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Brownian ratchets and Parrondo's games

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Parrondo's games present an apparently paradoxical situation where individually losing games can be combined to win. In this article we analyze the case of two coin tossing games. Game B is played with two biased coins and has state-dependent rules based on the player's current capital. Game B can exhibit detailed balance or even negative drift (i.e., loss), depending on the chosen parameters. Game A is played with a single biased coin that produces a loss or negative drift in capital. However, a winning expectation is achieved by randomly mixing A and B. One possible interpretation pictures game A as a source of ''noise'' that is rectified by game B to produce overall positive drift—as in a Brownian ratchet. Game B has a state-dependent rule that favors a losing coin, but when this state dependence is broken up by the noise introduced by game A, a winning coin is favored. In this article we find the parameter space in which the paradoxical effect occurs and carry out a winning rate analysis. The significance of Parrondo's games is that they are physically motivated and were originally derived by considering a Brownian ratchet—the combination of the games can be therefore considered as a discrete-time Brownian ratchet. We postulate the use of games of this type as a toy model for a number of physical and biological processes and raise a number of open questions for future research.
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Parrondo's paradox is the counter-intuitive situation where individually losing games "cooperate" to win. This can occur via deterministic or nondeterministic mixing of the games. Although counter-intuitive, it should not be surprising that losing strategies can be combined to win, as such effects are ubiquitous in physical and biological systems. For example, in the game of chess, pieces can be sacrificed to win the overall game. Also in evolutionary theory, the fitness landscape of a species can have a valley, i.e., fitness declines, before the species rises to a higher level of fitness. Here we analyze simple losing coin tossing games, that remarkably win when combined. This may be of interest in fields as diverse as economics, biogenesis, and social modeling. We raise a number of open questions for future investigation.

I. INTRODUCTION

Random motion or ''noise'' in physical systems is usually considered to be a deleterious effect. However, the rapidly growing fields of stochastic resonance[1–4] and Brownian ratchets[5] have brought the increasing realization that random motion can play a constructive role. Furthermore, noise also plays a constructive role in the creation of noise-induced patterns[6] and noise-induced phase transitions,[7,8] where it has been shown that noise can induce an ordered phase in a spatially extended system.

The apparent paradox that two losing games A and B can produce a winning expectation, when played in an alternating sequence, was devised by Parrondo as a pedagogical illustration of the Brownian ratchet.[9] However, as Parrondo's games are remarkable and may have important applications in areas such as electronics, biology and economics, they require analysis in their own right.

In this article, we first introduce the concept of the Brownian ratchet and then illustrate Parrondo's games. Graphical simulations of the outcomes of Parrondo's games are then explained, in terms of the Brownian ratchet model. In this article we focus on Parrondo's original games[9,10] where the rules depend on the player's capital. As we shall see, a rule based on modulo arithmetic is used to construct the required capital-dependence—this turns out to be a natural choice for mimicking the operation of a conventional Brownian flashing ratchet. However, this construction is not natural for exploring possible application in, say, biology or finance—for a discussion of Parrondo's history-dependent rules refer to Parrondo *et al.*[11] and for an analysis of cooperative games based on 1-D spatial neighbor-dependent rules see the work of Toral.[12]

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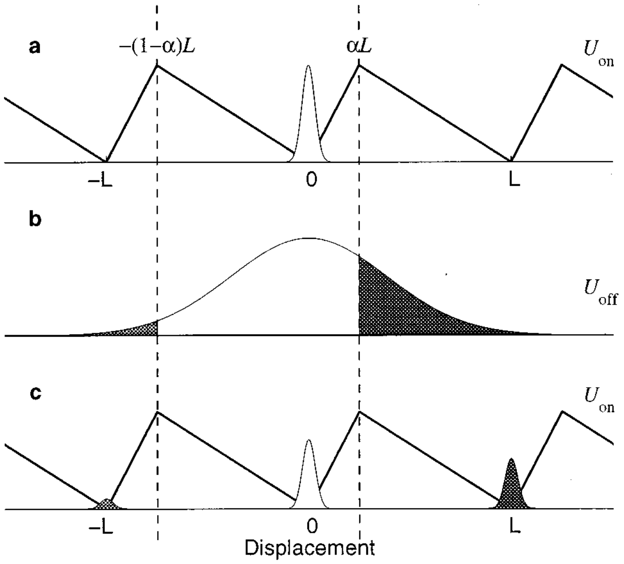

FIG. 1. Brownian ratchet mechanism. The sawtooth and flat potentials are labeled with U_{on} and U_{off}, respectively, while the distribution of Brownian particles is shown via the normal curves. This sequence of flashing between on and off potentials shows there is a net movement of particles to the right.

A. Brownian ratchets

A ratchet and pawl device was introduced in the early 20th century as a proposed perpetual motion machine—originally it was a thought experiment to try and harness the thermal Brownian fluctuations of gas molecules, by a process of rectification. An explanation of the mechanics for the ratchet and pawl device is given in *The Feynman Lectures on Physics*.[13]

In 1912, Smoluchowski[14] was the first to explain why it could not perform as a *perpetuum mobile*, showing that there is no net motion under equilibrium conditions for the ratchet and pawl device, which he called *Zahnrad mit einer Sperrklinke* in German. This device was later revisited by Feynman.[13] Even though Feynman's work was flawed,[15,16] it has been the source of inspiration for the "Brownian ratchet" concept.

The focus of recent research is to harness Brownian motion and convert it to directed motion or, more generally, a Brownian motor, without the use of macroscopic forces or gradients. This research was inspired by considering molecules in chemical reactions, termed molecular motors.[17] Recently, many man-made Brownian ratchets have been developed.[5] The roots of these Brownian devices trace back to Feynman's exposition of the ratchet and pawl system. By supplying energy from external fluctuations or nonequilibrium chemical reactions in the form of a thermal or chemical gradient, for example, directed motion is possible even in an isothermal system.[18,19] These types of devices have been shown to work theoretically,[17,20] even against a small macroscopic gradient.[21,22]

There are several mechanisms by which directed Brownian motion can be achieved.[23,24] We will consider one of the mechanisms, termed *flashing ratchets*,[17,21,22,25] which is shown in Fig. 1. This will prove fruitful when considering Parrondo's games later. Consider a system where there exist two one-dimensional potentials, U_{on} and U_{off}, as depicted in Fig. 1(a) and 1(b), respectively. Let there be Brownian particles present in the potential that diffuse to a position of least energy. Time modulating the potential U_{on} and U_{off} can induce motion, hence the term *flashing ratchets*.

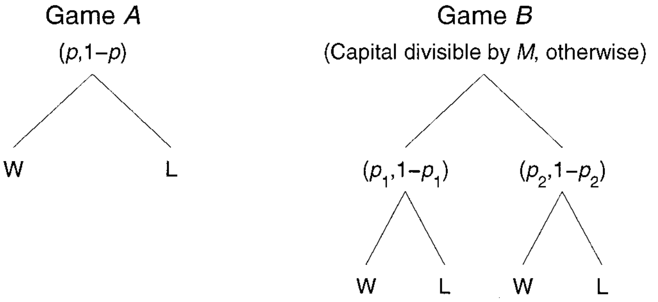

FIG. 2. Construction of Parrondo's games. The games could be formed using three biased coins, appropriately switching between them depending on the game being played and the value of the present capital.

When U_{on} is applied, the particles are trapped in the minima of the potential so the concentration of the particles is localized. Switching the potential off allows the particles to diffuse freely so the concentration is a set of normal curves centered around the minima. When U_{on} is switched on again there is a probability P_{fwd} that is proportional to the darker shaded area of the curve that some particles are to the right of αL. These particles move right to the minima located at L. Similarly there is a probability P_{bck} (lightly shaded) that some particles are to the left of $-(1-\alpha)L$; these move to the left minima located at $-L$. Since $\alpha<\frac{1}{2}$ in Fig. 1, then $P_{\text{fwd}}>P_{\text{bck}}$ and the net motion of the particles is to the right.

When a tilted periodic potential is toggled "on" and "off," by solving the Fokker–Planck equation for this system, Brownian particles are shown to move "uphill."[21] If the potential is held in either the "on" state or the "off" state, the particles move "downhill." This is the inspiration for Parrondo's paradox: the individual states are said to be like "losing" games and when they are alternated we get uphill motion or a "winning" expectation.

B. Parrondo's games

Here, we detail the construction of the games. Game A is straightforward and can be thought of as tossing a weighted coin that has probability p of winning. Game B is a little more complex and can be generally described by the following statement. If the present capital is a multiple of M, then the chance of winning is p_1; if it is not a multiple of M, the chance of winning is p_2.

The two games can be represented diagrammatically using branching elements as shown in Fig. 2. The notation (x,y) at the top of the branch gives the probability or condition for taking the left and right branch, respectively.

If we wish to control the three probabilities p, p_1 and p_2 via a single variable, a biasing parameter ϵ can be used to represent a subset of the parameter space. For example, one could have

$$p = 1/2 - \epsilon,$$
$$p_1 = 1/10 - \epsilon, \quad (1)$$
$$p_2 = 3/4 - \epsilon.$$

This parametrization along with $M = 3$ represents Parrondo's original numbers for the games.[9]

We will digress for a moment to discuss what constitutes a fair game. The behavior of game B differs from game A in that the starting capital affects whether we are likely to win or not. If the starting capital is a multiple of M, then we lose a little, and conversely win a little if the starting capital is not a multiple of M. For example, let the capital after the nth game be X_n. Then $E[X_1|X_0] < X_0$ if X_0 is a multiple of M. The concept of what it means for a game to be winning, losing or fair can be defined precisely in terms of hitting probabilities and expected hitting times of discrete-time Markov chains as is done in our analysis section. Before then we shall be a little looser with this terminology. We shall consider a game to be winning, losing or fair according to whether the probability of moving up n states is greater than, less than, or equal to the probability of moving down n states for some fixed large n.

Using the above criterion, both game A and game B are fair when ϵ is set to zero. This is true of game A because the probabilities of moving up and down n states are equal for all n. It is also true of game B even though the value of the starting capital influences the probability of going up and down n states for small values of n. Using this criterion and the parametrization given in (1), both games A and B lose when $\epsilon > 0$.

II. SIMULATIONS AND ANALYSIS

It can be deduced by a detailed balance analysis[26] and simulations that both game A and game B lose for a small positive biasing parameter (i.e., $\epsilon > 0$). However, when we start switching between the two losing games, e.g., play two games of A, two games of B, two of A, and so on, the result is quite counter-intuitive in that we start winning. That is, we can play the two losing games A and B in such an order as to produce a winning expectation. Furthermore, deciding which game to play by tossing a fair coin also yields a winning expectation. Figure 3 shows the progress when playing games A and B, as well as the effect of switching periodically and randomly between the games. The switching sequence affects the gain as the games are played, which is shown by the different finishing capitals in Fig. 3.

A. Parrondo game bounds

It would be desirable to develop a test, or a set of constraints, that can be applied to game parameters to determine if they form a Parrondo game. Such constraints were found for the specific case of $M = 3$ in Ref. 27; however, the generalized proof follows.

The analysis of game A is elementary and can be found in many textbooks (see, for example, Ref. 28), but we present it here in the interest of motivating our analysis of game B.

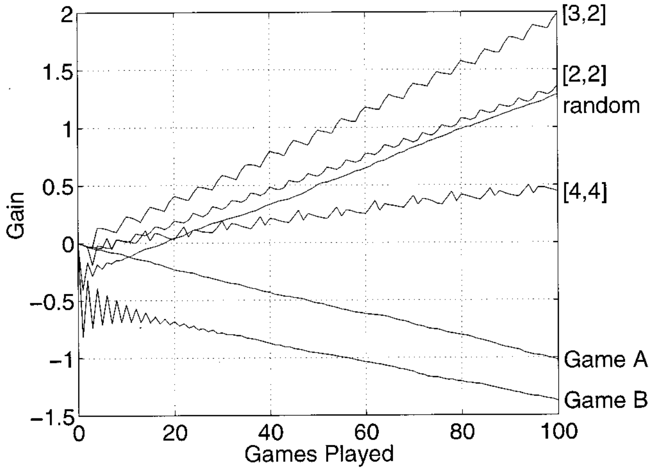

FIG. 3. The effect of playing A and B individually and the effect of switching between games A and B with Parrondo's original numbers (see text). The simulation was performed with $\epsilon = 0.005$ playing game A a times, game B b times, and so on until 100 games were played, which were averaged over 50 000 trials. The values of a and b are shown by the vectors $[a,b]$.

We win a single round of game A with probability p and lose with probability $q = 1 - p$. Assuming that we bet one unit on each round of the game, we wish to calculate the probability f_j that our capital ever reaches zero given that we start with a capital of j units. It is a consequence of Markov chain theory (Ref. 28, p. 93) that either

(1) $f_j = 1$ for all $j \geqslant 0$, in which case the game is either fair or losing, or
(2) $f_j < 1$ for all $j > 0$, in which case there is some probability that our capital will grow indefinitely and so the game is winning.

For $j \geqslant 1$, let $f_j^{(n)}$ be the probability that our capital reaches zero within the first n games, given that it starts at j. It is easy to see that $f_0^{(n)} = 1$ for all n. For each j, the sequence $\{f_j^{(n)}\}$ is increasing and thus must have a limit which is f_j, as defined earlier. By conditioning on what happens at the first time point, we derive the equation

$$f_j^{(n+1)} = p f_{j+1}^{(n)} + q f_{j-1}^{(n)}. \quad (2)$$

It follows that f_j is the minimal non-negative solution to the equation

$$f_j = p f_{j+1} + q f_{j-1}, \quad (3)$$

subject to the boundary condition $f_0 = 1$. This difference equation along with the boundary condition has a general solution of $f_j = K[(q/p)^j - 1] - 1$, where K is a constant.[29] From Ref. 27 we can write

$$f_j = \min(1, (q/p)^j), \quad (4)$$

and we observe that the game is winning if

$$q/p < 1. \quad (5)$$

For $j < 0$, it follows by analogy that the game is losing if $q/p > 1$ and is fair if $p = \frac{1}{2}$. This result, of course, accords with our intuition.

For game B, the probability that we win a single round depends on the value of our current capital. If the capital is a multiple of M, the probability of winning is p_1, whereas if the current capital is not a multiple of M, the probability of winning is p_2. The corresponding losing probabilities are $q_1=1-p_1$ and $q_2=1-p_2$, respectively. Let g_j be the probability that our capital ever reaches zero given that we start with j units. As with game A, Markov chain theory tells us that either

(1) $g_j=1$ for all $j\geqslant 0$, in which case the game is either fair or losing, or
(2) $g_j<1$ for all $j>0$, in which case there is some probability that our capital will grow indefinitely and so the game is winning.

Again following the derivation of game A, for $i\geqslant 0$ and $j\in\{1,\ldots,M-1\}$ the set of numbers $\{g_k\}$ satisfies the equations

$$g_{Mi}=p_1 g_{Mi+1}+q_1 g_{Mi-1} \tag{6}$$

and

$$g_{Mi+j}=p_2 g_{Mi+j+1}+q_2 g_{Mi+j-1} \tag{7}$$

subject to the boundary condition $g_0=1$. For $j\in\{1,\ldots,M-1\}$, the general solution to Eq. (7) for fixed i is

$$g_{Mi+j}=C_i(q_2/p_2)^j+D_i, \tag{8}$$

with

$$C_i=\frac{g_{Mi}-g_{M(i+1)}}{1-(q_2/p_2)^M} \tag{9}$$

and

$$D_i=\frac{g_{M(i+1)}-g_{Mi}(q_2/p_2)^M}{1-(q_2/p_2)^M}. \tag{10}$$

Substituting this into Eq. (6), we derive the equation

$$\begin{aligned}[1-(q_2/p_2)^M]g_{Mi}=p_1\{&g_{M(i+1)}[1-(q_2/p_2)]\\ &+g_{Mi}[(q_2/p_2)-(q_2/p_2)^M]\}\\ &+q_1\{g_{Mi}[1-(q_2/p_2)^{M-1}]\\ &+g_{M(i-1)}[(q_2/p_2)^{M-1}\\ &-(q_2/p_2)^M]\}\end{aligned}$$

for $i\geqslant 1$. After some tedious manipulation, for $i\geqslant 1$, this reduces to

$$\begin{aligned}0=[q_1q_2^{M-1}]g_{M(i-1)}-[p_1p_2^{M-1}+q_1q_2^{M-1}]g_{Mi}\\ +[p_1p_2^{M-1}]g_{M(i+1)}.\end{aligned} \tag{11}$$

This is in the same form as (3), thus equating gives

$$g_{Mi}=\min\left(1,\left(\frac{q_1q_2^{M-1}}{p_1p_2^{M-1}}\right)^i\right). \tag{12}$$

As for game A, we deduce from (12) that game B is winning, losing and fair if

$$\frac{q_1q_2^{M-1}}{p_1p_2^{M-1}} \tag{13}$$

is less than 1, greater than 1 or equal to 1.

Now consider the situation where we play game A with probability γ and game B with probability $1-\gamma$. If our capital is a multiple of M, the probability that we win the randomized game is $p_1'=\gamma p+(1-\gamma)p_1$, whereas if our capital is not a multiple of M, the probability that we win is $p_2'=\gamma p+(1-\gamma)p_2$. The probabilities of losing are $q_1'=1-p_1'$ and $q_2'=1-p_2'$, respectively. We observe that this is identical to game B except that the probabilities have changed. It follows from (13) that the randomized game is winning, losing and fair if

$$\frac{q_1'(q_2')^{M-1}}{p_1'(p_2')^{M-1}} \tag{14}$$

is less than 1, greater than 1 or equal to 1.

Thus, the existence of Parrondo's paradoxical games will be established if we can find parameters p, p_1, p_2, γ and M for which

$$\frac{1-p}{p}<1, \tag{15a}$$

$$\frac{q_1q_2^{M-1}}{p_1p_2^{M-1}}<1, \tag{15b}$$

and

$$\frac{q_1'(q_2')^{M-1}}{p_1'(p_2')^{M-1}}>1 \tag{15c}$$

are satisfied. For simplicity, set $\gamma=\frac{1}{2}$ and $M=3$. If we consider Parrondo's original probabilities as given in (1), then the above equations reduce to $\epsilon>0$, $\epsilon(80\epsilon^2-8\epsilon+49)>0$ and $320\epsilon^3-16\epsilon^2+229\epsilon-3\epsilon<0$. That is, we require $\epsilon>0$ for games A and B to lose, but $\epsilon<0.013\,11$ for the randomized games to win. So choosing any biasing parameter such that $0<\epsilon<0.013\,11$ leads to the paradoxical nature of the games being exhibited.

B. Parameter space

Now that the equations (15) have been established, it is possible to explore the range of probabilities that are possible. For simplicity and the ability to plot the results, we have fixed $M=3$ and the mixing rate $\gamma=\frac{1}{2}$.

Game A only depends on a single variable, thus only a single value exists for the game to be fair. This is clearly $p=\frac{1}{2}$. Game B adds an extra dimension by depending on two variables, p_1 and p_2, and there exists a continuous range of probabilities that gives rise to game B being fair. Combining games A and B, the randomized game depends on all p, p_1 and p_2. From the relations in (15), the surfaces separating winning and losing expectations for the games can be defined and are plotted in Fig. 4.

It is easy to determine that the region for losing in game A is below the plane Π_A and in game B to the right of the surface Π_B. The winning region in the randomized game is above the surface Π_R. Observing the surfaces in Fig. 4, there

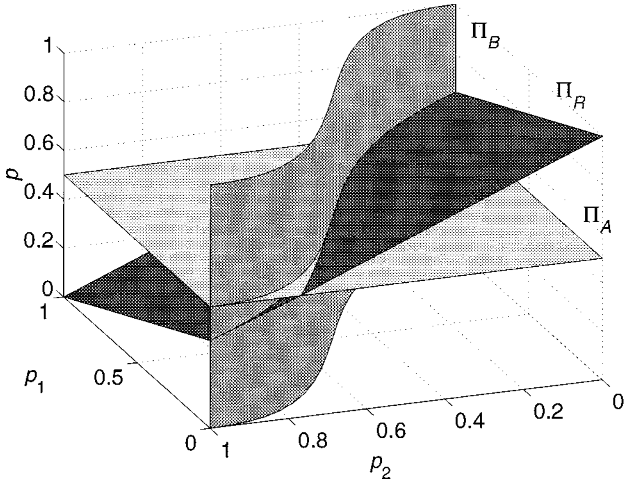

FIG. 4. The parameter space for the games. The three surfaces Π_A, Π_B and Π_R separate the winning and losing parameter spaces as determined by (15), respectively. The small volume at the front gives the parameter space where Parrondo's games exist.

exists a small volume that is both below and to the right of Π_A and Π_B, but above Π_R. This is the volume that contains all of the possible probabilities that lead to Parrondo's paradoxical games. Thus, choosing any point from that volume gives rise to a Parrondo's game. This volume accounts for 0.032% of the total volume of the parameter space, which is not large. Also note that there is a corresponding "inverse" volume at the opposite side of the parameter space that has the exact opposite properties of Parrondo's games, namely, two winning games combine to form a losing game. This shows that the games are symmetrical; swapping the winning and losing probabilities switches the characteristics of the games, as to be expected if the two players swapped places for example.

Another important principle that can be gleaned from the parameter space is that the set of (p_1, p_2) that corresponds to losing games is not convex and this is yet another viewpoint that explains how two losing games can combine to win. This idea is developed in more detail by Moraal.[30]

C. Rate of winning

The capital X_n decreases by 1 when we lose and increases by 1 when we win. Since the transition probabilities are periodic functions of the capital, we can associate the games with a state space $\{0,1,\ldots,M-1\}$ whenever X_n is equivalent to X_n modulo M. This forms a discrete-time Markov chain (DTMC) where the states represent the capital. Once in this form, the equilibrium (or stationary) distribution of the DTMC may be found. A method by Mihoc and Fréchet (see[31] for their results), which has been expanded for the periodic case in Ref. 31, gives the stationary probabilities in terms of the cofactors of the transition matrix **P**. In particular, the stationary probabilities $\boldsymbol{\pi}=(\pi_0,\ldots,\pi_{M-1})$ associated with **P** are proportional to the diagonal cofactors of $\mathbf{I}-\mathbf{P}$, where **I** is the identity matrix of appropriate size. Normalizing these diagonal cofactors gives the stationary distribution. For example, with $M=3$, the transition matrix for the randomized games is

$$\mathbf{P}=\begin{bmatrix} 0 & q_2 & p_2 \\ p_1 & 0 & q_2 \\ q_1 & p_2 & 0 \end{bmatrix}, \tag{16}$$

which leads to the stationary state being $\boldsymbol{\pi}=(0.3836,0.1543,0.4621)$ using the probabilities in (1) with $\epsilon=0$.

To calculate the rate of winning, we consider the distribution of state j after the nth game, $\pi_j(n)$. Intuitively it is defined as $E[X_{n+1}-X_n]$, which is always equivalent to $E[X_{n+1}]-E[X_n]$ no matter the dependency on X (Ref. 32, p. 143). The rate of winning is then

$$r(n)\equiv E[X_{n+1}]-E[X_n]=\sum_{j=-\infty}^{\infty} j[\pi_j(n+1)-\pi_j(n)]. \tag{17}$$

We can write the global balance equation (GBE) as

$$\pi_j(n+1)=P_{j-1,j}\pi_{j-1}(n)+P_{j+1,j}\pi_{j+1}(n), \tag{18}$$

where $P_{j,k}$ is the transition probability that the capital jumps from j to k in one run. More specifically, for these games the transitions are all one-step, that is, $k=j\pm1$. Using the GBE and the fact that $P_{j,j-1}=1-P_{j,j+1}$, one can find from (17) and (18) that

$$\begin{aligned} r(n)&=\sum_{j=-\infty}^{\infty}[2P_{j,j+1}-1]\pi_j(n) \\ &=2\sum_{j=-\infty}^{\infty}P_{j,j+1}\pi_j(n)-1. \end{aligned} \tag{19}$$

For game A, $P_{j,j+1}=p$ for all j and $r(n)$ reduces to $2p\Sigma\pi_j-1=2p-1$ as expected.

However, for game B, $P_{j,j+1}=p_1$ if j is a multiple of M and $P_{j,j+1}=p_2$ otherwise, and so the slope is given by

$$r(n)=2p_1\pi_0(n)+2p_2[1-\pi_0(n)]-1. \tag{20}$$

Note that this only depends on the stationary probability of the first state, π_0. Now, if n is large enough, we can use the stationary probabilities to find the slope as

$$r_{st}\equiv\lim_{n\to\infty} r(n)=2p_1\pi_0+2p_2[1-\pi_0]-1, \tag{21}$$

which is valid for all M and in agreement with Ref. 31. The tricky part is finding π_0, which is algebraically tractable for small M, but best done numerically on a computer for higher M. Notice that $r_{st}>0$ when the game is winning, $r_{st}=0$ when it is fair, and $r_{st}<0$ when it is losing.

For the case of $M=3$ we can find $\pi_0=(1-p_2q_2)/(3-p_1q_2-p_2q_1-p_2q_2)$ and hence calculate the slope for game B as

$$r_{st}=\frac{3(p_1p_2^2-q_1q_2^2)}{3-p_1q_2-p_2q_1-p_2q_2}. \tag{22}$$

This agrees with the slope found for game A by setting $p_1=p_2=p$.

For the probabilities given in (1), the slope for game B is

$$r_{st}^{(B)}=\frac{-6\epsilon(49-8\epsilon+80\epsilon^2)}{169-16\epsilon+240\epsilon^2}, \tag{23}$$

which is negative for $\epsilon>0$. The same calculation holds for the combination of possibly biased games A and B by replacing p_1 and p_2 by q_1 and q_2, respectively. Thus the slope for the random game with $\gamma=\frac{1}{2}$ is

$$r_{st}^{(\mathrm{rand})}=\frac{6(3-229\epsilon+16\epsilon^2-320\epsilon^3)}{709-32\epsilon+960\epsilon^2}, \tag{24}$$

which is positive for small ϵ.

Finally, for $\epsilon=0.005$, one has $r_{st}^{(A)}=-0.0100$, $r_{st}^{(B)}=-0.008\,70$ and $r_{st}^{(\mathrm{rand})}=0.0157$ that are also in agreement with Ref. 31. To find the slope from simulations, the transient effects that are caused when starting the games need to be removed to achieve a reliable measure. Thus, averaging a number of games played out to 2000 iterations and ignoring the first 100 games gives results that are in close agreement with the above theoretical result. They are $-0.009\,99$, $-0.008\,69$ and 0.0157, respectively.

III. COMPARING BROWNIAN RATCHETS AND PARRONDO'S GAMES

With some insight, one may see the analogy between the games and the Brownian ratchet. Here, we offer two explanations: comparing the games to the potentials, and comparing the distributions of the capital in the games to the particles in the potentials.

We have two similar systems: (i) the Brownian ratchet that requires the energy profile be flashed on and off to get directed movement of particles, and (ii) Parrondo's games that require switching between games in order to win. We can use the mechanics of the Brownian ratchet to explain how Parrondo's games work. Game A is well known, and after playing a number of times, the capital has a normal distribution. This is equivalent to when the potential is off in Brownian ratchets, seen by the particle distribution in Fig. 1. Thus, a reasonable assumption would be that game B has a potential associated with it like that of the ratchet. With a little more investigation it is possible to find the potential associated with game B.[9] Although the potential is a little more complicated, it works in a very similar fashion to energy profiles shown in Fig. 1.

An alternative explanation of the two systems can be given in terms of localization of particles or capital at system "ceilings." Considering game B alone with $M=3$, the capital tends to localize between the $3n-1$ and $3n$ states for some integer n. This is due to the chosen probabilities of p_1 and p_2. At $3n-1$, there is a high probability (p_2) that our capital will increase to $3n$. At that state there is an even higher probability ($1-p_1$) that the capital will be pushed back down to $3n-1$. So, we have a localization of capital at these $3n$ ceilings. In the same way, the particles in the ratchet teeth are localized in the pits, just before the steep edge.

Adding game A to the playing sequence improves the situation due to the fact that, in game B, the capital is localized at these ceilings. Switching to an approximately fair game allows about half the capital at these ceilings to move up to the next subsystem (i.e., from n to $n+1$), while the other half moves down a state (i.e., n to $n-1$). When game B is played again, the capital that moved down gets forced back to the $3n$ ceilings. This is exactly what happens when the ratchet teeth are made to "disappear" in the Brownian ratchet—about half of the particles can easily move over the steep edge into the next pit, while the remaining fall back into the same pit via the gentle edge when the ratchet teeth appear again.

TABLE I. This shows the relationship between quantities used for Parrondo's paradox and the Brownian ratchet.

Quantity	Brownian ratchet	Parrondo's paradox
Source of potential	Electrostatic, Gravity	Rules of games
Duration	Time	Number of games played
Potential	Potential field gradient	Parameter ϵ
Switching	U_{on} and U_{off} applied	Games A and B played
Switching durations	τ_{on} and τ_{off}	a and b
Measurement/output	Displacement x	Capital or gain
External energy	Switching U_{on} and U_{off}	Alternating games
Potential asymmetry	Depends on α	Branching of B to p_1 or p_2
Thermodynamic law	Work done < energy in	Total gain < gain with p_2 alone
Mode of analysis	Fokker–Planck equation	Discrete-time Markov chains

Although there are certain similarities between the two systems, there are also subtle differences worth exposing. The Brownian ratchet is continuous in time and space; the particles can exist at any real displacement along the potential, which can be 'flashed' on or off at any real time. This is in contrast to Parrondo's ratchet, which is discrete in both the analogous time and space. The capital of the games is quantized, and only integer numbers of games can be played. This is highlighted by the mode of analysis. The Brownian ratchet is analyzed via continuous variables in the Fokker–Planck equation whereas Parrondo's ratchet is via discrete-time Markov chain analysis. The analogy between various quantities in the two types of ratchet are conjectured in Table I.

When we consider the ratchet and pawl machine, directed motion is only achieved when energy is added to the system, as in a heat engine. Similarly for a flashing Brownian ratchet, energy is taken up by switching between two states to produce "uphill" motion of Brownian particles. From the simulations and mathematical analysis of Parrondo's games, we see that two losing games can obtain a winning expectation, without any apparent cost. This creates a paradox: "money for free." Where is the "energy" coming from in Parrondo's games? Of course, the money itself is conserved in that the winnings of the player are at the expense of the losing opponent—but this is not what we are talking about—when we say "money for free," we are saying there is no switching cost. On the face of it, this is strange as it does cost energy to operate a physical flashing ratchet.

One viewpoint is to say the answer lies in the context in which Parrondo's games are applied. For instance, in stock market models, the ‘‘switching energy’’ could be thought of as the buying and selling transaction cost. However, in the case of two individuals gaming, the interpretation of switching energy becomes problematic as there is no apparent ‘‘cost’’ in the process of switching—this appears truly paradoxical. Another possible view is to note that ‘‘winning’’ is dependent on one player being ignorant of the games—hence there is an ignorance ‘‘gradient’’ between the two players that will eventually equilibrate over time. There may be a heuristic analogy to quantum mechanics, in that a full description of the discrete ratchet could be dependent on the players/observers. A third, and perhaps more accurate, viewpoint is to say that the analogy between Parrondo’s discrete-time ratchet and the conventional physical flashing ratchet breaks down at this point. The thermodynamic law for the flashing ratchet is that the work done on pushing the particles uphill is less than the external energy used to flash the potentials (i.e., engine efficiency is less than unity). The corresponding ‘‘thermodynamic law’’ for the discrete-time ratchet is somewhat different: here we can say that the gain in capital created by randomly mixing games A and B is less than a game composed of tossing coin p_2 on its own. We can think of the ratio of gain from the mixed AB game and gain from p_2 alone as an ‘‘engine efficiency’’ for the discrete-time ratchet. An open question now is to ask how we can increase this efficiency and how it compares to other game versions. In summary, although the ‘‘transaction cost’’ and ‘‘ignorance gradient’’ viewpoints are interesting, the better solution to the switching energy problem is to say the analogy between the two systems simply breaks down when it comes to the question of cost of switching between subprocesses. This is costly in the physical system, but not in the games. However, we have shown how it is possible to modify the ‘‘thermodynamic law’’ to come up with a concept of ‘‘engine efficiency’’ for the games.

IV. CONCLUSIONS AND OPEN QUESTIONS

So far we have used models of the flashing Brownian ratchet to help explain what is happening in Parrondo’s games. Now that we have a reasonable idea of what is happening in Parrondo’s discrete Brownian ratchet, we can maybe use this information to infer back some characteristics to the continuous Brownian ratchet.

The flashing model is not the only type of Brownian ratchet.[5,17,18,22] There is also the ‘‘changing force ratchet’’ model, for instance. Both of these Brownian ratchets have their own variations. Is it possible to devise games that emulate other types of Brownian ratchets?

During the simulations we have only used one combination of p_1 and p_2 for each value of M. With the help of the DTMC analysis, we have found a continuous range of probabilities to keep game B fair. Changing p_1 and p_2 affects the potentials, which may affect the result of the games. We speculate that M changes the length of the teeth in the ratchet potential while the values of p_1 and p_2 change the slope of the teeth, like the value of α in Fig. 1.

Another type of ratchet, not to be confused with Parrondo’s discrete ratchet, is Muller’s ratchet.[33–35] This describes a process where asexual populations would necessarily decline in fitness (or reproductive success) over time if their mutation rate were high, as they would accumulate harmful mutations. This process only proceeds in one direction, each new mutation irreversibly eroding the population’s fitness—it is the irreversibility that is likened to a ratchet. Flashing ratchets differ in that they use external energy to work against a gradient, not with it like Muller’s ratchet—crudely speaking Muller’s ratchet goes ‘‘downhill’’ whereas the flashing ratchet goes ‘‘uphill.’’

It would appear therefore that Muller’s ratchet is a misnomer. The introduction of sexual reproduction into a species is said to ‘‘break Muller’s ratchet,’’ as recombination allows selection of beneficial mutations. It is this process of breaking Muller’s ratchet that can be likened to a real ratchet, as we are now moving against disorder or a natural gradient.

Parrondo’s ratchet involves two games, to emulate the two potentials in the Brownian ratchet. What would happen if we introduced more games? Observing Fig. 3, we see that as the values of a or b in $[a,b]$ increase, the gain reduces. In other words ‘‘fast’’ switching produces the best gain. So, introducing more games $[a,b,c,\ldots]$ would slow the overall switching rate and reduce the gain. Could this class of model be used to explain partially why there are two sexes and not more? Two sexes allow faster recombination and so the act of breaking Muller’s ratchet is more efficient—this corresponds to the higher gain in Parrondo’s discrete ratchet model, when two games and not more are used. This argument is appealing, but remains an open question until further investigation. The question of why there are two sexes is a major field of research, with multidisciplinary implications.[36]

Another biological conundrum is that of animal signals used to attract mates. The signal can be accentuated, the more fit (and hence attractive) an animal is (e.g., greater adornment). However, what is to stop genetically weaker rivals faking a particular signal? A classic example is that of the tail of a peacock, where the larger it is the more attractive it is to potential mates. One conjecture is that the tail is a losing game because it makes the bird more vulnerable to prey. But this losing game guarantees success, because then it becomes impossible for weaker rivals to mimic it and thus eliminates the ‘‘fakes.’’ These and similar scenarios can readily be found in the biological literature under what is termed the theory of *costly signaling*.[37] Clinton’s rise in popularity (winning game), despite the Lewinsky affair (losing game), was cited by *The New York Times* (p. D5, Jan. 25th, 2000) as a possible example of Parrondo’s paradox. The problem with both of the above examples is that there is no sense of alternation between games. However, an open question is: are there some general principles involving, say, asymmetry or convex parameter spaces that apply to these types of example and that of Parrondo’s games? Is there a unifying principle?

The relevance of generalized Parrondo games to biology should not be surprising. One reasonable way to illustrate this is to note that many biological systems can exhibit behavior that is Markovian, with respect to transitions between two individual states, having noise inevitably associated with these transitions—and they can be state dependent, in the sense that the probability of being in a certain state is dependent on past events. A basic molecular example might be an enzyme, which is activated by its own product at a site distinct from the "active site," where chemical catalysis takes place (i.e., an allosterically activated enzyme). If we postulate that the product molecule is itself subject to fluctuations in concentration (due to the effects of another enzyme or a transport system), it is easy to imagine the biosynthesis of the product molecule controlled by the interplay of noise and a state dependence.

Let us turn to another speculative example in molecular biology. Firstly, recall that the idea of Parrondo's game B is that it has branches that lead to an unfavorable outcome and branches that lead to a favorable outcome. A state-dependent rule in game B creates a bias towards the unfavorable or "bad" branch. The noisy effect of a mixture with game A breaks up the state dependence to create bias towards the favorable or "good" branch. Now, in the same way, we can perhaps conjecture the coding regions of DNA to be like game B. The position of each gene in the DNA sequence is a form of spatial state dependence and each gene is subjected to a "bad game" in that activators and suppressors of the neighboring genes can interfere (i.e., give rise to crosstalk). So this "game" is in a "bad" branch. Adding in intergenic "junk" DNA is like adding spatial randomness (like game A) to isolate genes from each other, hence breaking up their spatial state dependence. The isolating effect of mixing in intergenic junk hence is a winning game. In Toral's cooperative version of Parrondo's games,[12] it was observed that the game mixing reduced correlations between neighbors, i.e., neighbors did indeed become more isolated. So the open question here is to ask if it is possible to construct a suitable set of games that can reflect the state dependencies found in DNA.

Let us now turn to sexual reproduction. Sexual reproduction uses recombination as a mechanism for enhancing genetic diversity. When an organism produces gametes (sperm or eggs), the paternal and maternal versions of each chromosome are lined up. The chromosomes are then broken and rejoined at random locations,[38] so that hybrid chromosomes are produced. It is this process that is called *recombination*. Sometimes recombination is not precise and a few nucleotides may be lost or gained. A single nucleotide difference can totally disrupt a protein code within a gene—but a nucleotide difference in an intron or an intergenic DNA region is usually not a problem. As these "junk" regions can tolerate a good deal of change, they are safe places for recombination. So can we liken sexual recombination to a form of game B, and the spatial randomness introduced by the junk regions to be like game A? Both games A and B are losing games individually, but taken together errors are reduced and more information is correctly transferred.

Another example due to Clark[39] is to consider the GCN4 protein, which is found in baker's yeast. When yeast are starved of amino acids they make GCN4, which turns on gene transcription to produce amino acids. It is interesting that the transcription of the GCN4 gene itself is not activated by amino acid starvation. However, the translation of GCN4 mRNA to make GCN4 protein is activated by amino acid starvation and is inhibited under amino acid rich conditions, which is rather counter-intuitive. When rich amino acid conditions prevail most proteins are synthesized, but GCN4 is not. When poor amino acid conditions exist, overall protein production is lower, but GCN4 is made—this sets up the pathway for the yeast to make its own amino acids and hence relieve starvation. This is an example of what is called *translational control* of gene expression. The idea here is that the GCN4 protein is only produced when two inhibitory influences are combined (see the Appendix)—can this be modeled by the mixing of suitable losing games that win?

Evolution itself is a prime example of a Brownian ratchet. Natural selection increases the representation in the population of genes that contribute to above average fitness and reduces the representation of genes with below average fitness. This is a winning game. However, it arises due to the combination of losing games such as death and fluctuations in the environment and fluctuations in the fitness of mutations.

Other promising application areas for investigation of Parrondo's paradox have been suggested to be in biogenesis,[40] spin systems,[30] stochastic signal processing, economics, sociological modeling, game theory and quantum game theory.[41] Further technical open questions about the games themselves are the following.

(i) It is common for mathematicians to use a martingale as the definition of a fair game (Ref. 42, p. 299). However, game B, on its own, is not a martingale and yet is in a sense balanced/fair. How should the definition of "fairness" be extended to include such cases?

(ii) For randomized M, game B becomes a martingale and the mixed AB game then becomes balanced. To produce a gain, in the mixed game, M must be state dependent. Can the states be chosen in a chaotic way so that M is pseudo-random?

(iii) What happens if M is not dependent on capital but on some other parameter, such as game sequence number? What if M is allowed to vary in some fashion during play?

(iv) Where does the correspondence between the *continuous* Brownian ratchet and the *discrete* Parrondo ratchet break down? What would these points of departure teach us?

(v) What happens if games A and B are recast with qubits, where negative quantum probability amplitudes allow cancellation effects? This quantum Parrondo game opens up a number of questions. If game A can be interpreted as noise, can we devise a quantum Parrondo game where decoherence pushes the system in a preferred direction? Can game A be replaced by a measurement, as this is a form of decoherence? Can noise be counter-intuitively used to push the system into a decoherence free subspace (DFS)? Classically we know that the winning rate of the random AB mixed game divided by the winning rate of coin p_2 alone is the "engine efficiency" of the discrete-time ratchet—classically this efficiency cannot exceed unity. The question is, can a quantum Parrondo game achieve greater efficiency than the classical case? We doubt that the quantum case could give rise to efficiencies greater than unity, but the formal proof of this is another interesting question. Classically, the state-dependent rule acts as coupling between games A and B via the capital—so another interesting open question for quantum Parrondo games would be to investigate if the coupling can be achieved via rules based on quantum entanglement, rather than on the capital.

(vi) With reference to Brownian ratchets, it is possible for a probability current, J, to be reversed.[43,44] This means that by changing some characteristics of the ratchet system (switching rates or type of fluctuations for example), the Brownian particles can be made to travel in the opposite direction. The open question is to explore this phenomenon further in Parrondo's games.

(vii) In the last section we gave a heuristic expression for "engine efficiency" of the games as the ratio of the rate of winning in the mixed AB game to that of the coin with bias p_2 played alone. This is not rigorous, and the open question is to come up with a formal expression for engine efficiency for generalized games played with n biased coins.

(viii) Another interesting question is to ask if it is possible to recast the games as an *inference problem*. For example, consider a version of game B composed of n biased coins, which is hidden from Bob. Alice has access to the coins and only tells Bob the outcome of each state of play. Is it possible for Bob to construct an optimal set of different game A's so that he can infer game B, based on the information Alice gives him when A and B are mixed? If we think of the A games as "keys" and of a message as encoded in the bias values of the n coins in game B, could a quantum version of this game have consequences for cryptography?

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APPENDIX

Here we clarify the mechanism of GCN4 productions[45] and, in particular explain which inhibitory mechanisms are combined. First let us define a few terms. Every mRNA usually has three regions—the middle region is the section that does the work and encodes to a protein, a beginning region and an ending region. The middle encoding region is called the "open reading frame" or ORF. So, in general, an ORF would be translated by a ribosome and then encoded to a protein. Now, reading of the code happens in a particular direction—the starting end of the molecule is labeled 5′ and the end is labeled 3′. (Why 5′ and 3′ are used as labels has to do with chemical bond positions, and is not relevant to this discussion.) As the beginning and ending regions are not encoded to a protein, they are called untranslated regions (UTRs). So the three regions of the mRNA are called 5′UTR, ORF and 3′UTR.

In eukaryotes (i.e., yeasts, plants, animals, but not bacteria), mRNA is usually translated as follows: a ribosome binds to the 5′ end of the mRNA and scans through the 5′UTR until it finds a start codon. It then translates the ORF until it reaches a stop codon, and then it releases the manufactured protein. The ribosome may continue scanning along the 3′UTR for a little while until it unbinds from the mRNA. In general, a ribosome will not go through a process of *reinitiation*—that is, if it finds another ORF downstream from one that it has just translated, it will not translate again.

The interesting thing about GCN4 mRNA is that it has four small ORFs embedded in its 5′UTR. Let us call these mini ORFs "upstream ORFs" or uORFs. Now, even one of these uORFs should be deadly to the translation of GCN4, because the ribosome would see it first, then translate it, and then be inactive by the time it reaches the real ORF that encodes to GCN4 protein. However, it turns out that the uORFs (in particular the first and fourth uORFs) are critical for the proper regulation of GCN4—i.e. its repression in amino acid rich medium and activation during starvation.

We stated that a ribosome generally does not reinitiate translation after translation of an ORF. This remains true for the uORF4 in GCN4. If the ribosome translates it, then it will not translate the main ORF region. For an unknown physical reason, some ribosomes can reinitiate after translating uORF1. So what happens is that the ribosome first reaches uORF1 and translates it. It skips over uORF2 and uORF3 and is ready to reinitiate by the time it gets to uORF4. It translates the uORF4 and then is deactivated. So, the ribosome never (or very rarely) gets to the ORF to manufacture GCN4 protein.

This is the case under normal amino acid rich conditions. When amino acids are low and starvation sets in, something new happens. The uORF1 is still translated, but now the ribosome does not reinitiate as efficiently as before. Due to this weak reinitiation, the ribosomes tend to skip by the remaining uORFs, so that now a greater number of ribosomes are ready to reinitiate by the time they reach the GCN4 ORF. The result is that GCN4 protein is produced, which then triggers the production of amino acids.

So there are two inhibitory influences or two losing games here that are combining (to win) to produce GCN4. The first losing game is that of reinitiation at uORF4 that inhibits any production of GCN4. The second losing game is the inhibition of reinitiation under starvation conditions (intuitively we would hope for stronger reinitiation so that the ribosome can reach the ORF to produce GCN4). However, the combination of these two losing games results in fewer reinitiations at uORF4 and more at the ORF, which is a win. The open question is can games be constructed to model this process? The sequence of uORFs is very reminiscent of a ratchet[46] and the reinitiation behavior provides the necessary asymmetry.

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