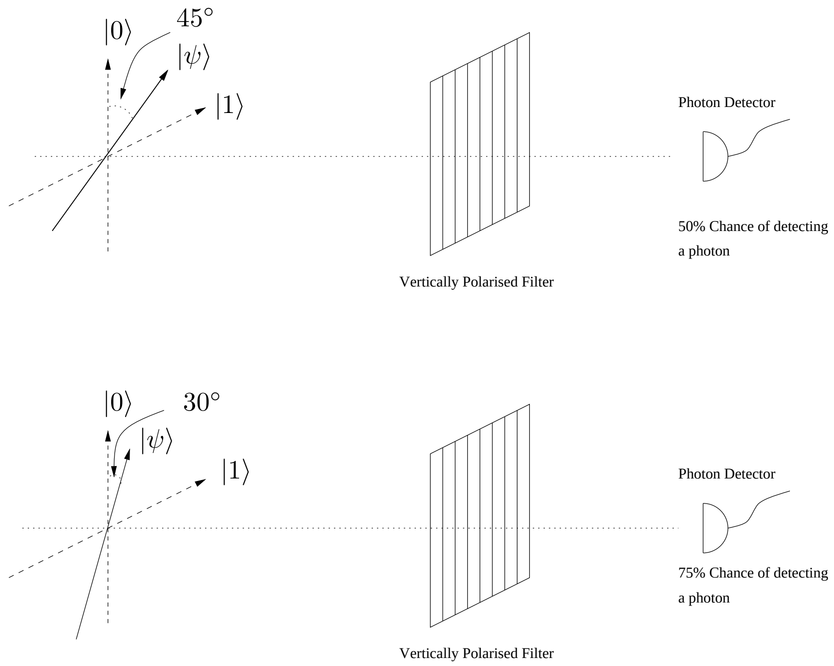

FIGURE 34.1. Using photon polarisation as a qubit. For the 45° polarised photon, the photon detector behind the filter has a 50% chance of detecting that the photon has passed through the filter. For the 30° polarised photon, the chance is increased to 75%.

$|\psi_f\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$, we rotate this ket by 45°. In our example, this can be achieved by rotating a vertically (or horizontally) polarised photon by 45° through a non-linear medium or waveguide.

This can be thought of as a simple gambling game, where one person can bet on whether the detector behind the filter will register a photon or not. Since a 45° polarised photon has a 50% of passing through the filter, this is a fair gamble. A 30° photon on the other hand represents uneven odds, and so this is equivalent to, say, tossing a biased, weighted coin.

34.4 Quantum Parrondo's Paradox

We have chosen to simulate the HD Parrondo's paradox game. This is because the feedback loop required for the CD game will generally, but not necessarily, be irreversible. In quantum terms, irreversibility means that some information must be taken out of the system, which can be regarded as a measurement. As noted earlier, a measurement on a quantum superposition will cause the superposition to collapse into one of the eigenstates depending on the probability amplitudes and thus lose its strange quantum properties.

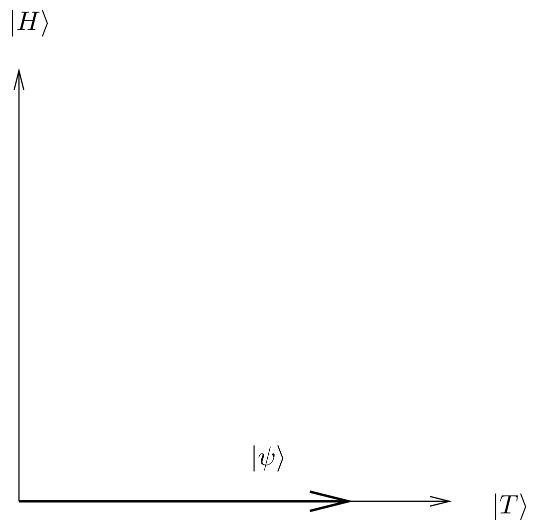

FIGURE 34.2. $|\psi\rangle = |T\rangle$ **in 2-D Hilbert Space. When we measure the system, we will get** $|T\rangle$ **with probability 1.**

34.4.1 Simulating Game A

Suppose we have a ket, $|\psi\rangle = |T\rangle$, representing a single coin initially in the "Tails" state. Fig. 34.2 shows this in the two-dimensional Hilbert Space.

If we do nothing to it, $|\psi\rangle$ will always be in the eigenstate $|T\rangle$ every time we measure it. This is analogous to leaving a coin sitting on the table (or having a vertically or horizontally polarised photon); it is either heads or tails (in this case, tails), and will stay that way until we do something to it.

A fair coin toss will be to rotate $|\psi\rangle$ by $0.5 * \frac{\pi}{2} = \frac{\pi}{4}$ (Fig. 34.3). Fig. 34.4 shows this in a quantum circuit form. If we want to bias the coin, we just change the probability from 0.5 to some other probability, but we will assume an unbiased coin at the moment for simplicity. This puts $|\psi\rangle$ into the state $\frac{1}{\sqrt{2}}|T\rangle + \frac{1}{\sqrt{2}}|H\rangle$ and so if we measure the system now, the superposition has a probability $\left|\frac{1}{\sqrt{2}}\right|^2 = 0.5$ of collapsing into the eigenstate $|T\rangle$ and a 0.5 chance of collapsing into the eigenstate $|H\rangle$. So this is a fair coin toss. A simplified analogy is if we sit the coin on its side, representing a superposition of heads and tails, and then slap a hand onto the coin and see what we are left with (measurement). Under fair circumstances, when we lift our hand, half the time, we find that we have heads, and tails the other half of the time. Also, if we manipulate the coin no further, it does not matter how many more times we slap our hand onto the coin, it will remain as heads or tails[5].

[5] It must be stressed that the example given is only an analogy and not a true quantum superposition. A coin is a classical object. A 45° polarised photon, however, is a true quantum superposition.

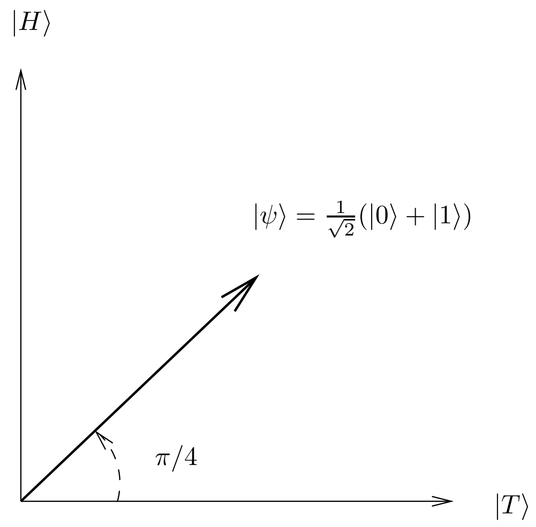

FIGURE 34.3. Rotating $|\psi\rangle$ by $\frac{\pi}{4}$ to create a superposition.

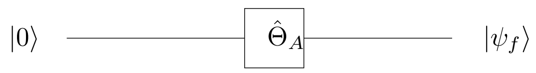

FIGURE 34.4. Quantum circuit representation of a single coin toss. For Parrondo's Game A, $\hat{\Theta}_A$ will rotate the qubit by $(0.5-\epsilon)*\frac{\pi}{2}$, assuming $\epsilon = 0$, so $|\psi_f\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

Algebraically this is a multiplication of the state ket, $|\psi\rangle = \begin{array}{cc} \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \begin{array}{c} |T\rangle \\ |H\rangle \end{array} \end{array}$ by an unitary rotation matrix $\hat{\Theta}(\theta) = \begin{bmatrix} \cos(\theta_A) & \sin(\theta_A) \\ -\sin(\theta_A) & \cos(\theta_A) \end{bmatrix}$

where the parameter θ_A gives the angle of rotation of the ket. Applying $\hat{\Theta}(\theta)$ on $|\psi\rangle$ results in,

$$
\begin{aligned}
\hat{\Theta}(\theta)|\psi\rangle &= a|T\rangle + b|H\rangle \\
\begin{bmatrix} \cos(\theta_A) & \sin(\theta_A) \\ -\sin(\theta_A) & \cos(\theta_A) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} &= \begin{bmatrix} a \\ b \end{bmatrix}
\end{aligned}
$$

where $a = \frac{1}{\sqrt{2}}, b = -\frac{1}{\sqrt{2}}$ for $\theta_A = \pi/4$. To simulate the Game A described by Parrondo *et al.* [7], we simply choose $\theta_A = (0.5 - \epsilon) * \pi/2$. We can see that in this example, the probability amplitude b is negative. As mentioned earlier, probability amplitudes are, in general, complex quantities. This reflects that they have both magnitude and phase components. The only limiting factor, in this case, being that $|b|^2 = \frac{1}{2}$.

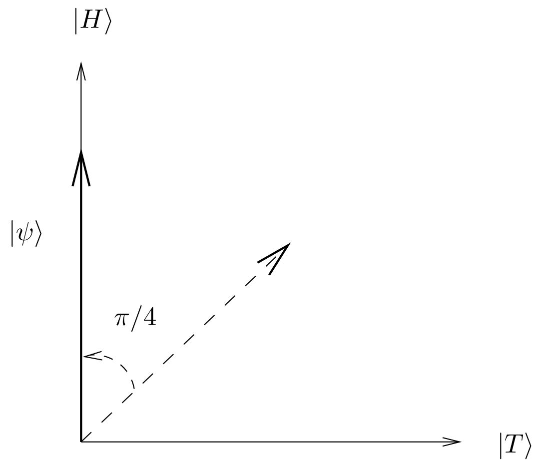

FIGURE 34.5. Rotating $|\psi\rangle$ by $\frac{\pi}{4}$ again will place the quantum coin into the $|H\rangle$ state. So when we now measure the system, we will get Heads with probability 1.

34.4.2 Two or more coin tosses

For two tosses of the coin, we cannot just use the rotation matrix on the same qubit again. If we did, we would put $|\psi\rangle$ into the eigenstate $|H\rangle$, which is obviously not representative of tossing the coin twice (Fig. 34.5).

What we need is to use another qubit, $|\psi_2\rangle$, and rotate that one instead. Now the total state of the system can be described as $|\psi\rangle = |\psi_1\psi_2\rangle$ in the 4-dimensional Hilbert space with $|TT\rangle$, $|TH\rangle$, $|HT\rangle$, $|HH\rangle$ as its base kets. The rotation matrices in this case would be the tensor product of $\hat{\Theta}(\theta)$ and $\hat{I}$, i.e.

for the 1st qubit,

$$\begin{aligned}
\hat{\Theta}_1(\theta) &= \hat{\Theta}(\theta) \otimes \hat{I} \\
&= \begin{bmatrix} \cos(\theta_A) & \sin(\theta_A) \\ -\sin(\theta_A) & \cos(\theta_A) \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \cos(\theta_A) & 0 & \sin(\theta_A) & 0 \\ 0 & \cos(\theta_A) & 0 & \sin(\theta_A) \\ -\sin(\theta_A) & 0 & \cos(\theta_A) & 0 \\ 0 & -\sin(\theta_A) & 0 & \cos(\theta_A) \end{bmatrix},
\end{aligned}$$

and for the 2nd qubit,

$$\begin{aligned}
\hat{\Theta}_2(\theta) &= \hat{I} \otimes \hat{\Theta}(\theta) \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} \cos(\theta_A) & \sin(\theta_A) \\ -\sin(\theta_A) & \cos(\theta_A) \end{bmatrix} \\
&= \begin{bmatrix} \cos(\theta_A) & \sin(\theta_A) & 0 & 0 \\ -\sin(\theta_A) & \cos(\theta_A) & 0 & 0 \\ 0 & 0 & \cos(\theta_A) & \sin(\theta_A) \\ 0 & 0 & -\sin(\theta_A) & \cos(\theta_A) \end{bmatrix}.
\end{aligned}$$

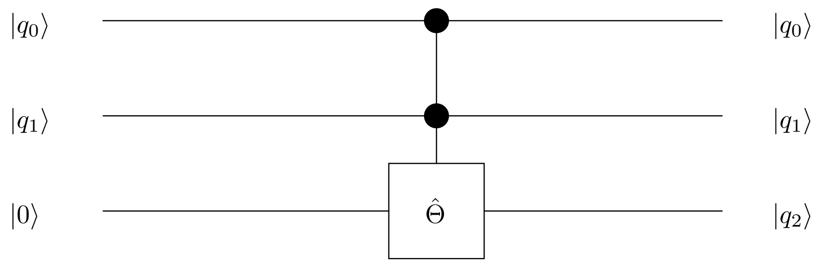

FIGURE 34.6. A Controlled-Controlled-Rotation gate/matrix. If qubit 2, $|q_2\rangle$, starts off in the $|0\rangle$ state, it will be rotated if both $|q_0\rangle$ and $|q_1\rangle$ are in the $|1\rangle$ state.

So the total system, $|\psi\rangle$, becomes $|\psi\rangle = \hat{\Theta}_2\hat{\Theta}_1|\psi_1\psi_2\rangle$, which is a superposition of the base kets. i.e. $a|TT\rangle + b|TH\rangle + c|HT\rangle + d|HH\rangle$. As before, $|a|^2$, $|b|^2$, $|c|^2$, $|d|^2$ represent the classical probability of obtaining the respective states if we measure the system and $|a|^2 + |b|^2 + |c|^2 + |d|^2 = 1$.

So what does it all mean if we find that the system is in the state, say, $|TH\rangle$?

Now the T represents the 1st qubit and the H, the 2nd qubit. As we have defined qubit 1 as the result of the 1st toss, and qubit 2 as the 2nd, $|TH\rangle$ means that we have a tail at the 1st toss, followed by a head. It gives us the toss history of the set of games. So, if a head is considered a win, and tails, a lost, then $|b|^2$ is the probability of losing the first game, and then winning the second game[6].

34.5 Simulating Game B

For Game B, we employ a very similar approach to Game A. However, the difference is that we will now use a Controlled-Controlled-Rotation (*CCRot*) matrix. A *CCRot* gate is a 3-qubit gate, where the 3rd bit is rotated when the first 2 qubits are 1 (Fig. 34.6). The truth table of a *CCRot* gate is given in Table 34.2:

TABLE 34.2. The truth table of a *CCRot* gate. The rotation is given by θ.

$\|q_0\rangle$	$\|q_1\rangle$	$\|q_2\rangle$
$\|0\rangle$	$\|0\rangle$	$\|0\rangle$
$\|0\rangle$	$\|1\rangle$	$\|0\rangle$
$\|1\rangle$	$\|0\rangle$	$\|0\rangle$
$\|1\rangle$	$\|1\rangle$	$\cos(\theta)\|0\rangle + \sin(\theta)\|1\rangle)$

In matrix form, this is

[6] From now on, we shall represent Heads as 1, and Tails as 0, i.e. $|00\rangle \equiv |TT\rangle$, $|11\rangle \equiv |HH\rangle$ etc.

$$\begin{array}{c} |000\rangle \\ |001\rangle \\ |010\rangle \\ |011\rangle \\ |100\rangle \\ |101\rangle \\ |110\rangle \\ |111\rangle \end{array} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cos(\theta_B) & \sin(\theta_B) \\ 0 & 0 & 0 & 0 & 0 & 0 & -\sin(\theta_B) & \cos(\theta_B) \end{bmatrix}.$$

The *CCRot* matrix is perfect for what we need to do because in Game B of the HD game, the choosing of the coin for the 3rd toss (qubit) is dependent on the previous 2 results (qubits). But Game B is a little more than the above. As mentioned earlier, the above matrix will rotate the 3rd bit if the first 2 qubits are 1. In our context, this means that the state of the system is only changed if we won the previous 2 games, i.e. this simulates choosing and tossing coin 4 in Game B. What we need is to obtain 3 other variations of the *CCRot* gate to simulate the other coins for the different possible histories. So for coins 2, 3 and 4, their respective matrices are:

$$\hat{\Theta}_{\text{win,win}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cos(\theta_{B_4}) & \sin(\theta_{B_4}) \\ 0 & 0 & 0 & 0 & 0 & 0 & -\sin(\theta_{B_4}) & \cos(\theta_{B_4}) \end{bmatrix}$$

$$\hat{\Theta}_{\text{win,lose}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos(\theta_{B_3}) & \sin(\theta_{B_3}) & 0 & 0 \\ 0 & 0 & 0 & 0 & -\sin(\theta_{B_3}) & \cos(\theta_{B_3}) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\hat{\Theta}_{\text{lose,win}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cos(\theta_{B_3}) & \sin(\theta_{B_3}) & 0 & 0 & 0 & 0 \\ 0 & 0 & -\sin(\theta_{B_3}) & \cos(\theta_{B_3}) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\hat{\Theta}_{\text{lose,lose}} = \begin{bmatrix} \cos(\theta_{B_2}) & \sin(\theta_{B_2}) & 0 & 0 & 0 & 0 & 0 & 0 \\ -\sin(\theta_{B_2}) & \cos(\theta_{B_2}) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Now Game B is obtained by putting these 4 *CCRot* matrices one after another (Fig. 34.7). i.e.

$$\begin{aligned} \hat{\Theta}_B &= \hat{\Theta}_0\hat{\Theta}_1\hat{\Theta}_2\hat{\Theta}_3 \\ &= \begin{bmatrix} B_2 & 0 & ... & 0 \\ 0 & B_3 & 0 & \vdots \\ \vdots & 0 & B_3 & 0 \\ 0 & ... & 0 & B_4 \end{bmatrix} \end{aligned}$$

where

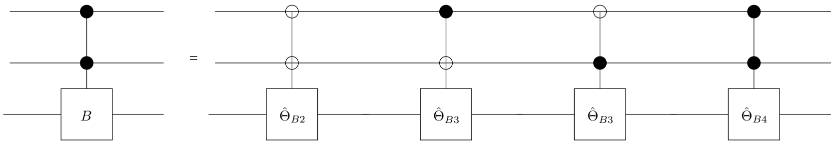

FIGURE 34.7. Combining the different rotation matrices to form the Game B quantum gate. A solid dot means that the control qubit must be in the $|1\rangle$ state for the gate to rotate the target qubit. An open circle means that the control qubit must be in the $|0\rangle$ state.

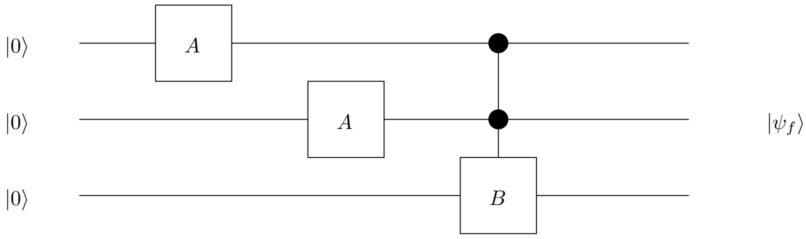

FIGURE 34.8. A 3-toss game, where we play two games of A followed by one game of B

$$\begin{aligned}
B_2 &= \begin{bmatrix} \cos(\theta_{B2}) & \sin(\theta_{B2}) \\ -\sin(\theta_{B2}) & \cos(\theta_{B2}) \end{bmatrix} \\
B_3 &= \begin{bmatrix} \cos(\theta_{B3}) & \sin(\theta_{B3}) \\ -\sin(\theta_{B3}) & \cos(\theta_{B3}) \end{bmatrix} \\
B_4 &= \begin{bmatrix} \cos(\theta_{B4}) & \sin(\theta_{B4}) \\ -\sin(\theta_{B4}) & \cos(\theta_{B4}) \end{bmatrix}.
\end{aligned}$$

As the $\hat{\Theta}_i$ matrices commute with each other, the order is not important. All we need to do is to vary the amount of rotation for each $\hat{\Theta}_i$, this gives us the required matrix for simulating Game B which we will denote $\hat{\Theta}_B$.

34.5.1 Combining Games A and B

To combine the two games, all we need to do is to decide on the number of games, create the correct rotation matrices for these games, and then apply these matrices to an initial state ket, $|\psi_0\rangle = |00...0\rangle$. For example, to play two games of Game A and a game of Game B, the final state of the system is $|\psi_f\rangle = \hat{\Theta}_1\hat{\Theta}_2\hat{\Theta}_3|000\rangle$ (see Fig. 34.8), which will be a superposition of all possible outcomes, so $|\psi_f\rangle = a|000\rangle + b|001\rangle + c|010\rangle + ... + h|111\rangle$. This means that we can now plot a graph of probability vs. outcome, and thus work out the most likely histories if we play the game infinite times (Fig. 34.9). Fig. 34.10 shows the results for playing two games of A's followed by two games of B's followed by two games of A's etc, for 10 games.

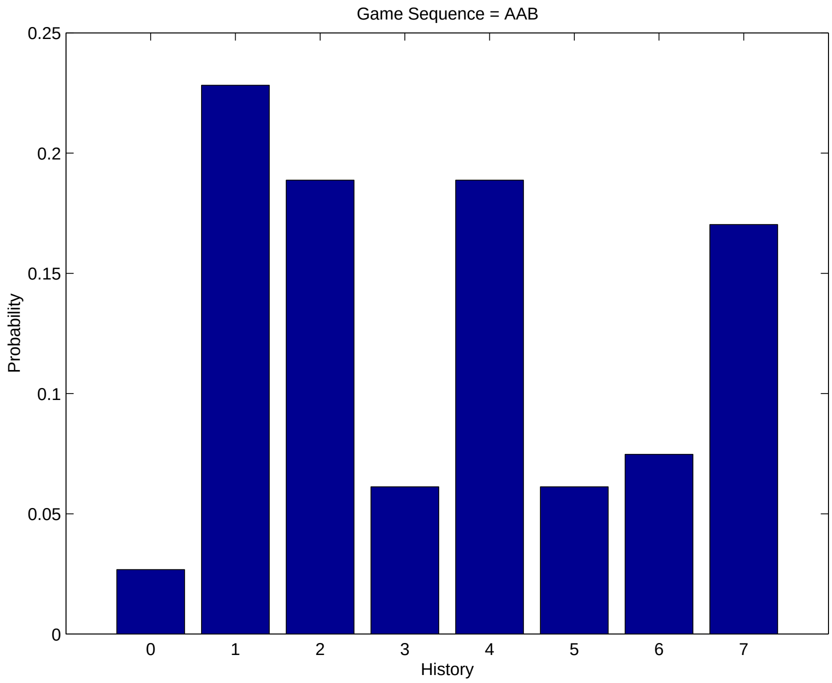

FIGURE 34.9. Classical Probability vs History plot of playing two games of A's followed by one game of B. The indices are binary strings converted into decimals. For example, the bar for 2 (010) is the probability of losing the first game, winning the second, and losing the third.

34.6 Discussion

What we have done so far is essentially the same as calculating each of the possible outcomes by multiplying the respective probabilities. So what we have here is a quantum game that produces classical results, where the two losing games combine to create a winning game (Fig. 34.11). It is a quantum simulation of a classical system. However, on a classical computer, to calculate every possible history for n games require 2^n bits. On a quantum computer, on the other hand, only n qubits are required: an improvement of $\log(n)$. The rotation matrices are 2^nx2^n because it is a classical way of representing, calculating and simulating quantum processes.

Looking back at Fig. 34.2, it is natural to ask what happens if we extend the axes to allow the coefficients of $|0\rangle$ and $|1\rangle$, a and b, to be negative and/or complex? These complex amplitudes are taken into account by a phase factor $e^{i\varphi}$ which is inserted into the rotation matrices. So the basic rotation matrix, $\hat{\Theta}(\theta)$, becomes $\hat{\Theta}(\theta,\varphi) = \begin{bmatrix} e^{i\varphi}\cos(\theta) & \sin(\theta) \\ \sin(\theta) & e^{-i\varphi}\cos(\theta) \end{bmatrix}$.

Since the actual probabilities are the lengths of the state kets, these complex amplitudes will still produce classical results under normal circumstance. How-

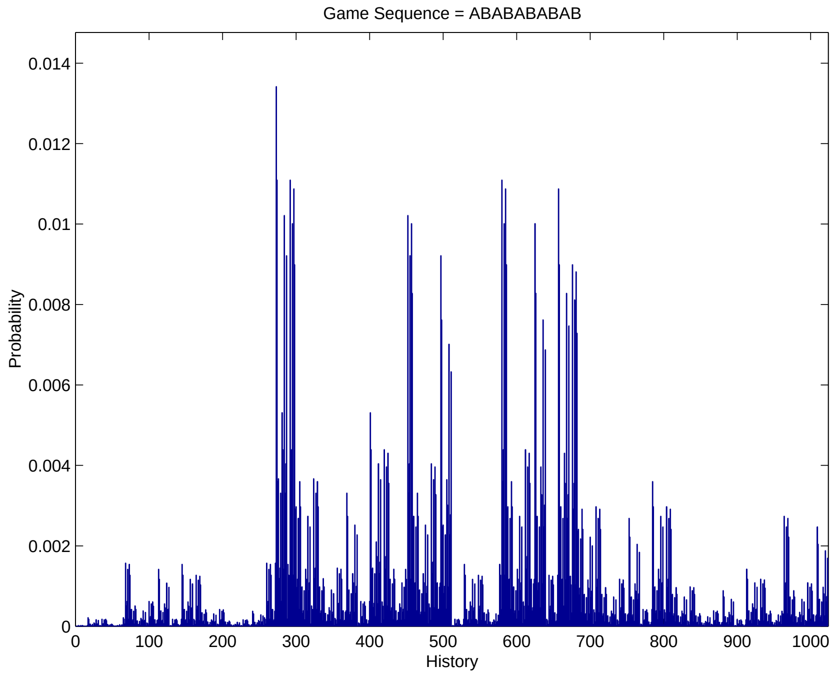

FIGURE 34.10. Probability vs History plot of playing 10 games. These games are played by playing two games of A's followed by two games of B's and so on.

ever, if made to interact with each other, they can produce radically different results. Two probability amplitudes of the same magnitude and different signs can cancel each other out, resulting in destructive interference. This will not happen in classical game theory because classical probabilities are always real and positive.

So how do we cause these quantum probability amplitudes to interact? Eisert's approach [2] was to employ an entangling gate, J, which calculates the payoff of the two parties. By varying the entangling parameter (which is essentially a phase parameter) in J, Eisert's result showed that the classical problem of Prisoner's dilemma is a subset of the quantum game, and there is no longer a dilemma when the game is fully explored in the quantum regime.

Parrondo's game can be seen as a competition between two players: Casino(C) and Parrondo(P). Both Parrondo's and Casino's aim is to maximise the winnings or minimise the losses. As mentioned above, despite the Casino's Game A and Game B being originally unfairly biased against Parrondo, he can construct a combined winning game by playing the games in certain sequences. However, one of the Casino's business strategists has read Parrondo's published paper, and brought the issue up at the next Casino board meeting. At that meeting, it was decided that the Casino should employ quantum mechanics to help them turn the tables back in their favour. This was done by implement-

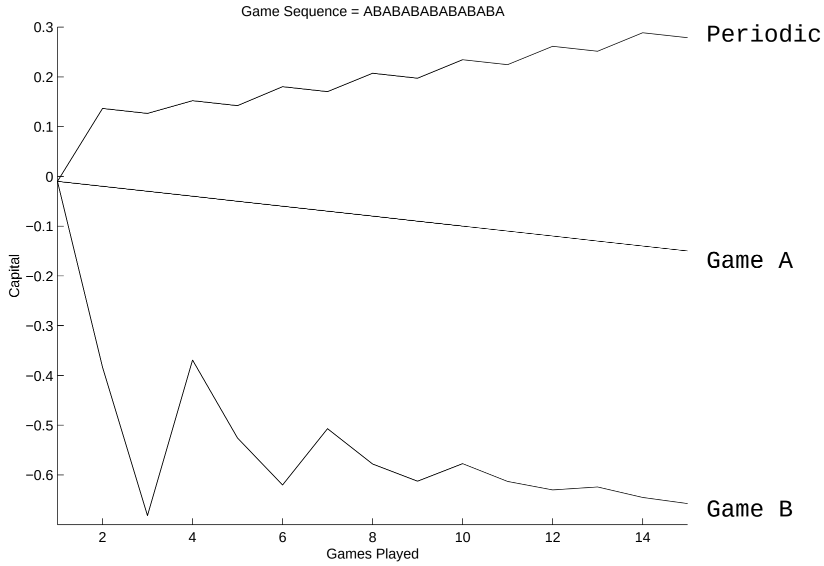

FIGURE 34.11. When we combine the two losing games (Game A and Game B) in a periodic fashion (alternate Game A and B), a winning game results. This result is identical to that of classical simulations [7].

ing a Casino Gate (Fig. 34.12), $C(\varphi)$, and a "de-Casino" gate $C^{\dagger}(\varphi)$, where $C(\varphi) = C^{\dagger}(\varphi) = \begin{bmatrix} \cos(\gamma) & \sin(\gamma) \\ -\sin(\gamma) & \cos(\gamma) \end{bmatrix}$. For n games, the resultant state is $|\psi_f\rangle = C^{\dagger}(\gamma)G(n)C(\gamma)|0\rangle^{\otimes n}$ where $G(n)$ is the collection of quantum gates that describes the sequence of games played. This can be thought of as the player walking through the doors from the classical world into the quantum Casino, and later, from the casino back into the classical world. By setting $\gamma = \frac{\pi}{2}$, Game A remains the same but Game B becomes a winning game, yet through Parrondo's strategy, the combined game is now a losing game (Fig. 34.14). Interestingly, although Game B wins faster than Game A, the combined game is still losing. In fact, it loses faster than just playing Game A on its own. This is a different paradox to the original!

However, it didn't take long for Parrondo to realise that this sudden change of fortune is not simply a statistical abnormality, but rather, due to the Casino's quantum strategy. So he decides to beat the Casino at their own game again, and adopts a quantum strategy as well. This is done with a phase-shift gate (Fig. 34.13), $P(\varphi) = \begin{bmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{bmatrix}$. Now, the resultant state is $|\psi_f\rangle = C^{\dagger}(\gamma)G(n)P(\varphi)C(\gamma)|0\rangle^{\otimes n}$. This causes both Game A and Game B to become winning games, and combine to create a winning overall game. So when both the Casino and Parrondo adopt quantum strategies, the original Parrondo game is no longer a paradox.

So as can be seen, Parrondo's best strategy lies in employing a quantum strategy. He is guaranteed to win regardless of the strategy used by the casino (Fig. 34.15). Unfortunately (or fortunately, depending on how one prefers to see

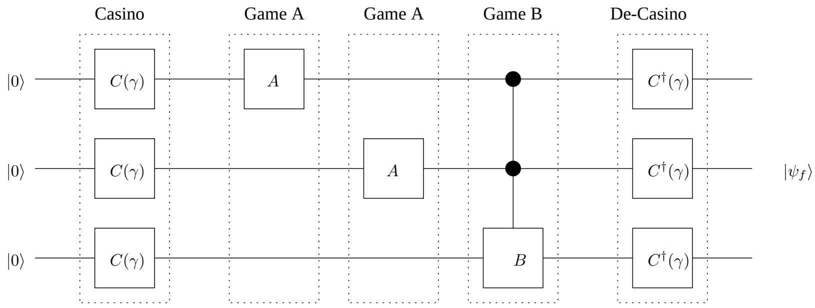

FIGURE 34.12. The Casino adopts a quantum strategy by employing entangling gates, $J(\gamma)$. The degree of entanglement is determined by γ, with $\gamma = 0$ representing no entanglement while $\gamma = \frac{\pi}{2}$ representing maximum entanglement.

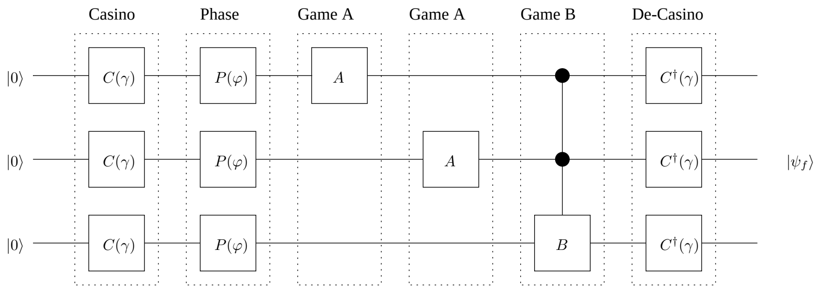

FIGURE 34.13. The player also adopts a quantum strategy by employing phase gates, $P(\varphi)$.

the situation), the same cannot be said for the Casino however. If the casino adopts a quantum strategy, Parrondo can choose a classical strategy and play only the winning Game B or a quantum strategy and still win.

34.7 Conclusion

Parrondo's games are of general interest as they illustrate how two losing coin tossing games can win when combined either in deterministic or non-deterministic sequences. For this phenomenon to occur, there must be coupling between the games. In Section 34.2 we saw that the CD games couple via capital-based state-dependence and the HD games couple via history-based state-dependence. The open question is, can a quantum Parrondo game be designed such that the coupling is via quantum entanglement?

For the case of non-deterministic sequences of games A and B, Game A can be thought of as "noise" that breaks up the state-dependent rules that are biassing Game B to lose - and this is why the combination of A and B wins ("the Boston Interpretation"). So another open question for quantum Parrondo games is, can the effect of Game A be in fact replaced by some form of decoherence such as

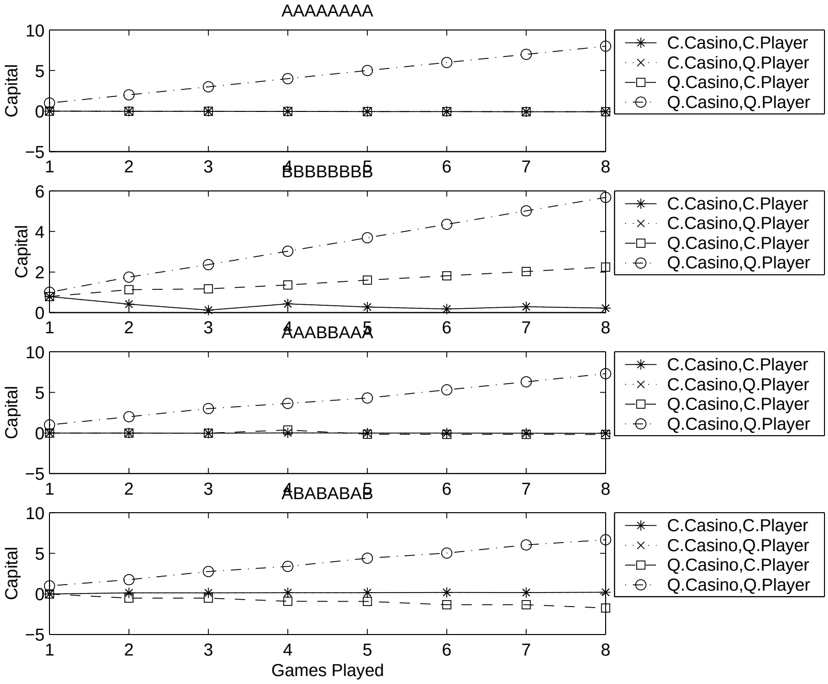

FIGURE 34.14. Plotting Capital vs Games for varying γ and φ and varying game sequences (as labeled). Classical Casino uses $\gamma = 0$, Quantum Casino uses $\gamma = \frac{\pi}{2}$. Classical Parrondo uses $\varphi = 0$, Quantum Parrondo uses $\varphi = \frac{\pi}{2}$. As can be seen, for a classical casino, the results are the same regardless of whether Parrondo uses a quantum coin or not.

a measurement?

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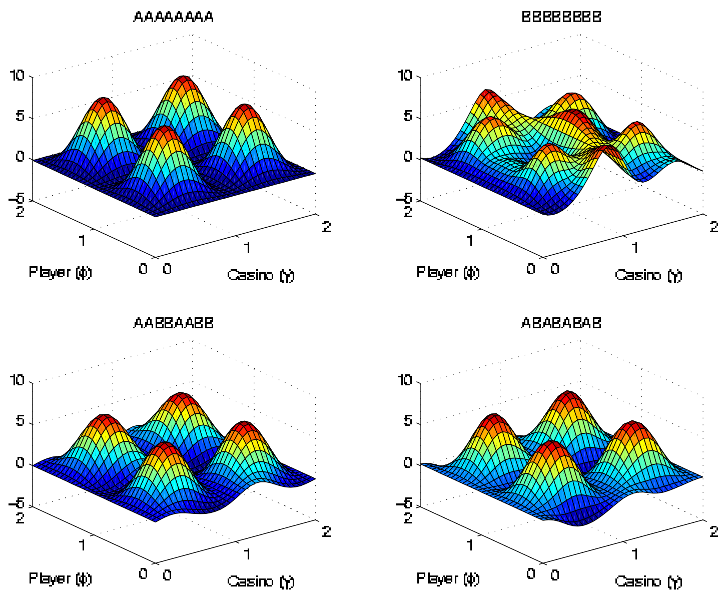

FIGURE 34.15. Plotting the capital after playing 8 games as labeled. The axes are plotted from 0 to 2π. γ is the casino's parameter, while φ is Parrondo's parameter. At $\gamma = \varphi = 0$, we have an entirely classical game. In fact, for all $\gamma = 0$, the results are the same as classical results, so in the plots, we have a straight line at $\gamma = 0$, regardless of what φ is.

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