Weak signal detection: Condition for noise induced enhancement

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1. Introduction

Recently, the employment of noise in enhancing the performance of signal processors has emerged as a topic of significant interest [1–11]. This notion is rooted in the concept of stochastic resonance (SR) that was first elucidated in the area of climate dynamics [12]. The attraction of SR is that an appropriate non-zero resonance (SR) that was first elucidated in the area of climate dynamics, the noise level can improve, rather than degrade, the performance of signal processors has emerged as a topic of significant interest [1–11]. This notion is rooted in the concept of stochastic resonance (SR) [12]. The attraction of SR is that an appropriate non-zero resonance (SR) that was first elucidated in the area of climate dynamics [12]. The attraction of SR is that an appropriate non-zero resonance (SR) that was first elucidated in the area of climate dynamics [12].

2. Noise enhancement of weak signal detection

2.1. Model

Consider the observation vector \( X = (X_1, X_2, \ldots, X_N) \) of real-valued components \( X_n \) defined by

\[
X_n = \theta s_n + Z_n, \quad n = 1, 2, \ldots, N, \tag{1}
\]

where the components \( Z_n \) form a sequence of independent and identically distributed \((i.i.d.)\) random variables with probability density function (PDF) \( f_Z \) and variance \( \sigma_Z^2 \), and the known signal components \( s_n \) have signal strength \( \theta \) [31]. The average signal power satisfies \( 0 < P_s = \frac{\sum_{n=1}^{N} s_n^2}{N} < \infty \) [31]. The detection problem can be formulated as a hypothesis-testing problem for deciding a null hypothesis \( H_0 (\theta = 0) \) and an alternative hypothesis \( H_1 (\theta > 0) \) associated with the joint probability densities

\[
H_0: \quad f_X(X) = \prod_{n=1}^{N} f_Z(X_n) \quad \text{for } \theta = 0, \tag{2}
\]

\[
H_1: \quad f_X(X) = \prod_{n=1}^{N} f_Z(X_n - \theta s_n) \quad \text{for } \theta > 0. \tag{2}
\]

In order to decide \( H_0 \) or \( H_1 \) on the basis of \( X \), consider a generalized correlation detector

\[
T_{GC}(X) = \sum_{n=1}^{N} g(X_n) s_n \overset{H_1}{\geq} \gamma, \tag{3}
\]

where the memoryless nonlinearity \( g \) has zero mean under \( f_z \), i.e. \( E_z[g(x)] = \int_{-\infty}^{\infty} g(x) f_z(x) \, dx = 0 \) and the test threshold is \( \gamma \) [31]. In the asymptotic case of \( \theta \to 0 \) and for a sufficiently large...
observation size $N$, the test statistic $T_{GC}$, according to the central limit theorem, converges to a Gaussian distribution with mean $E_z[T_{GC}|H_0] = 0$ and variance $\text{var}[T_{GC}|H_0] = E_z[T_{GC}^2|H_0] = N P_s E_z[g^2(x)]$ under the null hypothesis $H_0$ [31]. Similarly, under the hypothesis $H_1$, $T_{GC}$ is also asymptotically Gaussian with mean $E_z[T_{GC}|H_1] = \theta N P_s E_z[g^2(x)]^1$ and variance $\text{var}[T_{GC}|H_1] = [\text{var}[T_{GC}|H_0]]$ [31]. Here, the derivatives $g'(x) = d g(x)/d x$ and $f_z'(x) = df_z(x)/d x$ exist for almost all $x$. Given a false alarm probability $P_F$, the detection probability $P_D$ of the generalized correlation detector can be expressed as

$$P_D = Q \left[ Q^{-1}(P_F) - \theta \sqrt{N P_s} \xi_{GC} \right]$$

$$= Q \left[ Q^{-1}(P_F) - \theta \sum_{n=1}^{N} \frac{z_n^2}{\xi_{GC}} \right],$$

where $Q(x) = \int_{x}^{\infty} \exp[-t^2/2]/\sqrt{2\pi} dt$ and its inverse function is $Q^{-1}$ [31]. Thus, for fixed $N$ and $\theta$, since the signal is known, $P_D$ is a monotonically increasing function of the normalized asymptotic efficacy $\xi_{GC}$ given by [31]

$$\xi_{GC} = \lim_{N \to \infty} \frac{\{d E_z[T_{GC}](x)\}_{\theta = 0}}{\sqrt{N P_s} \text{var}[T_{GC}(x)]_{\theta = 0}} = \frac{E_z^2[g'(x)]}{E_z[g^2(x)]}$$

$$\leq E_z \left[ \frac{f_z'^2(\lambda)}{f_z^2(\lambda)} \right] \equiv I(f_z),$$

where the expectation $E_z[f_z'^2(\lambda)/f_z^2(\lambda)]$ is the Fisher information $I(f_z)$ of $f_z$ and the equality occurs as

$$g(x) = C \frac{f_z'(\lambda)}{f_z(\lambda)} \triangleq g_{LO}(x),$$

by the Cauchy–Schwarz inequality for a constant $C$. Here, $g_{LO}(x)$ represents the locally optimal nonlinearity [31].

It is noted that $P_D$ of (4) is a monotonically increasing function of $\xi_{GC}$. Thus, as the noise level $\sigma_z$ increases, the positive derivative

$$\frac{\partial \xi_{GC}}{\partial \sigma_z} > 0$$

(7)

indicates the occurrence of the noise-enhanced detection phenomenon. When the inequality of (7) holds for $0 < \sigma_z < \sigma_{opt}$ and the equality

$$\frac{\partial \xi_{GC}}{\partial \sigma_z} = 0$$

(8)

has only one solution $\sigma_z = \sigma_{opt}$, then $\sigma_{opt}$ is the optimal noise level that maximizes $\xi_{GC}$. It is noted that the signal strength $\theta$ is small enough to allow us to use the first-order approximations leading to the detection probability of (4), and the noise-enhanced detection performance indicated by (Eq. (7)) is valid for arbitrary small signal level $\theta > 0$.  

In the following, we assume that the scaled noise $Z(t) = \sigma_z Z_0(t)$ has PDF $f_z(z) = f_{Z_0}(z/\sigma_z)/\sigma_z$ and the cumulative distribution function $F_z(z) = F_{Z_0}(z/\sigma_z)$ [10,31]. Here, $Z_0(t)$ has a standardized PDF $f_{Z_0}$ with unity variance $\sigma_{Z_0} = 1$, the cumulative distribution function is $F_{Z_0}(x) = \int_{-\infty}^{x} f_{Z_0}(u) du$ and the Fisher information $I(f_{Z_0}) = E_{Z_0}[f_{Z_0}'(\lambda)/f_{Z_0}^2(\lambda)]$. Then, the Fisher information $I(f_z) = I(f_{Z_0})/\sigma_z^2$.

2.2. Noise enhancement by noise tuning

**Corollary 1.** No noise-enhanced detection phenomenon will occur in the locally optimal detector

$$T_{LO}(X) = \sum_{n=1}^{N} g_{LO}(X_n) s_n \xi_{LO} \gamma,$$

with the nonlinearity $g_{LO}$ defined in Eq. (6).

**Proof.** From Eqs. (5) and (6), the locally optimal detector in (9) has the normalized asymptotic efficacy $\xi_{LO} = I(f_z) > 0$. Then, for $\sigma_z > 0$, we have

$$\frac{\partial \xi_{LO}}{\partial \sigma_z} = \frac{\partial I(f_z)}{\partial \sigma_z} = -2I(f_{Z_0}) < 0.$$  

(10)

Thus, no noise-enhanced detection phenomenon will occur. □

**Corollary 2.** The dead-zone limiter detector

$$T_{DZ}(X) = \sum_{n=1}^{N} g_{DZ}(X_n) s_n \xi_{DZ} \gamma,$$

employs the characteristic

$$g_{DZ}(x) = \begin{cases} -1 & \text{for } x < -\lambda, \\ 0 & \text{for } -\lambda \leq x \leq \lambda, \\ +1 & \text{for } x > \lambda, \end{cases}$$

with response threshold $\lambda > 0$. Given the threshold $\lambda$, the noise-enhanced detection effect will occur in the interval $\sigma_z \in (0, \sigma_{opt}^D)$, where the optimal noise level $\sigma_{opt}^D$ is the non-zero solution of

$$\frac{\sigma_z}{\lambda} = \frac{g_{LO}(\lambda/\sigma_z)}{f_{Z_0}(\lambda/\sigma_z)} - \frac{f_{Z_0}(\lambda/\sigma_z)}{2[1 - F_{Z_0}(\lambda/\sigma_z)]},$$

(13)

with the nonlinearity

$$g_{LO}^{DZ}(x) = \frac{f_z'(\lambda)}{f_{Z_0}(\lambda/\sigma_z)}$$

(14)

**Proof.** From Eq. (5), the normalized asymptotic efficacy $\xi_{DZ}$ of the dead-zone limiter detector is [31,32]

$$\xi_{DZ} = \frac{E_z^2[g_{DZ}'(x)]}{E_z[g_{DZ}^2(x)]} = \frac{f_z^2(\lambda)}{1 - F_z(\lambda)}.$$  

(15)

Since

$$\frac{\partial f_z(\lambda)}{\partial \sigma_z} = \frac{\partial f_{Z_0}(\lambda/\sigma_z)}{\partial \lambda} \frac{1 - f_z(\lambda)}{f_z(\lambda)} - \frac{f_z(\lambda)}{\sigma_z^2} = \frac{\partial f_{Z_0}(\lambda/\sigma_z)}{\partial \lambda} \frac{1 - f_{Z_0}(\lambda/\sigma_z)}{f_{Z_0}(\lambda/\sigma_z)} \geq 0,$$

we obtain

$$\frac{\partial \xi_{DZ}}{\partial \sigma_z} = 4 f_z(\lambda) \frac{\partial f_{Z_0}(\lambda/\sigma_z)}{\partial \lambda} \frac{1 - f_z(\lambda)}{f_z(\lambda)} \geq 0,$$

(17)

$$\Rightarrow \frac{\partial f_z(\lambda)}{\partial \sigma_z} \frac{\lambda}{\sigma_z^2} f_z^2(\lambda) \leq \frac{\lambda}{\sigma_z^2} f_{Z_0}(\lambda/\sigma_z) \\ \Rightarrow -\frac{\lambda}{\sigma_z^2} f_{Z_0}(\lambda/\sigma_z) \leq \frac{\lambda}{\sigma_z^2} f_z(\lambda) \geq 0,$$

(18)

$$\Rightarrow \frac{\sigma_z}{\lambda} \leq \frac{g_{LO}(\lambda/\sigma_z)}{f_{Z_0}(\lambda/\sigma_z)} - \frac{f_{Z_0}(\lambda/\sigma_z)}{2[1 - F_{Z_0}(\lambda/\sigma_z)]}.$$  

(19)

where the equality of Eq. (20) gives the non-zero solution $\sigma_{opt}^D$. The numerical solution of $\sigma_{opt}^D$ can refer to [33]. When the noise level $0 < \sigma_z < \sigma_{opt}^D$, the derivative $\partial \xi_{DZ}/\partial \sigma_z > 0$, and the noise-enhanced effect will appear in the dead-zone limiter detector of Eq. (11). □
The corresponding nonlinearity of Eq. (14) is
\[
g_{\text{LO}}(x) = \alpha c_2 |x|^{\alpha-1} \text{sign}(x). 
\] (22)
For the dead-zone limiter detector of Eq. (11), Eq. (13) becomes
\[
\sigma_{\text{LO}} = \frac{\alpha c_2 \lambda}{\sigma_x} \left[ \frac{1}{\sigma_x} - \frac{1}{\sigma_y} \right] - \frac{c_1}{2} \left[ 1 - F_\alpha \left( \frac{1}{\sigma_y} \right) \right]. 
\] (23)

Without loss of generality, the response threshold takes \( \lambda = 1 \), and the optimal noise level \( \sigma_{\text{LO}}^{\text{opt}} \) is shown in Fig. 1(a) as a function of the exponent \( \alpha \). It is illustrated in Fig. 1(b) that, as the noise level \( \sigma_x \) increases from zero to \( \sigma_{\text{LO}}^{\text{opt}} \), the normalized asymptotic efficacy \( \xi_{\text{LO}} \) is enhanced to its maximum for different exponents \( \alpha = 0.5, 1, 2, 5 \) and \( \infty \). Fig. 1(a) also shows that, as the exponent \( \alpha \) increases, the optimal level of \( \sigma_{\text{LO}}^{\text{opt}} \) tends to a constant value of \( 1/\sqrt{3} \), which is just the optimal noise level \( \sigma_{\text{LO}}^{\text{opt}} \) corresponding to \( \alpha = \infty \) (uniform noise), as shown in Fig. 1(b).

**Example 1.** The non-Gaussian noise is often useful for modeling practical noisy environments where signals and systems are operated [31,32]. For example, a non-Gaussian model is the generalized Gaussian noise with PDF
\[
f_z(x) = \frac{c_1}{\sigma_x} \exp \left( -c_2 \left| \frac{x}{\sigma_x} \right|^\alpha \right). 
\] (21)
where \( c_1 = \frac{\Gamma(\frac{\alpha}{2})}{\sqrt{\pi} \Gamma(\frac{\alpha}{2} + \frac{1}{2})} \) and \( c_2 = \left( \frac{1}{\Gamma(\frac{\alpha}{2})} \right)^{\frac{1}{2}} \). A positive exponent \( \alpha \) allows us conveniently consider a spectrum of densities ranging from the Gaussian to those with relatively much faster or slower rates of exponential decay of their tails [31]. The corresponding nonlinearity of Eq. (14) is
\[
g_{\text{LO}}^g(x) = \alpha c_2 |x|^{\alpha-1} \text{sign}(x). 
\] (22)

**Corollary 3.** No noise-enhanced detection phenomenon will occur for the sign detector of Eq. (11) with threshold \( \lambda = 0 \) and characteristic \( g_{\text{LO}}(x) = \text{sign}(x) \).

**Proof.** From Eq. (5), the normalized asymptotic efficacy \( \xi_{\text{LO}} \) of the sign detector is
\[
\xi_{\text{LO}} = \frac{\mathbb{E}_w \left[ g_{\text{LO}}^2(x) \right]}{\mathbb{E}_w \left[ g_{\text{LO}}^2(x) \right]} = 4 f_2^2(0) = \frac{4 f_2^2(0)}{\sigma_x^2}. 
\] (24)

Then, we find
\[
\frac{\partial \xi_{\text{LO}}}{\partial \sigma_x} = -\frac{8 f_2^2(0)}{\sigma_x^3} < 0, 
\] (25)
for \( \sigma_x > 0 \). Therefore, no noise-enhanced detection phenomenon will occur. \( \square \)

**2.3. Noise enhancement by adding noise**

The received signal is often corrupted by noise before it arrives at the detector. We now add additional noise to a given observation vector \( X \) in the context of SR. The updated components
\[
\hat{X}_n = \theta s_n + Z_n + Y_n = \theta s_n + W_n, 
\] (26)
where the added i.i.d. random variables \( Y_n \) are with PDF \( f_y \) and variance \( \sigma_y^2 \). Then, the composite components \( W_n \) have a convolved PDF \( f_w(x) = \int_{-\infty}^{\infty} f_y(x-u) f_z(u) du \). In this case, the normalized asymptotic efficacy of Eq. (5) is updated as
\[
\hat{\xi}_{\text{LO}} = \frac{\mathbb{E}_w \left[ g_{\text{LO}}^2(x) \right]}{\mathbb{E}_w \left[ g_{\text{LO}}^2(x) \right]} = \tilde{I}(f_w), 
\] (27)
with the Fisher information \( I(f_w) \) of \( f_w \). Here, the equality is achieved by an updated locally optimal detector
\[
\hat{T}_{\text{LO}}(\hat{X}) = \sum_{n=1}^{N} \hat{g}_{\text{LO}}(\hat{X}_n) s_n \geq \gamma, 
\] (28)
based on the locally optimal nonlinearity
\[
\hat{g}_{\text{LO}}(x) = C f_w'(x). 
\] (29)
Furthermore, we assume \( f_w(x) = f_w(x/\sigma_w)/\sigma_w \) and \( f_w(x) \) is the standardized noise PDF with unity variance. Then, we have the following corollaries.

**Corollary 4.** No noise-enhanced detection phenomenon will occur in the updated locally optimal detector of Eq. (28).

**Proof.** For the composite noise components \( W_n \), the noise variance \( \sigma_w^2 = \sigma_y^2 + \sigma_z^2 \) and the initial noise variance \( \sigma_z^2 \) is fixed. Then, we have
\[
\frac{\partial \hat{\xi}_{\text{LO}}}{\partial \sigma_y} = \frac{\partial \hat{I}(f_w)}{\partial \sigma_y} = \frac{\partial I(f_w)}{\partial \sigma_w} \frac{\sigma_y}{\sqrt{\sigma_y^2 + \sigma_z^2}} = -\frac{2\sigma_y I(f_w)}{\sigma_w^2} < 0, 
\] (30)
where \( I(f_w) > 0 \) is the Fisher information of \( f_w \). Then, Corollary 4 is deduced. \( \square \)
Corollary 5. When the noise level $0 < \sigma_y < \sigma_{yo}$, the noise-enhanced detection phenomenon will occur for the dead-zone limiter detector of Eq. (11). Here, for a fixed noise level $\sigma_z$, the optimal noise level
\[
\sigma_{yo} = \sqrt{\left(\sigma_w^{opt}\right)^2 - \sigma_z^2},
\]
and $\sigma_w^{opt}$ is the non-zero solution of
\[
\frac{\sigma_w}{\lambda} = \frac{\hat{Q}_{LO}(\lambda)}{\hat{Q}_{LO}(\sigma_w)} = \frac{f_{w0}(\frac{\lambda}{\sigma_w})}{2 \left(1 - f_{w0}(\frac{\lambda}{\sigma_w})\right)},
\]
with the nonlinearity
\[
\hat{\xi}_{LO}(x) = -\frac{f_{w0}(x)}{f_{w0}(\lambda)}.
\]

Proof. For the composite noise components $W_n$, the normalized asymptotic efficacy of the dead-zone limiter detector of Eq. (11) can be calculated as
\[
\hat{\xi}_{DZ} = \frac{E_w[\xi_{DZ}(x)]}{E_w[\xi_{DZ}(x)]} = \frac{2 f_w(\lambda)}{1 - F_w(\lambda)},
\]
where $F_w$ represents the cumulative distribution function of $W_n$. Then, the noise-enhanced detection effect will occur as
\[
\frac{\partial \hat{\xi}_{DZ}}{\partial \sigma_y} + \frac{\partial \hat{\xi}_{DZ}}{\partial \sigma_w} \frac{\partial \sigma_w}{\partial \sigma_y} = \frac{\partial \hat{\xi}_{DZ}}{\partial \sigma_w} \frac{\sigma_w}{\sigma_y} > 0 \Rightarrow \frac{\partial \hat{\xi}_{DZ}}{\partial \sigma_w} > 0.
\]
The demonstration is similar to the proof of Corollary 2, and the occurrence condition is indicated by Eq. (32). Correspondingly, the optimal added noise level $\sigma_y^{opt}$ and $\sigma_w^{opt}$ can be solved by Eqs. (31) and (32). □

Example 2. Assume the initial Gaussian noise components $Z_n$ are with PDF $f_z(x) = \frac{1}{\sqrt{2\pi}\sigma_z} \exp\left(-\frac{x^2}{2\sigma_z^2}\right)$ and fixed variance $\sigma_z^2$. The added uniform random variables $Y_n$ have PDF $f_y(x) = 1/(2b)$ for $-b \leq x \leq b$ and zero otherwise. The composite random variables $W_n$ have PDF
\[
f_w(x) = \frac{Q\left(\frac{x+b}{\sigma_z}\right) - Q\left(\frac{x-b}{\sigma_z}\right)}{2b}.
\]
For the dead-zone limiter detector in Eq. (11), the normalized asymptotic efficacy of Eq. (34) can be expressed as
\[
\hat{\xi}_{DZ} = \frac{\int_{-\infty}^{\infty} b \left[Q\left(\frac{x+b}{\sigma_z}\right) - Q\left(\frac{x-b}{\sigma_z}\right)\right] dx}{\lambda}.
\]
Then, the noise-enhanced effect will occur for $\partial \hat{\xi}_{DZ}/\partial b \geq 0$, this is
\[
2 \int_{\lambda} f_z(\lambda - b) f_z(\lambda + b) \left[Q\left(\frac{x-b}{\sigma_z}\right) - Q\left(\frac{x+b}{\sigma_z}\right)\right] dx - \left[Q\left(\frac{\lambda-b}{\sigma_z}\right) - Q\left(\frac{\lambda+b}{\sigma_z}\right)\right] \\
\times \left\{\int_{\lambda} \left[Q\left(\frac{x-b}{\sigma_z}\right) - Q\left(\frac{x+b}{\sigma_z}\right)\right] dx + b \left[f_z(x-b) + f_z(x+b)\right]\right\} \geq 0,
\]
\[
\Rightarrow 4b^2 \left[f_z(\lambda - b) + f_z(\lambda + b)\right] \int_{\lambda} f_w(x) dx > 0.
\]

The optimal noise level $b^{opt}$ of added uniform noise versus the initial Gaussian noise level $\sigma_z$. Here, the dead-zone detector is with the response threshold $\lambda = 1$. The normalized asymptotic efficacy $\hat{\xi}_{DZ}$ of Eq. (37) for the dead-zone limiter detector as a function of the added uniform noise level $b$. Here, the initial Gaussian noise level is fixed as $\sigma_z = 0.3$.

\[
-2b^2 f_w(\lambda) \left\{2 \int_{\lambda} f_w(x) dx + \left[Q\left(\frac{\lambda-b}{\sigma_z}\right) + Q\left(\frac{\lambda+b}{\sigma_z}\right)\right]\right\} \geq 0.
\]

Thus, the optimal uniform noise level $b^{opt}$ can be solved by
\[
f_w(\lambda) \left[Q\left(\frac{\lambda-b}{\sigma_z}\right) + Q\left(\frac{\lambda+b}{\sigma_z}\right)\right] + 2 f_w(\lambda) \left[1 - F_w(\lambda)\right] \\
= 2 \int_{\lambda} f_w(x) dx \geq 0.
\]

The noise-enhanced effect will occur as the uniform level $0 < b < b^{opt}$. Without loss of generality, the response threshold takes $\lambda = 1$, and the optimal uniform noise level $b^{opt}$ is plotted in Fig. 2(a) as a function of the initial Gaussian noise level $\sigma_z$. For instance, when the initial Gaussian noise level $\sigma_z = 0.3$, the corresponding normalized asymptotic efficacy of Eq. (15) is $\hat{\xi}_{DZ} = 0.1232$ without the addition of uniform noise ($\hat{\xi}_{DZ} = 2.0922$ at $b^{opt} = 1.02$, as illustrated in Fig. 2(b)).

An important issue is that, for a given noise level $\sigma_z$, we can tune the threshold $\lambda$ to maximize the normalized asymptotic efficacy $\hat{\xi}_{DZ}$. Michels et al. demonstrated that the normalized asymptotic efficacy of the tuned dead-zone limiter detector with optimal threshold $\lambda^{opt}$ cannot be improved by adding noise to the signal [30] (Section 5.3, pp. 33–35). For the case of where the threshold is not optimal, they further proved that the optimal
detection performance can be achieved by adding independent dichotomous noise [23]. For a fixed threshold \( \lambda \), Corollaries 2 and 5 apply our general characterization to the dead-zone limiter detector for any type of scaled noise. The optimal noise level can be solved by Eq. (13) and (Eq. (32).

For the scaled noise PDF \( f_\sigma(x) = f_\text{sn}(x/\sigma_\text{sn})/\sigma_\text{sn} \) with a given noise level \( \sigma_\text{sn} \) and based on Eq. (15), the optimum threshold \( \lambda_{\text{opt}} \) can be solved by

\[
\frac{\partial \xi_{\text{DZ}}}{\partial \lambda} = 0,
\]

\[
4 f_\sigma(\lambda) f'_\sigma(\lambda) \left( 1 - F_\sigma(\lambda) \right) + 2 f_\sigma^2(\lambda) f_\sigma(\lambda) = 0,
\]

\[
2 g^2_{\text{GC}}(\lambda) f_\sigma(\lambda) \left( 1 - F_\sigma(\lambda/\sigma_\text{sn}) \right) - f_\text{sn}(\lambda) = 0.
\]

In Example 2, the initial Gaussian noise is with a given noise level \( \sigma_\text{sn} \). Eq. (43) yields the optimal threshold \( \lambda_{\text{opt}} = 0.612 \sigma_\text{sn} \). The fixed threshold \( \lambda = 1 \) is optimal for the initial noise level \( \sigma_\text{sn} = 1.634 \). It is shown in Fig. 2(a) that, for the fixed threshold \( \lambda = 1 \), the non-zero solution of added uniform noise level \( b_{\text{opt}} \) only exits for the initial Gaussian noise level \( 0 < \sigma_\text{sn} < 0.61 \). In other words, for the given initial noise level \( \sigma_\text{sn} > 0.61 \) (including the optimal matching noise level \( \sigma_\text{sn} = 1.634 \) for threshold \( \lambda = 1 \)), no enhancement by noise can take place. In this respect, our results here accord with the conclusions of [29,30] that the normalized asymptotic efficacy of the dead-zone limiter detector with optimal threshold cannot be improved by adding noise, but the SR effect is possible when the threshold is not optimal for the initial noise level.

### 2.4. Noise enhancement in a parallel array of nonlinearities

The constructive role of internal noise has been adequately reappraised for improving the performance of an array of nonlinearities [3,4,7–10]. Compared with an isolated nonlinearity, the performance of an array can be much improved by the internal noise [3,4,7–10]. Moreover, the positive role of noise does not need to occur for an isolated nonlinearity, but can come into play in a parallel array of nonlinearities [4,7–10].

Let \( \tilde{X}_\text{sn} = (X_{n1}, \tilde{X}_{n2}, \ldots, \tilde{X}_{nM}) \) be the vector of \( N \) observation components at the \( m \)-th element of receiving array of M identical nonlinearities. In this observation model [4], \( \tilde{X}_\text{sn} = X_n + \tilde{Y}_\text{sn} = \theta_{SN} + Z_n + Z_{nM} = \theta_{SN} + W_{m,n} \). Here, in each nonlinearity \( m \), the M noise terms \( Y_m \) are assumed to be mutually independent with the same PDF \( f_\sigma \) and variance \( \sigma^2_\text{sn} \). Then, at the observed time \( n \), the array outputs are collected as \( \tilde{g}_n = \sum_{m=1}^M g(\tilde{X}_{m,n})/M \), and the generalized correlation detector can be constructed as

\[
T_{\text{GC}}(\tilde{x}) = \sum_{n=1}^N \tilde{g}_n S_n \geq \gamma. (44)
\]

The statistic \( T_{\text{GC}} \) is also asymptotically Gaussian for a sufficiently large observed size \( N \). Under the null hypothesis \( H_0 \), the mean \( E_w[T_{\text{GC}}|H_0] = E_w[g(w)] \sum_{n=1}^N s_n = 0 \) and the variance

\[
\text{var}[T_{\text{GC}}|H_0] = E_w[E_{n=1}^2[T_{\text{GC}}|H_0] - \left( E_{n=1}[T_{\text{GC}}|H_0] \right)^2]
\]

\[
= N P_y E_{y} \left\{ \frac{1}{M^2} \sum_{k=1}^M \sum_{k=1}^M E_y[g(W_m)g(W_k)] \right\}
\]

\[
= \frac{N P_y}{M^2} E_{y} \left\{ M E_y[g^2](W_m) \right\} + M(M-1) E_y[g(W_m)g(W_k)] \quad \forall m \neq k
\]

\[
+ \frac{N P_y}{M} \left[ E_w[g^2(w)] + (M - 1) E_{y} \left\{ E_y[g(y+z)] \right\} \right].
\]

where \( E_{y} [E_y[g(W_m)g(W_n)]] = E_{y} [E_y[g^2(w)]] = E_{y} [E_y[g(y+z)]] \). Under the hypothesis \( H_1 \) and as the signal strength \( \theta \to 0 \), the mean has the asymptotic form

\[
E_{w}[T_{\text{GC}}|H_1] = E_{w} \left[ \sum_{n=1}^N \left\{ \sum_{m=1}^M g(\theta s_n + W_{mn}) \right\} s_n \right]
\]

\[
= E_{w} \left[ \sum_{n=1}^N \left\{ \sum_{m=1}^M g(w) + \theta s_n g'(w) \right\} s_n \right]
\]

\[
= \theta N P_y E_{w}[g'(w)].
\]

and variance \( \text{var}[T_{\text{GC}}|H_1] = \text{var}[T_{\text{GC}}|H_0] \). Then, the normalized asymptotic efficacy of the detector in Eq. (44) is given by

\[
\xi_{\text{DZ}} = \lim_{N \to \infty} \frac{E_{w} [E_y[g^2](w)] + (M - 1) E_{y} [E_y[g(y+z)]]}{E_{w} [E_y[g^2](w)] + \frac{M - 1}{M} E_{y} [E_y[g^2](w)] + \frac{1}{M} E_{y} [E_y[g(y+z)]].}
\]

Example 3. We choose the characteristic \( g(x) = \text{sign}(x) \) in the detector of Eq. (44). The initial noise \( Z(t) \) is Gaussian distributed, and the \( M \) array noise terms \( Y_m(t) \) are uniformly random variables. The composite noise \( W_{m}(t) \) are with the convolved PDF \( f_w \) of Eq. (36), as indicated in Example 2. Therefore, the normalized asymptotic efficacy is computed as

\[
\xi_{\text{DZ}} = \frac{4 f^2_{\text{opt}}(0)}{N P_y E_{w} [E_y[g^2](w)] + \frac{M - 1}{M} E_{y} [E_y[g(y+z)]].}
\]

\[
= \frac{4 f^2_{\text{opt}}(0)}{N P_y E_{w} [E_y[g^2](w)] + \frac{M - 1}{M} E_{y} [E_y[g^2](w)] + \frac{1}{M} E_{y} [E_y[g^2](w)].}
\]

Since the noise-enhanced phenomenon occurs when \( \partial \xi_{\text{DZ}}/\partial b \geq 0 \), it is found that the optimal noise level \( b_{\text{opt}} \) is the solution of

\[
E_{w} \left[ \left( f_w(0) - f_w(b) \right) \left\{ 1 + (M - 1) E_y \left[ \frac{(|z+b| - |z-b|)^2}{4b^2} \right]\right\} \right]
\]

\[
= E_{w} \left[ \left( f_w(0) - f_w(b) \right) \left\{ \frac{(|z+b| - |z-b|)^2}{2b^2} \right\}
\]

\[
- (|z+b| - |z-b|) \text{sign}(z+b) + \text{sign}(z-b) \right\}. \]

For the array size \( M = 1 \), Eq. (49) yields \( f_w(0) - f_w(b) = 0 \) and the optimal uniform noise level \( b_{\text{opt}} = 0 \). Thus, there is no noise-enhanced effect in the detector of Eq. (44) with a single nonlinearity. For a fixed Gaussian noise level \( \sigma_\text{sn} = 0.3 \), the optimal added uniform noise level \( b_{\text{opt}} \) is illustrated as a function of the array size \( M \) in Fig. 3(a). It is shown in Fig. 3(b) that the normalized asymptotic efficacy \( \xi_{\text{DZ}} \) varies as a function of added uniform noise level \( b \) for different array sizes. For a single nonlinearity \( g \), it is seen that the added uniform noise is no use for the performance enhancement of the detector \( (M = 1) \). As \( M \geq 2 \), it is seen in Fig. 3(b) that the added uniform noise can enhance the normalized asymptotic efficacy \( \xi_{\text{DZ}} \), and the noise-enhanced effect does occur. Moreover, as the array size \( M \) increases, the peak value of \( \xi_{\text{DZ}} \) is also improved gradually by tuning the added uniform noise level into the corresponding optimal value of \( b_{\text{opt}} \), as shown in Fig. 3(b).
3. Conclusion

In this paper, we study the noise-enhanced detection of a weak known signal in additive white noise. For a sufficiently large observation size, the performance of a generalized correlation detector is determined by the normalized asymptotic efficacy $\xi_{GC}$. Then, the positive derivative of $\xi_{GC}$ with respect to the noise level indicates the occurrence of the noise-enhanced detection effect. According to this condition, we arrive at some interesting conclusions on whether the role of noise in a generalized correlation detector offers an enhancement or not.

We here only consider some analytical nonlinearities, e.g. the dead-zone limiter nonlinearity and the locally optimal nonlinearity. There are other interesting nonlinearities such as the saturation nonlinearity [34] and the soft-threshold nonlinearity [35], which can be of interest for further studies of weak signal detection in the context of SR.

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References


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