Noise-enhanced SNR gain in parallel array of bistable oscillators

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For a sinusoidal signal, buried in white Gaussian noise, as an input to a parallel array of bistable oscillators, it is reported that conditions exist where the signal-to-noise ratio gain exceeds unity, for both subthreshold and suprathreshold sinusoids. The performance of infinite arrays is closely approached by finite arrays of moderate size, representing a novel method of applying the stochastic resonance phenomenon to array signal processing.

Introduction: Stochastic resonance (SR) is a well-known noise-induced nonlinear phenomenon existing in a variety of systems, and by which detection of a periodic or aperiodic signal can be enhanced by the addition of noise. The measure most frequently employed for conventional (periodic) SR is the signal-to-noise ratio (SNR). The SNR gain defined as the ratio of the output SNR over the input SNR, also attracts much interest in exploring situations where it can exceed unity. The SNR gain has been studied in the less stringent condition of narrowband noise [1]. Here, we address the more stringent condition of broadband white noise and the SNR gain achievable by summing the output of a parallel array of bistable oscillators, wherein extra array noise can be tuned to maximise the SNR gain. For an infinite parallel array, a tractable realisation is proposed as an array of two bistable oscillators in view of the functional limit of the autocovariance function, and conditions where the SNR gain exceeds unity are demonstrated numerically for both subthreshold and suprathreshold sinusoids.

Model and method: The parallel array of over-damped bistable oscillators is considered as a model. Each oscillator is subject to the same signal-plus-noise mixture $s(t)+\xi(t)$, where $s(t)$ is a sinusoid with period T_s and amplitude A, and $\xi(t)$ is zero-mean Gaussian white noise, independent of $s(t)$, with autocorrelation $\langle\xi(t)\xi(0)\rangle = D_\xi\delta(t)$ and noise intensity D_ξ. At the same time, zero-mean Gaussian white noise $\eta_i(t)$, together with and independent of $s(t)+\xi(t)$, is applied to each element of the parallel array with size N. The N noise terms $\eta_i(t)$ are mutually independent and have autocorrelation $\langle\eta_i(t)\eta_i(0)\rangle = D_\eta\delta(t)$ with a same noise intensity D_η. The internal state $x_i(t)$ of each oscillator is described as

$$\tau_a\frac{dx_i(t)}{dt} = x_i(t) - \frac{x_i^3(t)}{X_b^2} + s(t) + \xi(t) + \eta_i(t) \tag{1}$$

for $i = 1, 2, \ldots, N$. Their outputs are summed as the response of the array $y(t) = \sum_{i=1}^{N} x_i(t)/N$. Here, τ_a and X_b are real tunable array parameters. We now rescale the variables according to (where each arrow points to dimensionless variables):

$$x_i(t)/X_b \to x_i(t), t/\tau_a \to t, A/X_b \to A, T_s/\tau_a \to T_s, \\ D_\xi/(\tau_a X_b^2) \to D_\xi, D_\eta/(\tau_a X_b^2) \to D_\eta \tag{2}$$

Equation (1) is then recast in dimensionless form as

$$\frac{dx_i(t)}{dt} = x_i(t) - x_i^3(t) + s(t) + \xi(t) + \eta_i(t) \tag{3}$$

Note that $s(t)$ is subthreshold if $A < A_c = 2/\sqrt{(27)} \simeq 0.385$, otherwise it is suprathreshold [2, 3]. In this Letter, we numerically integrate (3) using Euler-Maruyama discretisation with a sampling time step $\Delta t \ll \tau_a$ and T_s.

Since $s(t)$ is periodic, the array response $y(t)$ is a cyclostationary random signal. The nonstationary mean $E[y(t)] = \sum_{i=1}^{N} E[x_i(t)]/N = E[x_i(t)]$ is a deterministic periodic function of time t with period T_s, having the order n Fourier coefficient

$$\bar{Y}_n = \left\langle E[y(t)]\exp\left(-\imath 2\pi\frac{n}{T_s}\right)\right\rangle \tag{4}$$

where $\langle\cdots\rangle = (1/T_s)\int_0^{T_s}\cdots dt$. At time t and $N\to\infty$, we have

$$\begin{aligned}\lim_{N\to\infty} R_{yy}(\tau) &= \lim_{N\to\infty}\langle E[y(t)y(t+\tau)]\rangle \\ &= \lim_{N\to\infty}\left\langle\frac{E[x_i(t)x_i(t+\tau)] + (N-1)E[x_i(t)x_j(t+\tau)]}{N}\right\rangle \\ &= \langle E[x_i(t)x_j(t+\tau)]\rangle \\ &= R_{x_ix_j}(\tau)\end{aligned} \tag{5}$$

and

$$\begin{aligned}\lim_{N\to\infty} C_{yy}(\tau) &= \lim_{N\to\infty} R_{yy}(\tau) - \lim_{N\to\infty}\langle E[y(t)]E[y(t+\tau)]\rangle \\ &= \lim_{N\to\infty} R_{yy}(\tau) \\ &\quad - \lim_{N\to\infty}\left\langle\frac{E[\sum_{i=1}^{N} x_i(t)]E[\sum_{j=1}^{N} x_j(t+\tau)]}{N^2}\right\rangle \\ &= \langle E[x_i(t)x_j(t+\tau)]\rangle - \langle E[x_i(t)]E[x_j(t+\tau)]\rangle \\ &= C_{x_ix_j}(\tau)\end{aligned} \tag{6}$$

for $i \neq j$ and $i, j = 1, 2, \ldots, N$. Note that $E[x_i(t)] = E[x_j(t)]$. The output SNR is the power contained in the output spectral line $1/T_s$ divided by the power contained in the noise background in a small frequency bin $\Delta B = 1/T_s$ around $1/T_s$, i.e.

$$R_{\text{out}}(1/T_s) = \frac{|\bar{Y}_1|^2}{\langle\text{var}[y(t)]\rangle H(1/T_s)\Delta B} \tag{7}$$

where $C_{yy}(0) = \langle\text{var}[y(t)]\rangle$ is the stationary variance of $y(t)$, $C_{yy}(\tau) = \langle\text{var}[y(t)]\rangle h(\tau)$ and the correlation coefficient $h(\tau)$ has a Fourier transform $\mathcal{F}[h(\tau)] = H(\nu)$ [2]. In the same way, the mixture of $s(t)+\xi(t)$ has an input SNR as

$$R_{\text{in}}(1/T_s) = \frac{A^2/4}{D_\xi\Delta B} = \frac{A^2/4}{\sigma_\xi^2\Delta t\Delta B} \tag{8}$$

where σ_ξ is the rms amplitude of the discrete implementation of Gaussian noise $\xi(t)$. Thus, the SNR gain follows as

$$G(1/T_s) = \frac{R_{\text{out}}(1/T_s)}{R_{\text{in}}(1/T_s)} = \frac{|\bar{Y}_1|^2}{\text{var}[y(t)]H(1/T_s)}\frac{\sigma_\xi^2\Delta t}{A^2/4} \tag{9}$$

Since the indices i and j are different, but arbitrary in (5) and (6), we can adopt two bistable oscillators, each embedded with independent noise, to evaluate the SNR gain of a parallel array with size $N\to\infty$. This method is tractable and effective. Numerical evolution of SNR gains for generic arrays will be presented in future studies in detail.

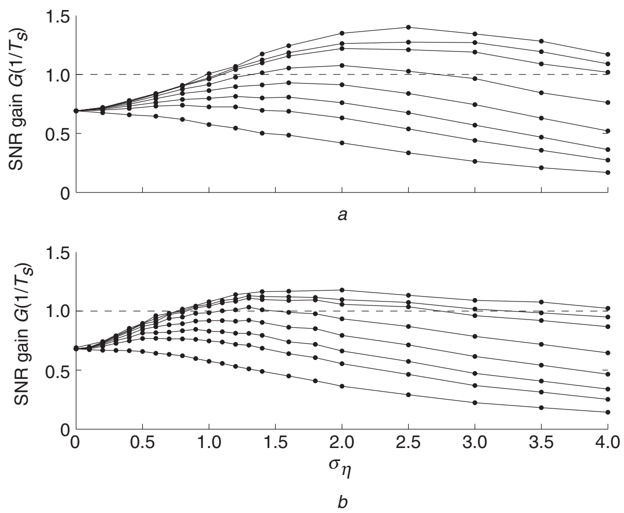

Fig. 1 *SNR gain $G(1/T_s)$ against rms amplitude σ_η of array noise $\eta_i(t)$*

a $A = 1.0 > A_c$ (suprathreshold), $\sigma_\xi = 1.8$ and $R_{\text{in}} = 77.16$
b $A = 0.38 < A_c$ (subthreshold), $\sigma_\xi = 1.8$ and $R_{\text{in}} = 11.14$
$T_s = 100$; $\Delta t = T_s\times 10^{-3}$; total evolution time of (1) is $T_s\times 10^5$ (arbitrary units).
SNR gain curves, from bottom up, correspond to $N = 1, 2, 3, 5, 10, 30, 60, \infty$

Results: Fig. 1 shows that, for a given noisy input, the SNR gain

$G(1/T_s)$ for $N \geq 2$ behaves as an SR-type function of the rms amplitude σ_η of the array noise $\eta_i(t)$. It is more a collective effect of the nonlinear array, and appears for not only suprathreshold inputs but also *subthreshold* signals, as illustrated in Fig. 1. Importantly, Fig. 1 reveals that the regions of SNR gains $G(1/T_s)$ rising above unity, via increasing the rms amplitude σ_η, are possible for moderately large array *N*. As the amplitude *A* increases, $G(1/T_s)$ reaches a larger maximal value for the same array size *N*. As $N \to \infty$, the maximal $G(1/T_s)$ is around 1.4 for $A = 1.0$, as seen in Fig. 1*a*, whereas the maximal $G(1/T_s)$ is about 1.2 for $A = 0.38$, as shown in Fig. 1*b*. We note that $\eta_i(t)$ are more controllable than the original input noise $\xi(t)$. Thus, this nonlinear collective characteristic of dynamical arrays provides a preferable strategy for processing periodic signals than linear systems.

Conclusions: We have studied the SNR gain of a parallel uncoupled array of bistable oscillators operating in a fixed mixture of sinusoidal signal and Gaussian white noise. Owing to the added array noise, the regions of SNR gains exceeding unity are observed for both suprathreshold and subthreshold inputs. We have shown that the performance of an infinite array can be closely approached by a finite array of moderate size, indicating a promising application in array signal processing. Interesting open questions are, for instance, at a given input SNR, to find the optimal array parameters and optimal added noise level, to maximise the output SNR gain.

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