Control systems with stochastic feedback

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In this paper we use the analogy of Parrondo's games to design a second order switched mode circuit which is unstable in either mode but is stable when switched. We do not require any sophisticated control law. The circuit is stable, even if it is switched at random. We use a stochastic form of Lyapunov's second method to prove that the randomly switched system is stable with probability of one. Simulations show that the solution to the randomly switched system is very similar to the analytic solution for the time-averaged system. This is consistent with the standard techniques for switched state-space systems with periodic switching. We perform state-space simulations of our system, with a randomized discrete-time switching policy. We also examine the case where the control variable, the loop gain, is a continuous Gaussian random variable. This gives rise to a matrix stochastic differential equation (SDE). We know that, for a one-dimensional SDE, the difference between solution for the time averaged system and any given sample path for the SDE will be an appropriately scaled and conditioned version of Brownian motion. The simulations show that this is approximately true for the matrix SDE. We examine some numerical solutions to the matrix SDE in the time and frequency domains, for the case where the noise power is very small. We also perform some simulations, without analysis, for the same system with large amounts of noise. In this case, the solution is significantly shifted away from the solution for the time-averaged system. The Brownian motion terms dominate all other aspects of the solution. This gives rise to very erratic and "bursty" behavior. The stored energy in the system takes the form a logarithmic random walk. The simulations of our curious circuit suggest that it is possible to implement a control algorithm that actively uses noise, although too much noise eventually makes the system unusable. © *2001 American Institute of Physics.* [DOI: 10.1063/1.1397769]

The starting point for the present line of inquiry is a pair of games of chance called Parrondo's games. It is possible to combine two losing games to create a new process which is winning. In this paper we extend the apparent paradox of Parrondo's games to the case of a real physical system, obeying the laws of conservation of energy and charge. The flow of "reward" in Parrondo's games is replaced with a flow of energy in a physical circuit. It should be clear that we have to be very careful when we assign labels, like "stable," to "games." Naive mental images may provide an initial motivation but are not enough to complete the analysis. We apply Lyapunov's direct method to illustrate that counterintuitive behavior does exist for some physical circuits.

I. INTRODUCTION

The Spanish physicist, Juan Parrondo, has devised a pair of games of chance[1] that can simulate the salient features of a Brownian ratchet.[2] These games also have interest in their own right. The curious feature of Parrondo's games is that an indefinite homogeneous sequence of either of the individual games gives rise to a process that is losing and yet a *mixed* random sequence of the two games gives rise to a process that is winning. There should be a clear distinction between the games themselves and the process that we create when we play a sequence of these games. The process has a dual effect on the internal state and on the immediate reward. This dual effect is nonlinear and gives rise to the apparent paradox.

In this paper we indicate the deep similarities between systems governed by randomly selected Markov operators, such as Parrondo's games,[1,3–6] systems governed by time-varying transition matrices, such as switched-mode circuits,[7–9] and physical systems with randomized control laws.[10–13] The defining property of Parrondo's games is that it is possible to combine two losing games to achieve a winning result.

The properties of "winning" and "losing," can be investigated by studying the geometric and topological properties of certain sets within the parameter space of the system. If we visualize Parrondo's games appropriately then it is apparent that boundary between the winning and losing regions of the parameter space is not planar. The winning and losing regions are not convex, as was first suggested by Moraal.[10] This is described in more detail in the Appendix. The analogous result for a switched-mode device is that it is possible to combine two unstable systems together to achieve a stable result. The unstable region is not necessarily convex.

The analogy between Parrondo's games and switched-mode systems can be made more rigorous if we consider the mathematical structures that they have in common:

(a) The macroscopic state of each system at each moment of time can be completely described by a state vector, $\mathbf{X_t}$.

(b) The time evolution of both systems is governed by an indefinite sequence of randomly selected transition operators,

$$\mathbf{X_{t+\Delta t}} = A_t \cdot \mathbf{X_t}, \tag{1}$$

where A_t is the transition operator that applied at time t.

(c) We can classify the responses of the systems in terms of asymptotic rates of flow of conserved quantities. In the case of a switched-mode system, we can consider power, $E[\dot{U}]$, as a flow of internal stored energy, U, where $E[x]$ is the expected value of x. If the mean rate of flow is always inwards, without bound, then system will accumulate an indefinite amount of energy and must be unstable. In the case of Parrondo's games, we must consider the flow of "reward" to determine whether the games are winning or losing. If the flow of reward is away from the player, without bound, then the game is losing.

(d) The effect of switching is to generate a new time-averaged system, which will be governed by a linear convex combination of the original transition operators.[14]

(e) The rate of flow associated with the time-averaged system is generally *not* the same as the time average of the flows associated with the original transition operators. The rate of flow is a nonlinear function of the transition operators.

The losing region of the parameter space for Parrondo's games is not convex. We show that it is possible to construct a simple "toy" switched-mode system which has a nonconvex unstable region in its parameter space. For the sake of simplicity, we limit the system to one free parameter which is a loop gain, K. The main body of this paper contains five key sections:

(1) The construction of a simple switched-mode system with a nonconvex unstable region in the parameter space.

(2) The formulation of this system in terms of a state vector $\mathbf{X_t}$ and two transition operators A_1 and A_2.

(3) The determination of the internal stored energy as a quadratic function of the state vector,

$$U = \mathbf{X}^T P \mathbf{X} \tag{2}$$

for some positive definite matrix P. This energy function can be used as a Lyapunov function.

(4) The proof of instability of processes governed by the original pure transition operators A_1 and A_2.

(5) The proof of stability, with probability one, of processes governed by a randomly selected mixed sequence of transition operators A_1 and A_2.

This shows that the Parrondo effect applies, with rigor, to at least one real switched-mode electronic system.

Simulations indicate that the particular system which we constructed has further interesting properties. We show the key results from the simulations and speculate, without proof, about some interesting open questions which (we believe) are worthy of future investigation.

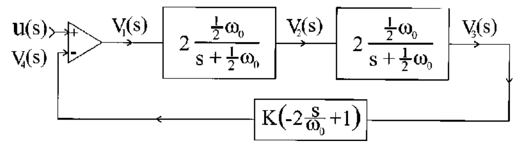

FIG. 1. General plan of a second-order system with one feedback loop.

II. CONSTRUCTION OF A SIMPLE SWITCHED-MODE SYSTEM

Our immediate aim is to design a simple "toy" system in the s or "Laplace" domain which has a nonconvex unstable region in the parameter space. We achieve this by constructing a system with a disjoint unstable region in the parameter space. For simplicity, we choose a parameter space with a single free variable, a loop gain, K.

If a linear system, or plant, is placed inside a feedback control loop then a new system, with new properties, is created. A simple system topology is shown in Fig. 1. We can write the equations for this system as

$$F(s)^{-1} = G(s)^{-1} + K \cdot H(s), \tag{3}$$

where $G(s)$ is called the open loop transfer function and $F(s)$ is called the closed loop transfer function. The loop gain, K, is a free parameter and $H(s)$ is the transfer function of the return path. For this particular system, we have

$$G(s) = \frac{(\omega_0)^2}{(s + \frac{1}{2}\omega_0)^2} \tag{4}$$

and

$$H(s) = K \cdot \left(-2\frac{s}{\omega_0} + 1 \right). \tag{5}$$

It is customary to analyze the stability of closed loop systems in terms of the poles of the closed-loop transfer function, $F(s)$, which are the zeros of $F(s)^{-1}$. These poles will generally move about in the complex plane in response to changes in the loop gain, K. A graph of the positions of the poles, as a function of gain, is called a "root locus" plot and is shown in Fig. 2. Some choices of gain may cause one, or

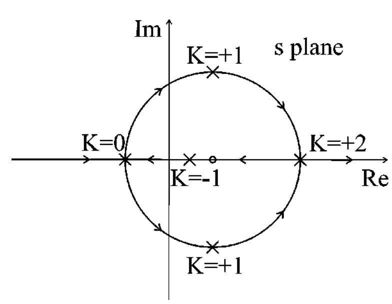

FIG. 2. Root locus plot for a second order system. The poles, in the s plane, for particular values of K are represented by crosses. The direction of movement of the poles, with increasing K, within the locus, is indicated by the arrows. The radius of the circle is ω_0. The neutral position for the plant corresponds to a pair of repeated poles at $s = -\frac{1}{2}\omega_0$.

more, of the poles to move into the unstable region, on the right-hand side of the s plane, which would mean that the closed loop system would then be unstable. This is the basis of the Hurwitz criterion.[15] In general, there will be stable and unstable values for the gain, K.

In this case, we can think of the ''neutral'' position of the system as being the case where $K=0$ and there are repeated poles at $s=-\frac{1}{2}\omega_0$. The system is stable in the ''neutral'' position. The general positions of the poles are given by the roots of the characteristic equation,

$$F(s)^{-1}=\left(\frac{s}{\omega_0}\right)^2+(1-2K)\left(\frac{s}{\omega_0}\right)+\left(\frac{1}{4}+K\right)=0. \tag{6}$$

Fortunately this is a quadratic function of s and we can readily calculate the loci of the roots,

$$s=\omega_0\cdot((K-\tfrac{1}{2})\pm\sqrt{K\cdot(K-2)}). \tag{7}$$

The loci of these roots of the characteristic equation, in the s plane, are shown on the root locus plot of Fig. 2. Some particular values of K have special interest. For $K=-1$ we get closed loop poles at $s=\omega_0(-\frac{3}{2}\pm\sqrt{3})$. The pole at $s=\omega_0(-\frac{3}{2}+\sqrt{3})$ is a positive real number and gives rise to the exponentially increasing response shown in Fig. 4. For $K=+1$ we get closed loop poles at $s=\omega_0(\frac{1}{2}\pm j)$ which have positive real parts and give rise to the exponentially increasing oscillations shown in Fig. 5.

It is clear that there is a range of stable values for K surrounded by two unstable ranges. Analysis of Eq. (7) reveals that the stable range of values for K is $(-\frac{1}{4}<K<+\frac{1}{2})$. The other intervals, $(-\infty<K<-\frac{1}{4})$ and $(+\frac{1}{2}<K<+\infty)$ are associated with unstable values of K. The unstable region, within the parameter space for K, is composed of two disjoint open intervals and is clearly not convex. Our choice for $G(s)$ was guided by the need to develop a very simple second order system with an appropriate root locus and a nonconvex unstable region in the parameter space for the loop gain K. We can think of the system with $K=-1$ as being unstable plant number 1. We can think of the system with $K=+1$ as being unstable plant number 2. The mean value of these two values of gain would be $K=0$ which corresponds to the neutral system, which is stable. We could switch rapidly between the two *unstable* control systems and we might expect that the result would be a *stable* control system that somehow corresponds to the neutral system.

We proceed to reformulate this simple switched-mode system in state space and to derive the necessary mathematical machinery to establish that the switched system actually is stable.

III. A SWITCHED STATE-SPACE FORMULATION

We formulate the system in terms of a state vector $\mathbf{X_t}$ and two transition operators A_1 and A_2. The choice of state variables is not unique. The strategy used here is to imagine the system $G(s)$ as being constructed of two function blocks in series,

$$G(s)=\left(\frac{+\omega_0}{(s+\frac{1}{2}\omega_0)}\right)\cdot\left(\frac{+\omega_0}{(s+\frac{1}{2}\omega_0)}\right). \tag{8}$$

The state variables, $\{V_1,V_2,V_3,V_4\}$ are the voltages at the outputs of the various function blocks shown in Fig. 1. Closer analysis reveals that only voltages V_2 and V_3 are needed to store the internal state of the system. All other variables can be written as linear combinations of these state variables. The state variables $\{V_2,V_3\}$ constitute a holonomic set of generalized coordinates for the system.

We can analyze the system using signal flow concepts which leads to a closed-loop state-space model for the whole system,

$$\dot{\mathbf{X}}=A\mathbf{X}+\mathbf{B}u, \tag{9}$$

where $\mathbf{X}$ is the state vector, A is the transition matrix, $\mathbf{B}$ is an input vector and u is an input voltage, shown in Fig. 1.

The state vector is composed of two state variables,

$$\mathbf{X}=\begin{bmatrix}V_2\\V_3\end{bmatrix}. \tag{10}$$

The transition matrix defines the way in which the system evolves over time,

$$A=\omega_0\cdot\begin{bmatrix}(+2K-\frac{1}{2}) & -2K\\ +1 & -\frac{1}{2}\end{bmatrix}. \tag{11}$$

We can think of A as being a function of K, $A=A(K)$. The input vector is

$$\mathbf{B}=\begin{bmatrix}+1\\0\end{bmatrix} \tag{12}$$

and the input voltage is $u(t)$. If we are only interested in the asymptotic stability of the system then can consider $u(t)$ to be simply a Dirac delta function, $u(t)=\delta(t)$. Alternatively, we could choose $u(t)=0$ and select initial conditions, $\mathbf{X}=\mathbf{X_0}$ at time, $t=0$. This approach leads to a homogeneous equation in time,

$$\dot{\mathbf{X}}=A\mathbf{X}. \tag{13}$$

All the simulations presented in this paper are for the homogeneous system described in Eq. (13), with nonzero initial conditions $\mathbf{X}=\mathbf{X_0}$.

We can make use of the two special values for A corresponding to the two special values of K discussed earlier, $K_1=-1$ and $K_2=+1$. We can define $A_1=A(K_1)$ and $A_2=A(K_2)$. We can also define the state transition matrix corresponding to the neutral position as $A_0=A(0)$. We note that A_0 is the average of A_1 and A_2 and $A_0=\frac{1}{2}(A_1+A_2)$. We can now imagine an inhomogeneous process where we switch at random with equal probability between the two systems defined by transition matrices A_1 and A_2 at regular time intervals, ΔT. The time evolution of such a system can be simulated using a discrete time model,

$$\mathbf{X_{t+\Delta T}}=\exp(A_t\cdot\Delta T)\mathbf{X_t}, \tag{14}$$

where $\exp(A_t\cdot\Delta T)$ is the matrix exponential function, applied to the matrix $A_t\cdot\Delta T$. We can compare Eq. (14) with Eq. (1). The matrix exponential function can be evaluated numerically, using power series, or algebraically, using Laplace transform techniques. The symbol A_t represents the transition operator that applies at time t, which will either be A_1 or A_2. In this sense, we now consider A to be a function of t although it only takes one of two values. The stability of this stochastic inhomogeneous system cannot be analyzed using linear techniques, like the Hurwitz criterion. We proceed to use Lyapunov's second, or direct method, to analyze this problem.

IV. INTERNAL STORED ENERGY

We can represent the internal stored energy of the system as a quadratic function of the state vector. Using notation from Levine,[15] we can define the internal stored energy as

$$U = \mathbf{X}^T P \mathbf{X}, \tag{15}$$

where X is the state vector and P is a positive-definite matrix, called an "energy matrix." If we differentiate the stored energy along the trajectories of the system, as defined by Eq. (13), then we get

$$\dot{U} = \mathbf{X}^T Q \mathbf{X}, \tag{16}$$

where

$$A^T P + PA = -Q, \tag{17}$$

and Eq. (17) is called the "Lyapunov equation." The choices of P and Q are related through the Lyapunov equation but we are *free* to choose one of them.

In order to construct a workable Lyapunov function, we begin with the stored energy in the feed forward path. We use the fact that the stored energy in a capacitor is $U = \frac{1}{2} C V^2$ and we find that the simplest possible construction will work. We can use Eq. (15) where

$$P = \begin{bmatrix} \frac{1}{2} C_{11} & 0 \\ 0 & \frac{1}{2} C_{22} \end{bmatrix}. \tag{18}$$

We can think of C_{11} and C_{22} as being physical capacitors in the feed forward path. The other circuit variables, $\{V_1, V_4\}$ are linear combinations of the state variables, $\{V_2, V_3\}$ and entire energy in the circuit, including the feedback path, can be expressed in terms of state variables only. The energy in the feedback path makes no fundamental difference to the stability argument.

If we use Eq. (17) to solve for the power matrix, Q, then we get

$$Q = \omega_0 \cdot \begin{bmatrix} (\frac{1}{2} C_{11} - 2KC_{11}) & (KC_{11} - \frac{1}{2} C_{22}) \\ (KC_{11} - \frac{1}{2} C_{22}) & (\frac{1}{2} C_{22}) \end{bmatrix}. \tag{19}$$

We require this matrix to be positive definite for some range of values of K. We can establish when the matrix is positive definite by evaluating all the top left hand minor determinants of Q. We get: $\Delta_1 = C_{11}(\frac{1}{2} - 2K)$ and $\Delta_2 = \frac{1}{4} C_{22}(C_{11} - C_{22}) - K^2 (C_{11})^2$. We can obtain the largest admissible range of values for K if we choose $C_{11} = 2C$ and $C_{22} = C$ for some standard capacitance C. This gives an admissible range of values of K as

$$-\tfrac{1}{4} < K < +\tfrac{1}{4}. \tag{20}$$

We can use this Lyapunov function to establish that the system is stable when K is in the admissible range. Since the unswitched system is linear, we can actually calculate a larger range of values for which the unswitched system is stable, using the Hurwitz criterion, $-\frac{1}{4} < K < +\frac{1}{2}$. This is larger than the admissible range for the present Lyapunov function, which we can only use when $-\frac{1}{4} < K < +\frac{1}{4}$. We know that the present Lyapunov function is adequate in the smaller range.

We can think of ω_0 as a characteristic frequency for the system and $R_0 = 1/(\omega_0 C)$ as a characteristic resistance. This implies that Eq. (16), describing the rate of change of stored energy, is dimensionally consistent with Joule's Law, $\dot{U} = \partial U / \partial t = V^2 / R_0$.

We can consider the system near its neutral position, when $K = 0$ and $A = A_0$. Lyapunov's theorem establishes that the system A_0 is stable since both P and Q are positive definite. It seems desirable to test this analytical result. We simulated the system using the values of $K = 0$, $A = A_0$, the value of P from Eq. (18) and the value of Q from Eq. (19). The results are shown in Fig. 3. The energy is always positive, since P is positive definite. The energy is always decreasing which is consistent with the fact that the power $\dot{U}$ is always negative. This is also consistent with the fact that Q is positive definite. This was found to be true for a variety of initial conditions, $\mathbf{X_0}$.

We note that there is no stochastic element in the simulation in Fig. 3. This is only a simulation of the time-averaged plant A_0 and is not sufficient to establish the stability of the stochastic inhomogeneous process where A_1 and A_2 are chosen at random.

V. PROOF OF INSTABILITY OF PLANTS: "A_1" AND "A_2"

The plants "A_1" and "A_2" were *designed* to be unstable. This is clearly supported by simulations. Figure 4 shows a simulation of the plant A_1. All variables clearly diverge exponentially to infinity. Figure 5 shows a simulation of the plant A_2. All variables diverge to infinity in an exponentially growing sinusoidal fashion. The formal proof for plants "A_1" and "A_2" is straightforward. Neither A_1 nor A_2 are Hurwitz matrices. This is clear if we examine the characteristic polynomials in Eq. (6) and evaluate the roots in Eq. (7). There is nothing stochastic about these equations. There are no averages or expected values involved.

VI. PROOF OF STABILITY OF THE STOCHASTICALLY MIXED PROCESSES

Simulations strongly suggest that the mixed process should be stable but this is not a proof. A sample path is shown in Fig. 6. The system clearly appears to converge to the point $\mathbf{X}^T = [0,0]$ in the state-space. We note that the instantaneous power may vary greatly and is often positive. We also note that the "average" power is always decreasing, this

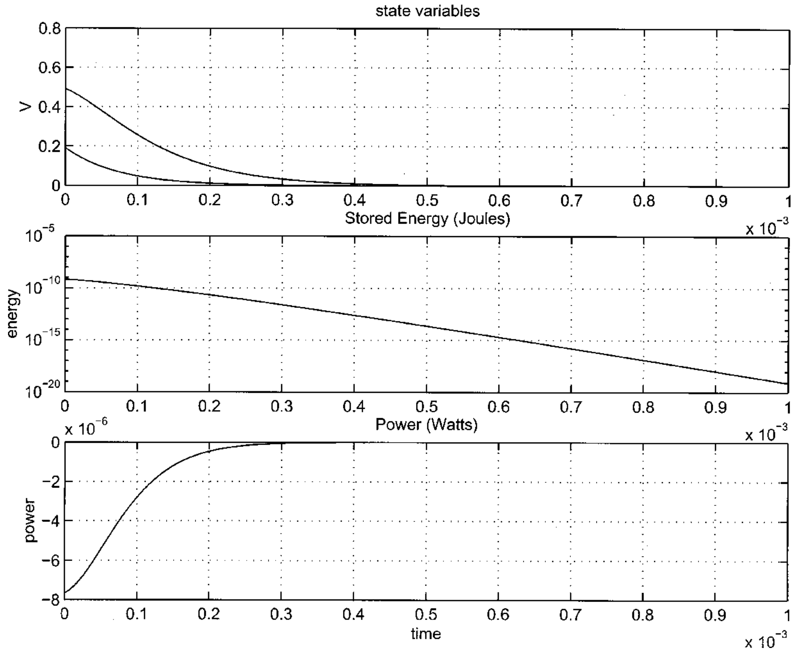

FIG. 3. Discrete state-space simulation of system A_0. The state variables V_2 and V_3 are shown in the top of the figure. The stored energy is shown on a logarithmic scale in the middle and the power dissipation is shown in the bottom graph. All units are SI and correspond to a characteristic frequency of about 2.2 kHz, and a characteristic resistance of 33 kΩ. There are two curves in the upper graph due to the two state variables.

is supported by the fact that the curve for stored energy is decreasing in some average sense. We need to make these ideas more precise.

There is a theorem due to Kushner, which is reproduced in Levine[15] on page 1108, which states that: "The mixed system is stable with probability one if: $\mathcal{L}U \leqslant 0$ and $U \geqslant 0$, where $\mathcal{L}$ is the infinitesimal generator for the process,

$$\mathcal{L}U(\mathbf{X_0}) = \lim_{\Delta t \to 0} \frac{E[U(\mathbf{X_{\Delta t}})] - U(\mathbf{X_0})}{\Delta t}, \tag{21}$$

where $E[\mathbf{X}]$ is the expected value of $\mathbf{X}$." We can make use of the fact that $E[U(\mathbf{X_0})] = U(\mathbf{X_0})$ when $U(\mathbf{X_0})$ is known so we can write

$$\mathcal{L}U(\mathbf{X_0}) = \lim_{\Delta t \to 0} \frac{E[U(\mathbf{X_{\Delta t}})] - E[U(\mathbf{X_0})]}{\Delta t} \tag{22}$$

$$= \lim_{\Delta t \to 0} E\left[\frac{U(\mathbf{X_{\Delta t}}) - U(\mathbf{X_0})}{\Delta t}\right]. \tag{23}$$

This reduces to

$$\mathcal{L}U(\mathbf{X_0}) = E\left[\frac{\partial U(\mathbf{X})}{\partial t}\right] = E[\dot{U}(\mathbf{X})] \tag{24}$$

wherever the limit exists, at the point in state-space,

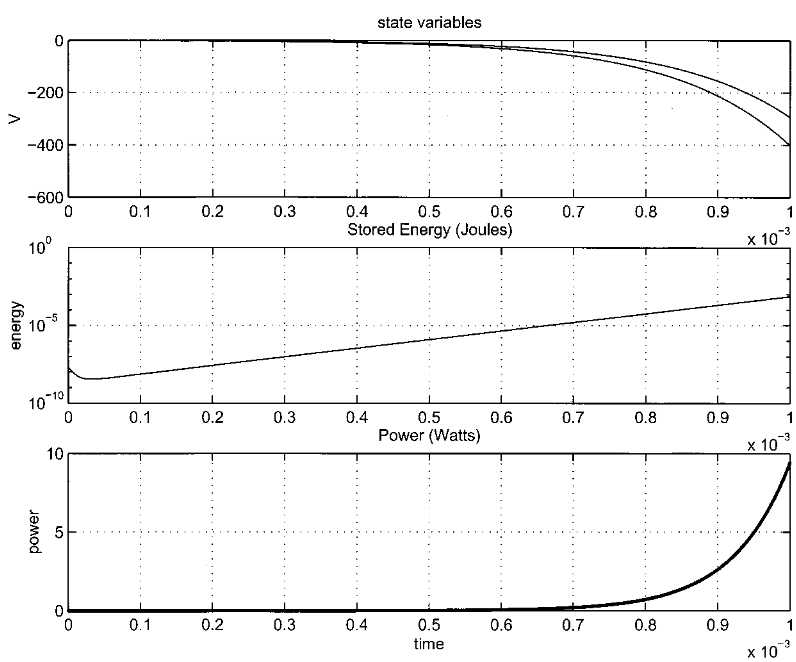

FIG. 4. Discrete state-space simulation of system A_1. The system clearly has a real exponential unstable mode.

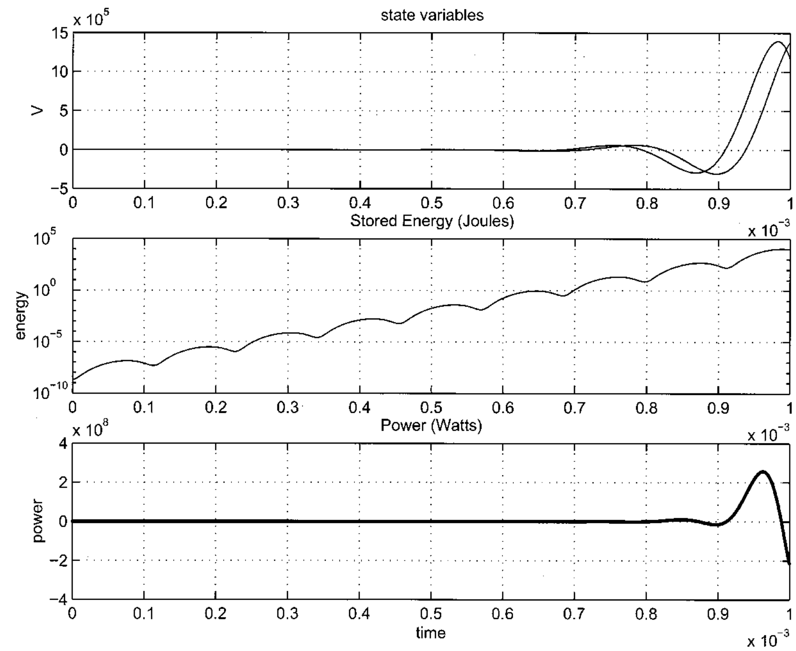

FIG. 5. Discrete state-space simulation of system A_2. The system clearly has a complex exponential unstable mode.

$\mathbf{X}=\mathbf{X_0}$. This raises the question of whether or not $E[\dot{U}(\mathbf{X})]$ converges uniformly. We recall Eq. (2) so we can write

$$E[U(\mathbf{X})]=E[\mathbf{X}^TP\mathbf{X}] \tag{25}$$

but $\dot{\mathbf{X}}=A\mathbf{X}$, where $A=A_1$ or $A=A_2$ so

$$E[\dot{\mathbf{X}}]=E[A\mathbf{X}] \tag{26}$$

$$=E[A]\mathbf{X}. \tag{27}$$

Equation (24) now reduces to

$$\mathcal{L}U(\mathbf{X})=E\left[\frac{\partial U(\mathbf{X})}{\partial t}\right] \tag{28}$$

$$=\mathbf{X}^T(E[A]^TP+PE[A])\mathbf{X}. \tag{29}$$

We are choosing $A=A_1$ or $A=A_2$ at random with equal probability so $E[A]=\frac{1}{2}(A_1+A_2)=A_0$ and we arrive at a simple expression for $\mathcal{L}U(\mathbf{X})$,

$$\mathcal{L}U(\mathbf{X})=+\mathbf{X}^T(A_0^TP+PA_0)\mathbf{X} \tag{30}$$

and we know from Eq. (20) that this is negative since $Q_0=-A_0^TP-PA_0$ corresponds to the case with $K=0$ and is positive definite. This can all be summarized by the following statement:

$$\mathcal{L}U(\mathbf{X})=E\left[\frac{\partial U(\mathbf{X})}{\partial t}\right]=-\mathbf{X}^TQ_0\mathbf{X}\leqslant 0. \tag{31}$$

We have $\mathcal{L}U\leqslant 0$ and $U\geqslant 0$ so, applying the theorem from Kushner, *the mixed system will be stable with probability of*

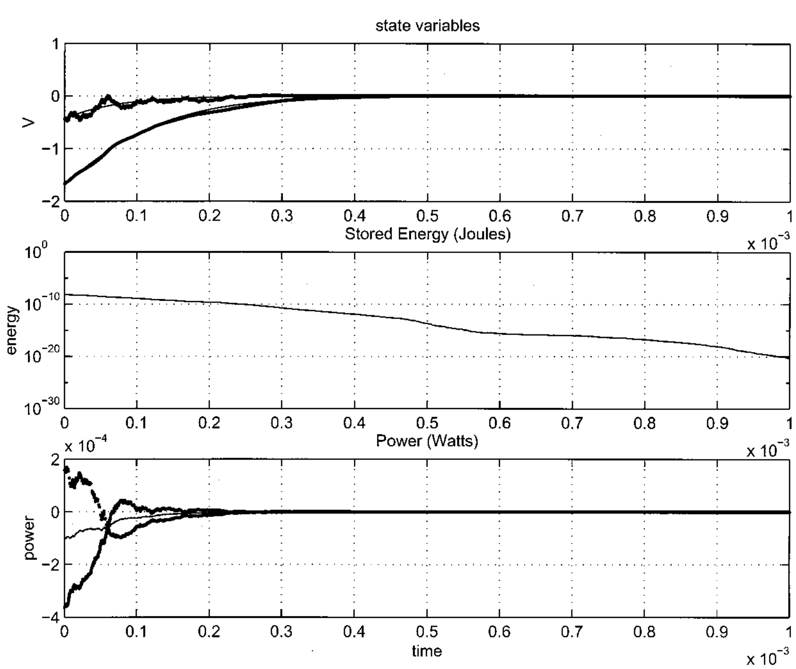

FIG. 6. Discrete state-space simulation of the randomly switched system. The response of the time-averaged system is included for comparison. The instantaneous power may diverge wildly from the expected value of the power.

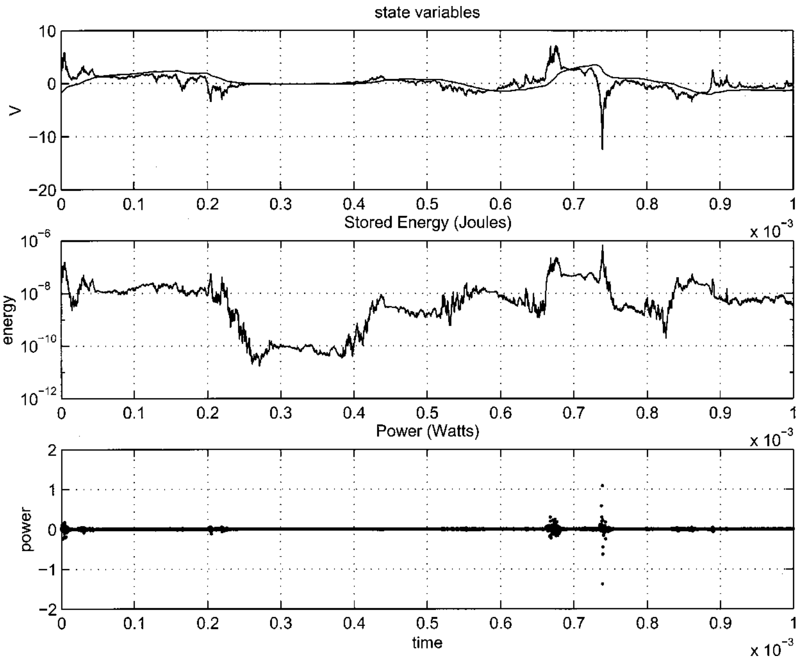

FIG. 7. System with large variance in K. The stored energy function is similar to a logarithmic Brownian motion.

one. A simulation of this process is shown in Fig. 6. The stored energy in the system does increase, about half of the time for a short intervals, but the overwhelming effect is a consistent reduction on stored energy. The presence of switching noise implies that the instantaneous power can be quite large even though the expected value is very small and negative.

We have constructed a switched-mode system in which both pure "modes" are unstable but the random mixture of the two modes is stable. This shows that the Parrondo effect can be applied to energy flow in at least one real physical system.

VII. SOME INTERESTING PROPERTIES OF THE MIXED PROCESS

Up to this stage, we have regarded the source of uncertainty as being a sequence of discrete choices, $K \in \{K_1, K_2\}$, at fixed sampling intervals, ΔT. This is somewhat artificial. It is likely that real-world systems with stochastic feedback would not be restricted to two values of gain and would not be clocked. This raises the interesting question of what would happen if the loop gain were a continuous random variable and the system operated in continuous time. The central limit theorem would suggest that the natural noise signal to consider would be Gaussian white noise. We would also expect that real physical systems would have finite noise power. We can represent the noise in the loop gain K using a stochastic model, $K = K_0 + \sigma dB$. The symbol dB represents white noise which is equivalent in measure to an infinitesimal increment of Brownian motion in the stochastic calculus of Itô.[15,16] We can think of σ^2 as being the noise power in the signal $K = K_0 + \sigma dB$.

If a signal with very large noise power, σ^2 is fed into a *linear* system then the result, at the output, will be a signal with large variance. The system will not suddenly become unstable. In contrast, we find that for our "toy" switched system; if the feedback variable, K, is chosen as a random Gaussian variable (and we make the variance large enough) then we can drive the system into instability using only variance, or noise power. This is qualitatively different from the linear system with noise at the input. In simulation, the output from the system appears to be made up of "bursts" of oscillation. The size of the "bursts" increases without limit if we allow the simulation to run for long enough.

If we choose a value of the variance which is near the limit of stability then we get the very complex output dynamic shown in Fig. 7. It is difficult to reconcile this type of output with the narrow band noise that we would expect from a linear system with stochastic input. In particular, the oscillations seem to "die" completely, only to return again in "bursts" at later times. It would appear that the Gaussian random variation in loop gain, K is a nonlinear element that fundamentally alters the behavior of the system.

The full analysis of this system is complicated. We sketch an approach here and support our ideas with some simulations.

If we apply a very small amount of noise to the loop gain K then the result is qualitatively very similar to additive white noise. This can be seen from the periodograms in Fig. 8. The stochastically switched system is *different* to the stationary filter with a white noise input, but it may be possible to use similar techniques to identify the open loop transfer function of an unknown system, provided that the variance, or noise power, is small. We can make some analogy between our two dimensional "toy" system and a system governed by a one-dimensional stochastic differential equation (SDE).

The classical one-dimensional linear SDE can be written as

$$dx(t) = a \cdot x(t)dt + \alpha x(t)dB, \tag{32}$$

where x is the dependent variable, which is analogous to $\mathbf{X}$.

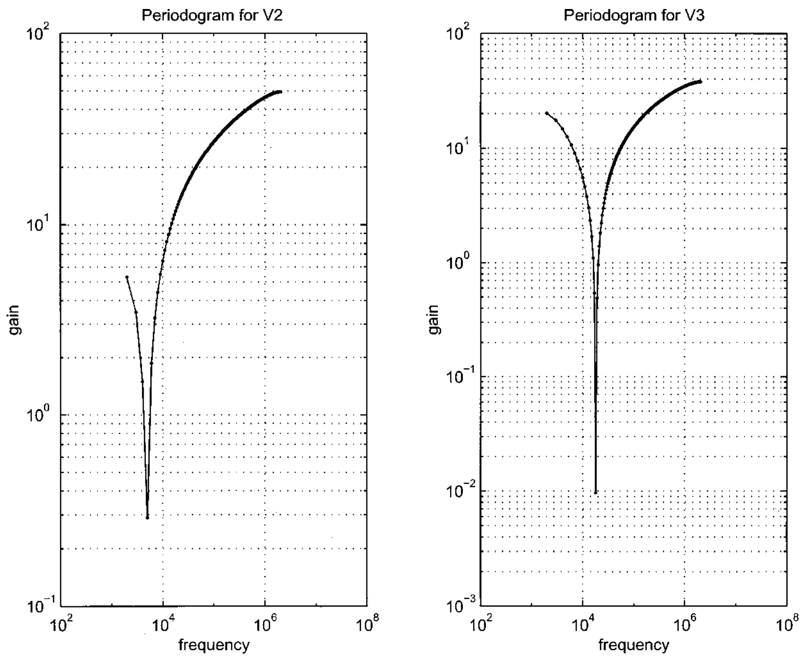

FIG. 8. Spectral analysis of the state variables with small amounts of white noise, $\alpha=0.01$. Small amounts of multiplicative noise are similar to small amounts of additive noise.

The independent variable is time, t. The state transition is governed by a rate a which is analogous to the state transition matrix A and there is a noise term αdB which is analogous to the noise in K. The notation dB represents an infinitesimal change in Brownian motion. It really represents a limiting process in the stochastic calculus of Itô. The solution to this simple SDE is given in Øksendal,[16]

$$x(t)=x_0\ \exp((a-\tfrac{1}{2}\alpha^2)t+\alpha B_t), \tag{33}$$

where B_t represents Brownian motion. If $\alpha \ll 1$, then $\frac{1}{2}\alpha^2 \to 0$ much faster than $\alpha \to 0$ so we can write

$$x(t)\approx x_0\ \exp(at+\alpha B_t) \tag{34}$$

$$=x_0\ \exp(at)\cdot\exp(\alpha B_t), \tag{35}$$

but if α is small then we can use the linear terms of a Taylor's expansion, $\exp(\alpha B_t)\approx 1+\alpha B_t$ and we can write

$$x(t)\approx x_0\ \exp(at)+x_0\ \exp(at)\cdot\alpha B_t. \tag{36}$$

The solution is approximately the solution to the "clean" or nonstochastic DE with an added term which looks like Brownian motion with a scaling factor. We can write

$$\frac{x(t)}{x_0\ \exp(at)}-1\approx\alpha B_t. \tag{37}$$

So the relative error between the solution to the SDE and the nonstochastic ODE should be similar to Brownian motion. We point out that dB has a white noise spectrum and that B is essentially the integral of dB which has a $1/f$ spectrum.

A simulation of the relative offsets for our "toy" second order system, with $\alpha=0.01$ is shown in Fig. 9. The relative offset for V_2 looks very similar to a sample path from Brownian motion. The offset for V_3 looks similar but has clearly been filtered. This should be clear from an examination of Fig. 1 and Eq. (8). We have

$$V_3(s)=\left(\frac{+\omega_0}{(s+\frac{1}{2}\omega_0)}\right)\cdot V_2(s). \tag{38}$$

The offsets for the two-dimensional system would appear to be filtered or conditioned Brownian motion.

If we have $K=\sigma dB$ then we can rewrite Eqs. (11) and (13) as

$$d\mathbf{X}=A_0\mathbf{X}dt+N_0\mathbf{X}\sigma dB, \tag{39}$$

where

$$A_0=\omega_0\cdot\begin{bmatrix}-\frac{1}{2} & 0\\ +1 & -\frac{1}{2}\end{bmatrix} \tag{40}$$

and

$$N_0=\omega_0\cdot\begin{bmatrix}+2 & -2\\ 0 & 0\end{bmatrix}, \tag{41}$$

and σdB is an infinitesimal increment in Brownian motion. The analytic solution to Eq. (39) is nontrivial but we suggest that the stochastic calculus of Itô would be the most appropriate tool for the full analysis of the system because it provides techniques for systematically handling noise terms. This task is still an open question for future work.

VIII. CONCLUSIONS AND OPEN QUESTIONS

We make the following claims for our simple system:

(1) We can synthesize a stable system by switching between two unstable systems. The system is even stable if it is switched at random.
(2) It is possible to implement a control algorithm that actually *uses* noise as a switching policy.

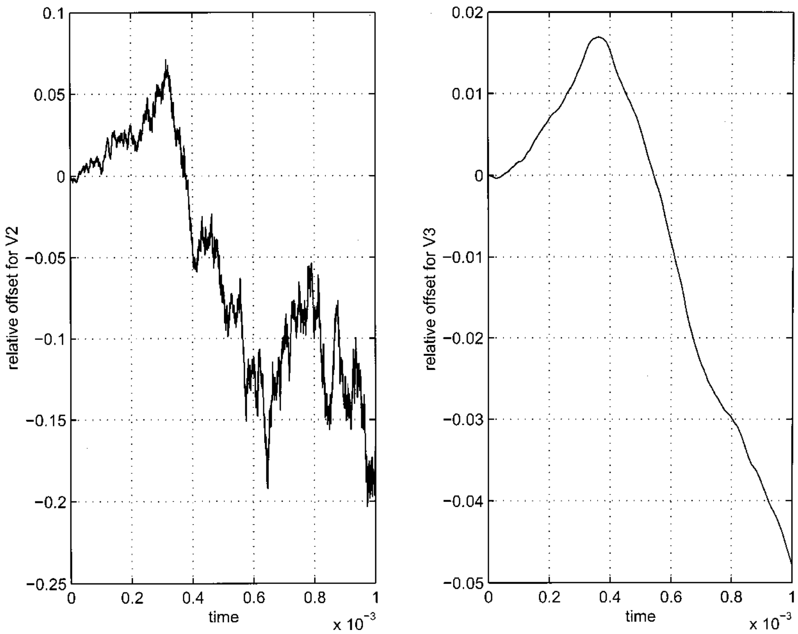

FIG. 9. Relative offset between SDE and ODE solutions. The difference between the solution to the SDE (with noise) and the solution to the ODE (without noise) looks like a scaled and filtered random walk.

(3) We can use random variations in a system parameter to inject small amounts of multiplicative noise into a system.
(4) We have presented an approximate analysis, and simulation, of the system with ''small'' amounts of noise. The solution to the SDE looks like the solution to the corresponding ODE with a scaled and filtered random walk added to the motion.

The following open questions require further investigation:

(1) Is it possible to derive exact criteria for the limits of stability as the mean and variance of the loop gain, K, are varied? The system was simulated using a state-space formulation. Sufficient conditions for the stability of switched state-space controller systems have been stated in the literature.[13]
(2) Can the theory of stochastic signal processing be applied to stochastically switched control in the case where the noise power is small? Given the similarity in the power spectral densities, it is quite possible that we can use autoregressive (AR) models to identify the closed loop system.[15] The Yule–Walker equations can be used to identify a system, given estimates of the autocorrelation functions.
(3) Can the theory of stochastic differential equations, embodied in the Itô calculus,[14] be applied to the state-space models in this paper? The Itô calculus would seem to provide a systematic approach to the system with large variance.
(4) Is this type of model useful for modeling systems with irregular feedback in the real world, such as climate or the business cycle? We suggest that many real-world systems include feedback which is dependent on random events. This has been carefully studied in the area of financial analysis. We would expect that these techniques would have application in the analysis of noise in electronic circuits.
(5) We have established an analogy between the flow of probability in Parrondo's games and the flow of energy in the herein Lyapunov stability analysis. Can these analogies be made rigorous to the point where they become an exact homomorphism? Can all observed effects be modeled and represented in both systems?

APPENDIX: THE NONCONVEX OR "CONCAVE" WINNING AND LOSING REGIONS IN PARRONDO'S GAMES

The issues of convexity and concavity arise in the analysis of Parrondo's games and in the analysis of the ''toy'' control system in this paper. The key concept in Parrondo's games is a flow of reward. We can construct a reward function, $R(\mathbf{P})$, of a parameter vector within a parameter space.

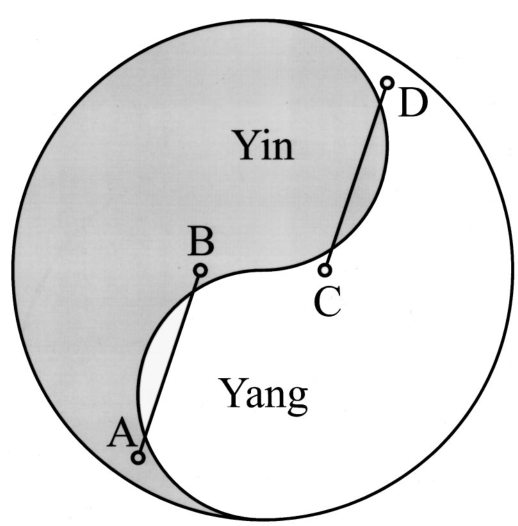

FIG. 10. Two well known complementary nonconvex sets. Sets may be locally convex but that does not make them convex.

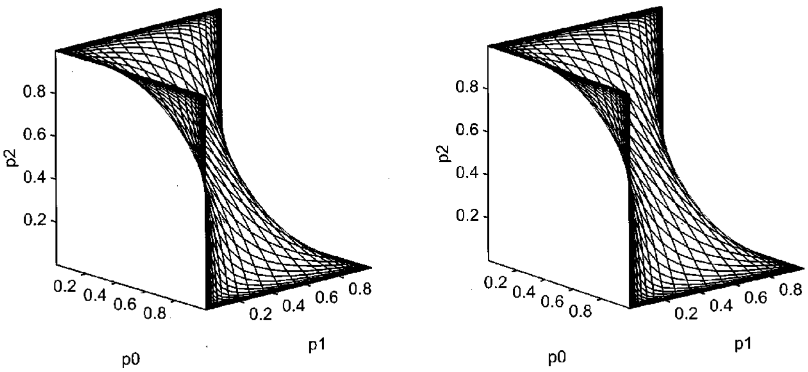

FIG. 11. The winning and losing regions of Parrondo's games: The losing region is below the surface and the winning region is above the surface. The parameter space is three dimensional. The zero-gain surface, that divides the two regions, has a topological dimension of two. It is possible to mentally reconstruct a three-dimensional image of the surface by viewing the stereo pair in the appropriate way.

For Parrondo's games the parameter vector is $\mathbf{P}^T = [P_0, P_1, P_2]$ and represents three conditional probabilities of winning under various circumstances.[4]

We can construct a reward function in terms of the parameter vector $\mathbf{P}$ and a time varying probability vector, $\mathbf{X}$. The time varying probability vector plays an analogous role to a state-vector in a dynamical system. Since the asymptotic limiting value of $\mathbf{X}$ is a function of $\mathbf{P}$, we can (ultimately) write the reward function as a function of the parameter vector: $R(\mathbf{P})$. Parrondo's paradox is a statement that we can find two parameter vectors $\mathbf{P_1}$ and $\mathbf{P_2}$ and a probability γ such that

$$\gamma R(\mathbf{P_1}) + (1-\gamma) R(\mathbf{P_2}) < 0 < R(\gamma \mathbf{P_1} + (1-\gamma)\mathbf{P_2}). \quad \text{(A1)}$$

The quantities $R(\mathbf{P_1})$ and $R(\mathbf{P_2})$ are the rewards from the losing games and $R(\gamma \mathbf{P_1} + (1-\gamma)\mathbf{P_2})$ is the reward from the winning linear convex combination of the two games. This is equivalent to saying that the reward, $R(\mathbf{P})$, is not a convex function over the parameter-space (or parameter manifold) for Parrondo's games. We could call $R(\mathbf{P})$ a "nonconvex" function. Some authors prefer to use the words "locally concave" to describe this property.

We can relate these concepts to the common sense meaning of the word "convex" if we imagine the parameter space to be partitioned into winning, $R(\mathbf{P}) > 0$, and losing, $R(\mathbf{P}) < 0$, regions. These are shown, slightly fancifully, in Fig. 10. The two partitions, "Yin" and "Yang" are both nonconvex in the usual sense. They partition a convex manifold, represented by a circular disk. Neither set is convex.

We should not confuse the convexity of a complete set with some notion of the local curvature of a boundary. In Fig. 10 the region labeled "Yin" has an outer boundary which looks convex but the complete set is nonconvex. This is clear because the line AB crosses the "Yang" region. The same argument applies to the "Yang" region since the line CD crosses the "Yin" region. The "Yin" and "Yang" regions are both nonconvex but, taken together, they form a complete partition of the entire circular disk. The winning and losing regions, within the parameter space, of Parrondo's games are of this type.

A stereo image of the surface that divides the winning and losing regions of the parameter space for Parrondo's original games is shown in Fig. 11. The zero-gain surface partitions a cube into two nonconvex regions.

The analogy with the control system is the rate of change in the stored energy satisfies a similar inequality to Parrondo's games,

$$\tfrac{1}{2}\dot{U}(K_1) + (1-\tfrac{1}{2})\dot{U}(K_2) > 0 > \dot{U}(\tfrac{1}{2}K_1 + (1-\tfrac{1}{2})K_2), \quad \text{(A2)}$$

where we consider the flow of stored energy $\dot{U}(K)$ as a function of the system parameter K. This is equivalent to saying that the energy flow, $\dot{U}(K_1)$, is not a convex function over the parameter-space of the gain, K, for our "toy" control system. (In our simple case, the unstable region is disjoint as well as nonconvex.) The relevance of this concept to control theory is that the stable and unstable regions, within the parameter space of a control system, can be nonconvex which can lead to counterintuitive behavior.

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