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Hilbert's forgotten equation, the equivalence principle and velocity dependence of free fall

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Abstract

Referring to the behavior of accelerating objects in special relativity, and applying the principle of equivalence, one expects that the coordinate acceleration of point masses under gravity will be velocity dependent. Also, using the Schwarzschild solution, we analyze the similar case of masses moving on timelike geodesics, which reproduces a little-known result by Hilbert from 1917, describing this dependence. We find that the relativistic correction term for the acceleration based on general relativity differs by a factor of two from the simpler acceleration arguments in flat space. As we might expect from the general theory, the velocity dependence can be removed by a suitable coordinate transformation, such as the Painlevé–Gullstrand coordinate system. The validity of this approach is supported by previous authors who have demonstrated vacuum solutions to general relativity producing true flat space metrics for uniform gravitational fields. We suggest explicit experiments could be undertaken to test the property of velocity dependence.

Keywords: general relativity, special relativity, equivalence principle, geodesics, velocity dependence

1. Introduction

General relativity provides the standard description of generalized motion and gravity, where free-fall particles follow geodesics within a given spacetime metric. Inhomogeneous gravitational fields are characterized by tidal forces described by the geodesic deviation equations

of a pseudo-Riemannian metric. The principle of equivalence, although not playing a highly prominent role in the modern version of the theory, is still useful if the limits of its applicability are clearly delineated. That is, the principle is believed to hold in the limit of small spacetime regions of the gravitational field. We therefore begin by using the principle of equivalence for simple cases of acceleration from a special relativistic viewpoint, in an attempt to elucidate the qualitative effect of velocity dependence on acceleration for timelike geodesics in gravitational fields.

Einstein reasoned that light would bend under gravity based on simple acceleration arguments in flat space and the principle of equivalence. He then calculated a more precise result using general relativity, taking into account the effect of curved spacetime. Using the same approach we deduce that acceleration under gravity is velocity dependent within accelerating frames under special relativity, which we then confirm with the Schwarzschild solution of general relativity—incidentally recovering a largely forgotten result by Hilbert.

1.1. Special relativity

We define a spacetime coordinate differential with a four-vector

$$\mathrm{d}x^{\mu} = (c\mathrm{d}t,\ \mathrm{d}x,\ \mathrm{d}y,\ \mathrm{d}z), \tag{1}$$

with contribution from three spatial dimensions and t is the time in a particular reference frame and c is the invariant speed of light [1]. In this paper we are able to focus exclusively on one-dimensional motion and so we can suppress two of the space dimensions writing a spacetime vector $\mathrm{d}x^{\mu} = (c\mathrm{d}t,\ \mathrm{d}x)$. We have the metric tensor $g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ that defines the covariant vector $\mathrm{d}x_{\mu} = g_{\mu\nu}\mathrm{d}x^{\nu} = (c\mathrm{d}t,\ -\mathrm{d}x)$. In the co-moving frame we have $\mathrm{d}x = 0$ and so $\mathrm{d}x^{\mu} = (c\mathrm{d}\tau,\ 0)$, which defines τ the local proper time. We define the four-velocity

$$v^{\mu} = \frac{\mathrm{d}x^{\mu}}{\mathrm{d}\tau} = \frac{\mathrm{d}t}{\mathrm{d}\tau}\frac{\mathrm{d}x^{\mu}}{\mathrm{d}t} = (\gamma c,\ \gamma v), \tag{2}$$

where $v = \mathrm{d}x/\mathrm{d}t$ and

$$\gamma = \frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{3}$$

We then have the magnitude of the velocity four-vector

$$\sqrt{v^{\mu}v_{\mu}} = \sqrt{\gamma^2c^2 - \gamma^2v^2} = c \tag{4}$$

that is a Lorentz invariant, where we have used the Einstein summation convention. We also have the four-acceleration [2]

$$a^{\mu} = \frac{\mathrm{d}v^{\mu}}{\mathrm{d}\tau} = \frac{\mathrm{d}^2x^{\mu}}{\mathrm{d}\tau^2} = (\gamma^4 a\frac{v}{c},\ \gamma^4 a), \tag{5}$$

where we have used the result that $\frac{\mathrm{d}\gamma}{\mathrm{d}\tau} = \frac{\mathrm{d}\gamma}{\mathrm{d}v}\frac{\mathrm{d}v}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\tau} = \gamma^4 a\frac{v}{c^2}$. Note that the special case of one-dimensional radial motion implies that v is parallel to $a = \frac{\mathrm{d}^2x}{\mathrm{d}t^2}$, which is not necessarily true in three-dimensions. We then find the magnitude squared of the four-acceleration

$$a^{\mu}a_{\mu} = \gamma^8v^2a^2/c^2 - \gamma^8a^2 = -\gamma^6a^2. \tag{6}$$

Now, in the momentarily co-moving inertial frame (MCIF) we have $v = 0$ giving the acceleration four-vector $a^{\mu'} = (0,\ \alpha)$ and the four-velocity $v_{\mu'} = (c,\ 0)$, which gives

$a^{\mu'}a_{\mu'} = -\alpha^2$, as well as the expected orthogonality $v^{\mu'}a_{\mu'} = 0$, [3, ch 6]. Thus $a^{\mu}a_{\mu} = -\alpha^2$ is an invariant for all frames[1]. Hence, comparing the magnitudes of the four-acceleration in equation (6) with the magnitude in the MCIF we find $\alpha = \gamma^3 a$. That is, in an alternate inertial frame to the MCIF, we observe a coordinate acceleration [4]

$$a = \alpha/\gamma^3. \tag{7}$$

1.1.1. Thought experiment. Consider a rocket out in space far from the effects of any gravitational influences. Within this effectively flat region of space, we place small frames of reference that individually can measure the acceleration of passing objects. We will call these types of frames PG1 for 'particle group 1'. The PG1 frames are currently at rest relative to the rocket and also with respect to each other and they are spread throughout the space surrounding the rocket. The rocket also has a hole at the top and bottom so that as the PG1 pass through they can measure the acceleration of the rocket. The rocket also has an inbuilt mechanism so that, when the rocket is accelerating, it will drop a second group of particles, labeled PG2, from the origin of the rocket's coordinate system, at predetermined fixed time intervals as measured by the rocket. Thus, each PG2 can also measure the rocket's acceleration, and at each instant when they are released will describe the MCIF for the rocket. Note, that as each PG2 are dropped at different times during the rocket's acceleration, then each will occupy a distinct inertial frame, whereas the PG1 all occupy the same inertial frame.

Now, for the sake of argument, let the rocket be accelerated at $9.8\ \mathrm{ms}^{-2}$ and as specified, PG2 will be dropping from the rocket. The rocket now accelerates with $\alpha = T/m = 9.8\ \mathrm{ms}^{-2}$ as measured by PG2, where m is the mass of the rocket and assuming T is an applied thrust in order to maintain a constant proper acceleration α. As the rocket continues its acceleration it will encounter PG1 lying in its path that will enter the hole at the top of the rocket and while passing through measure the acceleration of the rocket. Now, as the rocket is maintaining a steady acceleration, clearly the velocity of the rocket will be steadily increasing. Hence the rocket will be encountering the PG1 frame at higher and higher relative velocities.

There are two questions we now wish to consider. Firstly: *Will PG1 and PG2 measure the same acceleration for the rocket?*

Based on standard theory, we expect the answer to be in the negative. Intuitively, this will be because special relativity asserts that, as viewed by PG1, the rocket's velocity will converge to the light speed upper bound, and so the acceleration will appear to decrease. Since, this physical setting is described by equation (7), the one-dimensional relativistic equation for acceleration a, as measured in the PG1 frames, can be written as

$$a = \frac{\alpha}{\gamma^3} = \frac{T}{m}\left(1 - \frac{v^2}{c^2}\right)^{3/2}, \tag{8}$$

where α is the acceleration measured in the co-moving inertial frames PG2 and v is the velocity of the rocket relative to any inertial frame measuring it, such as PG1. We note that the velocity dependence of acceleration a, viewed here by PG1, is clearly a frame dependent effect, as we can simply boost to the MCIF in order to remove it. Of further interest is that due to the low acceleration of the rocket, we can ignore the effect of Einstein's time dilation occurring as a function of 'vertical' position in the rocket, and hence the rocket can consider

[1] Because the four-acceleration is a four-vector, we can apply a Lorentz boost to the MCIF four-acceleration $a^{\mu'} = (0, \alpha)$, with the transformation $t = \gamma(t' + vx'/c^2)$ and $x = \gamma(x' + vt')$. This produces $a^{\mu} = (\gamma v\alpha/c^2, \gamma\alpha)$ and so comparing this with equation (5) we have $\gamma\alpha = \gamma^4 a$ or $\alpha = \gamma^3 a$, confirming equation (7).

itself a satisfactory candidate for viewing the accelerations of PG1 and PG2. Observed accelerations are not necessarily reciprocal between frames, and we will show that when measured by the rocket the acceleration of PG1 is seen to be less than PG2.

So, given the above, we can now ask a second question: *Given the principle of equivalence will these results for accelerating observers, be replicated in a gravitational field?*

We presume that locally, in an approximately homogeneous section of a gravitational field, that the answer must be in the affirmative, provided we now replace the words 'acceleration of the rocket with respect to both PG1 and PG2' with the words 'fixed observer with respect to the source of gravitational field' and we replace the words 'acceleration of PG1 and PG2 with respect to the rocket' with 'PG1 and PG2 being freely falling particles on timelike geodesics'. In order to make this transition to gravitational fields, we now investigate the Schwarzschild solution of the general theory and the Rindler metric for accelerating frames.

1.2. Gravitational fields

The central role played by the equivalence principle in the general theory was stated by Einstein in 1907:

We [...] assume the complete physical equivalence of a gravitational field and a corresponding acceleration of the reference system.

Einstein's equivalence principle (EEP) was initially based on the well established equivalence of gravitational and inertial mass, also called the weak equivalence principle, which has now been confirmed by experiment [5–7] to an accuracy better than 1×10^{-15}. It is now recognized that the full Einstein equivalence principle requires a curved spacetime metric theory of gravity in which particles follow geodesics, in accordance with the general theory [3, 8].

Incorporating the equivalence principle, our proposition is that since equation (8) pertains to a reference frame described above with an accelerating rocket, then this will also apply to a frame in an homogeneous region of a gravitational field. That is, we write

$$a = g\left(1 - \frac{v^2}{c^2}\right)^{3/2}, \tag{9}$$

where g represents the free fall of an object equivalent to PG2 but now in a gravitational field. Note that, since we are taking a single point in the rocket, we do not need to consider the full rocket frame[2]. The proper acceleration that we have calculated refers to PG2, that is, the inertial frame instantaneously co-moving with the rocket. Note that the proper acceleration is the rate of change of proper velocity with respect to coordinate time, that is $\alpha = \gamma^3 a = \frac{\mathrm{d}(\gamma v)}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}x}{\mathrm{d}\tau}\right)$. Hence to switch to the rocket frame we need to convert the coordinate time t to the proper time τ of the rocket, and so we expect to pick up an extra factor of γ, however, because clocks and rulers vary with position in the rocket, we need to set up a

[2] For a general accelerating frame depicted by the rocket we have the acceleration

$$a(X) = \frac{g(1 + gX/c^2)}{[(gt/c)^2 + (1 + gX/c^2)^2]^{3/2}},$$

where X is the proper distance in the rocket, t is the coordinate time in the inertial frame and at $t = 0$ the rocket is instantaneously at rest relative to the inertial frame. Now, by specifying the weak field limit $gX/c^2 \ll 1$ we produce the simpler

$$a(X) \approx \frac{g}{[(gt/c)^2 + 1]^{3/2}} = g(1 - v^2/c^2)^{3/2},$$

as assumed in equation (8), where we have used $v = gt/\sqrt{(1 + g^2t^2/c^2)}$.

properly synchronized accelerating frame within which the acceleration is calculated, also called Rindler coordinates[3]. This frame will produce the following acceleration from the perspective of the rocket frame, of

$$a = g\left(1 - \frac{2v^2}{c^2}\right), \tag{10}$$

where v is the velocity measured at the origin of the rocket frame [9].

Hence, combining this result from special relativity with the principle of equivalence, we would predict that the acceleration of a free-falling object in a homogeneous region of a gravitational field will be velocity dependent, whether viewed from the perspective of the free-falling frame or the shell observer in the gravitational field. The shell observer is one remaining stationary in the field, perhaps by using rocket propulsion to maintain its position, for example. For the Schwarzschild metric, the shell observer is defined as one in the spherical shell with $r =$ constant, and with θ and ϕ also constant. The shell observer will record time at a rate $\left(1 - \frac{2\mu}{r}\right)^{1/2}\mathrm{d}t$ and infinitesimal lengths as $\left(1 - \frac{2\mu}{r}\right)^{-1/2}\mathrm{d}r$, using the coordinates defined by the observer at infinity; refer to equation (13).

To support our approach, it has been shown that constant acceleration and a homogeneous uniform region of the gravitational field can both be treated with a flat spacetime metric, consistent with general relativity and the Einstein vacuum field equations [10].

We also note from equation (8) that all inertial frames will agree on

$$\frac{a}{\left(1 - \frac{v^2}{c^2}\right)^{3/2}} = \gamma^3(v)\frac{\mathrm{d}v}{\mathrm{d}t} = \alpha. \tag{11}$$

Following Rindler [4] and integrating equation (11), where we choose $t = 0$ when $v = 0$ and $\alpha t = \gamma v$, we find

$$\tau^2 = x^2 - c^2t^2 = \frac{c^4}{\alpha^2} = \frac{c^4}{a^2}\left(1 - \frac{v^2}{c^2}\right)^3. \tag{12}$$

This shows, for inertial observers at least, a coordinate invariant view with respect to constantly accelerating points indeed exists.

2. Schwarzschild solution

For a static, non-rotating spherical mass the field equations of general relativity give the Schwarzschild solution

$$c^2\mathrm{d}\tau^2 = \left(1 - \frac{2\mu}{r}\right)c^2\mathrm{d}t^2 - \left(1 - \frac{2\mu}{r}\right)^{-1}\mathrm{d}r^2 - r^2\mathrm{d}\theta^2 - r^2\sin^2\theta\mathrm{d}\phi^2, \tag{13}$$

where $\mu = GM/c^2$ and r is measured from the center and outside the mass [3]. We also have the geodesic equation

[3] The Rindler metric for the rocket is given by

$$c^2\mathrm{d}\tau^2 = \left(1 + \frac{gX}{c^2}\right)^2 c^2\mathrm{d}T^2 - \mathrm{d}X^2.$$

$$a^{\alpha} = \frac{\mathrm{d}v^{\alpha}}{\mathrm{d}\tau} = -\Gamma^{\alpha}_{\mu\nu} v^{\mu} v^{\nu}. \tag{14}$$

The geodesic equation gives the path of a particle through spacetime, which is the path that minimizes the metric distance traveled, as given by the Schwarzschild metric in our case, in equation (13). In differential geometry, a geodesic is a curve $C(t)$ whose tangent vector is parallel transported along $C(t)$. On a Riemannian manifold, with the Levi–Civita connection, it gives the shortest curve connecting two points.

Now, from the metric we have $g_{rr} = -\left(1 - \frac{2\mu}{r}\right)^{-1}$ and $g_{tt} = \left(1 - \frac{2\mu}{r}\right)$ and so if we select purely radial motion, then we have the non-zero Christoffel symbols[4]

$$\begin{aligned} \Gamma^{r}_{rr} &= \frac{1}{2} g^{rr} \partial_r g_{rr} = -\frac{\mu}{r^2}\left(1 - \frac{2\mu}{r}\right)^{-1} = -\Gamma^{t}_{rt} = -\Gamma^{t}_{tr} = -\frac{1}{2} g^{tt} \partial_r g_{tt} \\ \Gamma^{r}_{tt} &= -\frac{1}{2} g^{rr} \partial_r g_{tt} = \frac{\mu}{r^2}\left(1 - \frac{2\mu}{r}\right). \end{aligned} \tag{15}$$

The radial coordinate acceleration is then

$$\begin{aligned} \frac{\mathrm{d}^2 r}{\mathrm{d}\tau^2} &= -\Gamma^{r}_{rr}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 - \Gamma^{r}_{tt}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 \\ &= \frac{\mu}{r^2}\left(1 - \frac{2\mu}{r}\right)^{-1}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 - \frac{\mu c^2}{r^2}\left(1 - \frac{2\mu}{r}\right)\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 \\ &= -\frac{\mu c^2}{r^2}\left(\left(1 - \frac{2\mu}{r}\right)\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 - \left(1 - \frac{2\mu}{r}\right)^{-1}\left(\frac{\mathrm{d}r}{c\,\mathrm{d}\tau}\right)^2\right). \end{aligned} \tag{16}$$

The last term in brackets is simply the metric in equation (13) and so equal to one, and so

$$a^{r} = \frac{\mathrm{d}^2 r}{\mathrm{d}\tau^2} = -\frac{\mu c^2}{r^2} = -\frac{GM}{r^2}. \tag{17}$$

For a rocket observer applying thrust in order to remain at a fixed radius r in the field, we have $a^{t} = 0$ and so this implies the magnitude of the four-acceleration is

$$\sqrt{g_{\mu\nu} a^{\mu} a^{\nu}} = \sqrt{g_{rr}}\frac{GM}{r^2} = \frac{1}{\sqrt{1 - 2\mu/r}}\frac{GM}{r^2}, \tag{18}$$

which is a Lorentz invariant[5].

Now, we can write equation (17) as

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{1}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 - \frac{GM}{r}\right) = 0 \tag{19}$$

[4] The Christoffel symbol is defined as $\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2} g^{\alpha\beta}\left(\frac{\partial g_{\beta\mu}}{\partial x^{\nu}} + \frac{\partial g_{\beta\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\beta}}\right)$.

[5] For the observer at fixed spatial coordinates we have the four-velocity $v^{\beta} = \left(c\left(1 - \frac{2\mu}{r}\right)^{-1/2}, 0\right)$ and the four-acceleration $a^{\beta} = \left(0, -\frac{GM}{r^2}\right)$, where we have suppressed the two angular coordinates. This observer then sees the four-velocity of the radial infalling particle $u^{\beta} = \gamma\left(c\left(1 - \frac{2\mu}{r}\right)^{-1/2}, v\left(1 - \frac{2\mu}{r}\right)^{1/2}\right)$.

and so

$$\frac{1}{c^2}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 - \frac{2GM}{c^2 r} = \text{constant} = \frac{E_0^2}{m^2c^4} - 1, \tag{20}$$

where E_0 can be shown to be the conserved total energy of the particle. Hence

$$\frac{\mathrm{d}r}{\mathrm{d}\tau} = \pm c\sqrt{\frac{2\mu}{r} + \frac{E_0^2}{m^2c^4} - 1}\,. \tag{21}$$

Now $\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\mathrm{d}r}{\mathrm{d}\tau}\frac{\mathrm{d}\tau}{\mathrm{d}t}$ and so using equation (13) we determine

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{E_0}{mc^2\left(1 - \frac{2\mu}{r}\right)} \tag{22}$$

and so we find

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \pm c\left(1 - \frac{2\mu}{r}\right)\sqrt{1 - \frac{m^2c^4}{E_0^2}\left(1 - \frac{2\mu}{r}\right)}. \tag{23}$$

Therefore, for particles entering the gravitational field with r approaching infinity with an initial velocity v, we have $E_0 = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}}$, as expected for flat space, where $v = \frac{\mathrm{d}r}{\mathrm{d}t}$. For the alternate case of bound particles we have $E_0 < mc^2$, also calculated from equation (23).

Using the chain rule, $\frac{\mathrm{d}^2r}{\mathrm{d}t^2} = \frac{\mathrm{d}(\mathrm{d}r/\mathrm{d}t)}{\mathrm{d}r}\frac{\mathrm{d}r}{\mathrm{d}t}$, we find

$$\frac{\mathrm{d}^2r}{\mathrm{d}t^2} = -\frac{\mu c^2}{r^2}\left(1 - \frac{2\mu}{r}\right)\left(3\left(1 - \frac{2\mu}{r}\right)\frac{m^2c^4}{E_0^2} - 2\right). \tag{24}$$

Then, using the relation between E_0 and $\frac{\mathrm{d}r}{\mathrm{d}t}$ we find

$$\frac{\mathrm{d}^2r}{\mathrm{d}t^2} = -\frac{\mu c^2}{r^2}\left(1 - \frac{2\mu}{r}\right)\left(1 - 3\left(\frac{\mathrm{d}r}{c\,\mathrm{d}t}\right)^2\left(1 - \frac{2\mu}{r}\right)^{-2}\right). \tag{25}$$

This shows the velocity dependence $\frac{\mathrm{d}r}{\mathrm{d}t}$ of the coordinate acceleration $\frac{\mathrm{d}^2r}{\mathrm{d}t^2}$ based on the Schwarzschild coordinates, for an observer at infinity. For the case of the weak field only we have $\frac{2\mu}{r} \to 0$ and so

$$\frac{\mathrm{d}^2r}{\mathrm{d}t^2} = -\frac{GM}{r^2}\left(1 - \frac{3v^2}{c^2}\right), \tag{26}$$

a result first derived by Hilbert [11–13] in 1917, for the special case of particles moving radially in the Schwarzschild metric. We note that even in the weak field with very small distortion of space and time of the order $\frac{2\mu}{r}$ we can create a velocity dependence of order unity for relativistic particles. We can perhaps see an obvious source for this with the velocity terms in the geodesic equation, in equation (14). However, more fundamentally, we see that velocity dependent effects of a similar magnitude arise within special relativity in flat space, as shown by equation (8). Hence, while the effect is seen in gravity under the Schwarzschild metric, as a similar effect is also observed with acceleration under SR, it should not be seen as a sole property of gravity, but more a generic property of relativistic acceleration. The result of Hilbert also appears to indicate 'gravitational repulsion', for $v > \frac{c}{\sqrt{3}}$, as indeed might also be claimed for equation (10). However we note that it applies to the case where the observer is at infinity, and as the coordinate velocity in equation (23) never changes sign this represents a

deceleration. However, for the case of a large outbound velocity, we would note an outward acceleration away from the central mass, rather than an attraction.

Now, a more direct comparison with the Rindler frame would be given by a shell observer located at a fixed r in Schwarzschild coordinates at the location of the falling particle. We can write the line element as [13]

$$c^2\mathrm{d}\tau^2 = c^2\mathrm{d}T^2 - \mathrm{d}R^2 - r^2\mathrm{d}\theta^2 - r^2\sin^2\theta\mathrm{d}\phi^2, \tag{27}$$

where the conversion between the observer at a coordinate r and the observer at infinity is now $\mathrm{d}T^2 = \left(1 - \frac{2\mu}{r}\right)\mathrm{d}t^2$ and $\mathrm{d}R^2 = \left(1 - \frac{2\mu}{r}\right)^{-1}\mathrm{d}r^2$. Therefore the radial velocity is

$$\frac{\mathrm{d}R}{\mathrm{d}T} = \left(1 - \frac{2\mu}{r}\right)^{-1}\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{mc^2}{E_0}\frac{\mathrm{d}r}{\mathrm{d}\tau}. \tag{28}$$

Therefore

$$\begin{aligned}\frac{\mathrm{d}^2R}{\mathrm{d}T^2} &= \frac{\mathrm{d}R}{\mathrm{d}T}\frac{\mathrm{d}}{\mathrm{d}R}\left(\frac{\mathrm{d}R}{\mathrm{d}T}\right) = \frac{m^2c^4}{E_0^2}\frac{\mathrm{d}r}{\mathrm{d}R}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)\frac{\mathrm{d}r}{\mathrm{d}\tau} \\ &= -\frac{\mu c^2}{r^2}\frac{m^2c^4}{E_0^2}\left(1 - \frac{2\mu}{r}\right)^{\frac{1}{2}} \\ &= -\frac{GM}{r^2}\left(1 - \frac{2\mu}{r}\right)^{-\frac{1}{2}}\left(1 - \left(\frac{\mathrm{d}R}{c\mathrm{d}T}\right)^2\right).\end{aligned} \tag{29}$$

This last relation shows that for a local observer, the acceleration while apparently weakening at high velocity, never changes sign as for the case of Hilbert's equation for the observer at infinity. This equation[6], is applicable to a terrestrial experiment located in the gravity field of the Earth measuring radially falling particles. We can see, therefore, that the Schwarzschild solution gives velocity dependent geodesics for all observers at rest with respect to the gravitational field coordinates. This could be interpreted as an apparent weakening of the field strength in gravity, for objects with radial velocity.

We also show in appendix B, that for the general time independent metric, subject to the condition that they approach flat space for $r \to \infty$, then we find this same velocity dependence, as in equation (26). However, we may attempt to remove the velocity dependence of the acceleration by using the Painlevé–Gullstrand coordinate system, which matches coordinate time to proper time and is spatially flat, and given by

$$c^2\mathrm{d}\tau^2 = c^2\mathrm{d}t^2 - (\mathrm{d}r - v\mathrm{d}t)^2, \tag{30}$$

where t will be the time on the free-fall clock and $v = -\sqrt{\frac{2GM}{r}}$ is defined as the escape velocity at each r. The transform between this metric and the Schwarzschild metric is given by

$$\mathrm{d}t \to \mathrm{d}t + \frac{v}{c^2(1 - v^2/c^2)}\mathrm{d}r. \tag{31}$$

It is interesting to note that the coefficient for $\mathrm{d}r$ differs from the derivative of γ with respect to velocity, by a factor of γ, which would give us equation (9), our original velocity dependence in flat space. In these coordinates we find the coordinate acceleration now given by $\frac{\mathrm{d}^2r}{\mathrm{d}t^2} = -\frac{Gm}{r^2}$ and so indeed now has no velocity dependence. However, we note that these

[6] Equation (29) is a special case of the general result [13] restricted to purely radial motion.

coordinates assume a particle beginning at rest at $r \to \infty$ and so more generally we will need to adjust the metric for particles with different input velocities to the field if we wish to remove the velocity dependence. This suggests that the velocity dependence has now moved to the metric itself.

In order to properly compare a Schwarzschild frame observer at some radius r_0 with the rocket frame observer at $X = 0$ in flat space, we need to expand the Schwarzschild metric around $r = r_0$ and adjust clock rates to measure proper time and rulers to measure proper distance [14, 15]. Thus, write the Schwarzschild metric factor as

$$\begin{aligned} 1 - \frac{2GM}{c^2 r} &= 1 - \frac{2GM}{c^2 r_0\left(1 + \frac{x}{r_0}\right)} \\ &\approx 1 - \frac{2GM}{c^2 r_0}\left(1 - \frac{x}{r_0}\right) \\ &= \left(1 - \frac{2GM}{c^2 r_0}\right)\left(1 + \frac{2\bar{g}x/c^2}{1 - \frac{2GM}{c^2 r_0}}\right), \end{aligned} \tag{32}$$

where the approximate expression is valid if $|x|/r_0 \ll 1$ and we have defined $\bar{g} = \frac{GM}{r_0^2}$. The relationship between x and proper distance X is given by

$$X \approx \left(1 + \frac{GM}{c^2 r_0}\right)x. \tag{33}$$

Also letting

$$\mathrm{d}T^2 = \left(1 - \frac{2GM}{c^2 r_0}\right)\mathrm{d}t^2, \tag{34}$$

finally gives a local transformed Schwarzschild metric of

$$c^2\mathrm{d}\tau^2 \approx \left(1 + \frac{2gX}{c^2}\right)c^2\mathrm{d}T^2 - \frac{\mathrm{d}X^2}{1 + \frac{2gX}{c^2}}, \tag{35}$$

where $g \approx GM/r_0^2$ in the weak field limit. As expected, equation (35) is Riemann flat [14, 15]. We note, therefore, that a metric can be Riemann flat without a spatial metric coefficient of unity[7]. Incidentally, we can obtain this metric from the Schwarzschild metric in equation (13), simply by making the substitution $\frac{GM}{r} \to -gX$ describing a field of constant acceleration g.

Now, making the coordinate transformation $X \to X + \frac{c^2}{2g}$ followed by $X \to \frac{g}{2c^2}X^2$, in equation (35), we obtain a form of the Rindler metric:

[7] A suitable metric for a homogeneous gravitational field was determined by Desloge [2] as

$$c^2\mathrm{d}\tau^2 = \mathrm{e}^{2gX/c^2}c^2\mathrm{d}T^2 - \mathrm{d}X^2, \tag{36}$$

$$\approx \left(1 + \frac{2gX}{c^2}\right)c^2\mathrm{d}T^2 - \mathrm{d}X^2, \tag{37}$$

when $|gX/c^2| \ll 1$. Note, though, that it does not satisfy the vacuum Einstein equations. Also, in the weak field, we obtain the same velocity dependence as the Rindler metric and so this metric is not used.

$$c^2 \mathrm{d}\tau^2 = \frac{g^2 X^2}{c^2} \mathrm{d}T^2 - \mathrm{d}X^2. \tag{38}$$

Alternatively, with a shift in the coordinate origin $X \to X - c^2/g$ we find

$$c^2 \mathrm{d}\tau^2 = \left(1 + \frac{gX}{c^2}\right)^2 c^2 \mathrm{d}T^2 - \mathrm{d}X^2, \tag{39}$$

producing another form of the Rindler metric. Hence, we see that a local region of the Schwarzschild metric indeed matches the Rindler metric, confirming the equivalence principle.

Interestingly, even though we have applied a coordinate transform from the metric of a local region in the Schwarzschild metric in equation (35) to the Rindler metric shown in equations (38) and (39), the calculations of the velocity dependence will give a factor of two as expected by equation (10) for Rindler but a factor of three for Schwarzschild metric shown in equation (26). Note that even though equation (26) is a local Riemann flat region of the field we nevertheless produce the same velocity dependence of equation (26) and so different from the Rindler frame. We can now see the cause of the discrepancy, that in converting from Schwarzschild metric in equation (35) to the Rindler metric of equation (38) we applied a quadratic coordinate transformation of $X \to \frac{g}{2c^2} X^2$ and so modified the velocity dependence of coordinate acceleration. Hence the variation in the velocity dependence of the coordinate acceleration indeed appears to be a spatial coordinate effect rather than spatial curvature. This also then explains why when we set the spatial coefficient to one in the Schwarzschild metric, as shown in appendix A and equation (A.10), we find the velocity dependence becomes $1 - \frac{2v^2}{c^2}$ in the weak field, and so now matching the result for the Rindler frame.

2.1. Non curved solutions to uniform gravitational fields and exact equivalence to accelerated frames

A key expectation of this paper is that there should be an analogous result in gravity for the velocity dependence of acceleration in special relativity. This approach, as we have seen above is not necessarily easily implemented. Although not applicable to the Rindler metric [16], one of the complicating factors is the insistence by some authors, such as Desloge, who disallow a flat space to exist in a uniform gravitational field. On the other hand Munoz and Jones and others [10, 17, 18] address Deloge's claims, finding solutions [19] that produce a flat space in uniform gravitational fields consistent with general relativity and the Einstein vacuum solutions $R_{\mu\nu} = 0$.

After considering the principles of isotropy, a static metric, translational invariance along planar spatial cross sections of the geometry, and orthogonal planes to the x-coordinate implying the metric coefficients depend only on x, Munoz and Jones begin with the metric

$$\mathrm{d}s^2 = \lambda^2(x)\mathrm{d}t^2 - \sigma^2(x)\mathrm{d}x^2 - \xi^2(x)(\mathrm{d}y^2 + \mathrm{d}z^2). \tag{40}$$

Then, after using Taub's theorem [20], which states that a spacetime with plane symmetry with $R_{\mu\nu} = 0$ admits a coordinate system where the line element is static, derives the following metrics which are equivalent to the Rohrlich–Tilbrook form [10]

$$\mathrm{d}s^2 = \lambda^2(x)\mathrm{d}t^2 - \sigma^2(x)\mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2, \tag{41}$$

with $\lambda' = g\sigma$.

Equation (41) is the only admissible one representing a singularity free, flat spacetime of a uniform gravitational field, consistent with general relativity.

These equations satisfy the Einstein vacuum solutions. Furthermore, after considering a free-falling observer in a gravitational field, it is possible to derive the fundamental time asymmetric acceleration equations that produces the usual hyperbolic motions of accelerated

systems as observed by inertial observers in flat space $T = \frac{\lambda(x)}{g}\sinh(gt)$. With this elegant result, and since the form of these equations can be recovered from equation (8) by a second integral[8], one is tempted to place the primary cause of the velocity dependence of a point particle, on the asymmetry in the time coordinates. Additionally this idea also confirmed in the last section, where we found going from equations (35) to (38) involved a spatial coordinate effect, rather than spatial curvature. However our other proposition that the velocity dependence is in the metric itself, further supported by the Munoz and Jones analysis, suggests that the additional curvatures of the metric, e.g. Schwarzschild, adds to the velocity dependence, 'on top' of the coordinate contribution. In view of this we would expect the results to match very closely for approximate uniform sections of the gravitational field and accelerations frames. We therefore propose the effect would make an interesting test of general relativity, testable under Earth's gravity.

3. Discussion

We show, in this paper, using the principle of equivalence to relate accelerating frames and gravity, the velocity dependence of coordinate acceleration under gravity for weak locally homogeneous regions of gravitational fields. Specifically, we have shown that the behavior of inertial free-fall particles in gravity are a function of initial particle velocity, as given by equation (9). We also note the comparison to a little-known result first derived by Hilbert in 1917, shown in equation (26). From the viewpoint of a shell observer, fixed with respect to Schwarzschild coordinates, this might also be interpreted as a weakening of the field. The discrepancy factor of two for velocity dependence of acceleration in the Rindler frame equation (10) of $1 - \frac{2v^2}{c^2}$ compared with the equivalent shell observer for Schwarzschild in equation (29) of $1 - \frac{v^2}{c^2}$, shows the difference in physical effects that a curved and flat space can manifest. This also points to the limits of applicability when attempting to apply the principle of equivalence.

We might expect the velocity dependence of acceleration, due to coordinate acceleration for timelike radial geodesic motion, can be obtained from the Christoffel symbols by contraction with two velocity vectors within the geodesic equation. Nevertheless, the acceleration in terms of proper time is the Newtonian relation in equation (17), which contains no velocity dependence. The velocity dependence only appears when we convert to a coordinate time, as shown in equation (25). Also, the relation of velocity dependence as found in a Rindler frame in special relativity in flat space, is comparable to that found under the Schwarzschild solution of general relativity. This appears to indicate that the source of the velocity dependence can be identified within special relativity, and specifically appears to relate to the use of a coordinate time parameterization of the path as opposed to proper time. Also, the fact that an exact correspondence is found between a Rindler frame in flat space and general relativity for arbitrary initial velocities, shows the generality of the equivalence principle as applying not only to particles falling from rest, but also for arbitrary initial radial velocities. Interestingly, we can remove the velocity dependence by changing to the Painlevé–Gullstrand coordinate system, although the velocity dependence is now in the metric itself. We could also see this in

[8] We have $a = g\left(1 - \frac{v^2}{c^2}\right)^{\frac{3}{2}} = \frac{\mathrm{d}v}{\mathrm{d}t}$ and so $g\mathrm{d}t = \mathrm{d}v\left(1 - \frac{v^2}{c^2}\right)^{-\frac{3}{2}}$ and so integrating gives $gt = \frac{v}{\sqrt{1 - \frac{v^2}{c^2}}}$. Hence $T = \int \frac{\mathrm{d}t}{\sqrt{1 - \frac{v^2}{c^2}}} = \int \frac{\mathrm{d}t}{\sqrt{1 + \left(\frac{gt}{c}\right)^2}} = \frac{c}{g}\sinh^{-1}\left(\frac{gt}{c}\right)$.

flat space where the velocity dependence can be viewed as a boost out of the co-moving frame. We have also shown in appendix B that independent of coordinate choices in the Schwarzschild metric, we recover in the weak field, the same velocity dependence as found in equation (26). We also identified the variation in the velocity dependence of coordinate acceleration, shown by equation (10) for the Rindler frame and equation (26) for the Schwarzschild observers, as due to the spatial coordinate transformation between the frames and not spatial curvature *per se*. Given that flat space solutions do exist in uniform gravitation that are in exact correspondence with acceleration where we know velocity dependence exists, we see that equations (8) and (A.10) should hold in such regions of gravity. This assists in understanding some of the discrepancies between several of our results, such as the coordinate dependencies for flat space versus spatial metrical effects when curvature is taken into account using the Schwarzschild metric. Additionally we also show our approach aids in giving insight into how geodesic deviation within general relativity might behave in appendix C. However in an attempt to shed further light on the discrepancy factor of 2, found in our results, one approach might be to use Rindler coordinates, near for example the event horizon. Another could involve using Fermi normal coordinates to elucidate the relation between inertial and gravitational effects in Earth-based laboratories [21].

Regarding experimental tests, we derive equation (29), which gives the acceleration expected for a terrestrial experiment for the velocity dependence of acceleration due to gravity. Currently, lunar and satellite laser ranging applied to the radial component of orbital motion, allows precise verification of GR and the EEP using the parameterized post Newtonian or PPN formalism. However, the velocity dependence effect can be explicitly tested [22, 23], such as by measuring free fall in a vacuum with high precision sapphire clocks now accurate to 1 part in 10^{-18} s [24]. This type of test would also provide further confirmation of the EEP. Additionally, we refer to experimental tests of relativistic gravity by testing of Hilbert's equation and black hole physics at the Large Hadron Collider [25, 26]. Also new precision space-based tests of GR are planned [27], as well continuing measurements of space probe trajectories with the NASA Deep Space Network [28] and new terrestrial measurements of the acceleration under gravity using cold atoms directed upwards and falling under gravity [29].

We believe our approach provides a natural pathway from accelerations in special relativity to the behavior of radial geodesics in gravitational fields, including tidal effects. Of course, when making these comparisons, since ruler lengths and clock rates can differ between the two frames, we need to be careful that we are identifying physical agreement rather than merely formal agreement.

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Appendix A. Geodesic motion using a Lagrangian approach

Geodesics follow paths of maximal proper time, and from the line element in equation (13) we produce a Lagrangian

$$\mathcal{L} = -g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} = \left(1 - \frac{2\mu}{r}\right)\dot{t}^2 - \frac{1}{c^2}\left(1 - \frac{2\mu}{r}\right)^{-1}\dot{r}^2 = 1, \tag{A.1}$$

where $\dot{t} = \frac{\mathrm{d}t}{\mathrm{d}\tau}$ and $\dot{r} = \frac{\mathrm{d}r}{\mathrm{d}\tau}$ and for purely radial motion we have assumed that the angular terms are zero. As we are assuming we are dealing with particles with mass we can parameterize their motion using the proper time τ and so we can use $\mathcal{L} = -g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}$ rather than $\mathcal{L} = \sqrt{-g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}}$.

Lagrange's equations are an equivalent form of the geodesic equation equation (14), and for t produce $\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial\mathcal{L}}{\partial\dot{t}}\right) - \frac{\partial\mathcal{L}}{\partial t} = 0$, giving

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\left(1 - \frac{2\mu}{r}\right)c^2\dot{t}\right) = \frac{\mathrm{d}\mathcal{L}}{\mathrm{d}t} = 0. \tag{A.2}$$

Hence we have a constant of the motion

$$\left(1 - \frac{2\mu}{r}\right)\dot{t} = \frac{E_0}{mc^2}, \tag{A.3}$$

where E_0 represents the conserved total energy for its motion. Substituting equation (A.3) into equation (A.1) we find

$$\frac{\mathrm{d}r}{\mathrm{d}\tau} = \pm c\sqrt{\frac{E_0^2}{m^2c^4} - \left(1 - \frac{2\mu}{r}\right)}. \tag{A.4}$$

Differentiating with respect to proper time τ gives

$$a^r = \frac{\mathrm{d}^2r}{\mathrm{d}\tau^2} = \frac{\mathrm{d}r}{\mathrm{d}\tau}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 = -\frac{\mu c^2}{r^2}, \tag{A.5}$$

as measured by a co-moving observer.

Now, using equation (A.3) we find

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\mathrm{d}\tau}{\mathrm{d}t}\frac{\mathrm{d}r}{\mathrm{d}\tau} = \pm c\left(1 - \frac{2\mu}{r}\right)\sqrt{1 - \frac{m^2c^4}{E_0^2}\left(1 - \frac{2\mu}{r}\right)}. \tag{A.6}$$

Then, using the chain rule $\frac{\mathrm{d}^2r}{\mathrm{d}t^2} = \frac{\mathrm{d}(\mathrm{d}r/\mathrm{d}t)}{\mathrm{d}r}\frac{\mathrm{d}r}{\mathrm{d}t}$, we find

$$\frac{\mathrm{d}^2r}{\mathrm{d}t^2} = -\frac{\mu c^2}{r^2}\left(1 - \frac{2\mu}{r}\right)\left(3\left(1 - \frac{2\mu}{r}\right)\frac{m^2c^4}{E_0^2} - 2\right). \tag{A.7}$$

Finally, using the relation between E_0 and $\frac{\mathrm{d}r}{\mathrm{d}t}$ in equation (A.6) we find

$$\frac{\mathrm{d}^2r}{\mathrm{d}t^2} = -\frac{\mu c^2}{r^2}\left(1 - \frac{2\mu}{r}\right)\left(1 - \frac{3}{c^2}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2\left(1 - \frac{2\mu}{r}\right)^{-2}\right), \tag{A.8}$$

in agreement with equation (25).

If we set $g_{rr} = -1$ and repeat the derivation above, then we find $\frac{\mathrm{d}t}{\mathrm{d}\tau}$ is unaffected but we now have

$$\begin{aligned}\frac{\mathrm{d}r}{\mathrm{d}\tau} &= \pm c\left(1 - \frac{2\mu}{r}\right)^{-1/2}\sqrt{\frac{E_0^2}{m^2c^4} - \left(1 - \frac{2\mu}{r}\right)}\\ \frac{\mathrm{d}^2r}{\mathrm{d}\tau^2} &= -\frac{GM}{r^2}\frac{E_0^2}{m^2c^4}\left(1 - \frac{2\mu}{r}\right)^{-2}\end{aligned} \tag{A.9}$$

and

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = -\frac{\mu c^2}{r^2}\left(1 - 2\left(\frac{\mathrm{d}r}{c\mathrm{d}t}\right)^2\left(1 - \frac{2\mu}{r}\right)^{-1}\right). \tag{A.10}$$

Hence in the weak field limit, the coefficient on the velocity dependence reduces from 3 to 2 in this case, now agreeing with the flat space Rindler frame in equation (10). We note, though, that a spatially flat space can be achieved with $g_{rr} \neq 1$.

Appendix B. General form of the Schwarzschild metric

The Schwarzschild metric in its most general form, which is the most general spherically symmetric vacuum solution of the Einstein field equations for a non-rotating uncharged mass, can be written

$$\begin{aligned} c^2\mathrm{d}\tau^2 = &\left(1 - \frac{2\mu}{D}\right)c^2\mathrm{d}t^2 - D'^2\left(1 - \frac{2\mu}{D}\right)^{-1}\mathrm{d}r^2 \\ &- D^2\mathrm{d}\theta^2 - D^2\cos^2\theta\mathrm{d}\phi^2. \end{aligned} \tag{B.1}$$

The four common variants, which are time independent, are as follows: Schwarzschild's original metric with $D = (r^3 + 8\mu^3)^{1/3}$, isotropic coordinates with $D = r\left(1 + \frac{\mu}{2r}\right)^2$, Brillouin coordinates with $D = r + 2\mu$ and the more conventional form of the metric with $D = r$. These coordinate choices all enforce a flat space condition, such that for $r \to \infty$, $D \to r$, which ensures the circumference of a circle $2\pi D \to 2\pi r$. This also implies that $D' \to 1$ and $D'' \to 0$ as $r \to \infty$.

Now, given alternative spatial coordinates $\rho = D(r)$, we have $d\rho = D'\mathrm{d}r$, where $D' = \frac{\mathrm{d}D}{\mathrm{d}r} = \frac{\mathrm{d}\rho}{\mathrm{d}r}$, then

$$\frac{\mathrm{d}^2\rho}{\mathrm{d}t^2} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\frac{\mathrm{d}\rho}{\mathrm{d}r}\right) = \frac{\mathrm{d}\rho}{\mathrm{d}r}\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} + \left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\mathrm{d}\rho}{\mathrm{d}r}\right). \tag{B.2}$$

Hence, in the weak field with $\frac{\mathrm{d}\rho}{\mathrm{d}r} =$ constant the second term is zero and so we recover the same velocity dependence, as found in equation (26). Hence, we have shown, independent of coordinate choices in the Schwarzschild metric we have recovered, in the weak field, the same velocity dependence as found in equation (26).

We also note the general Christoffel symbols are as follows:

$$\Gamma^r_{rr} = \frac{D''}{D'} - \frac{\mu D'}{D(D - 2\mu)},\ \Gamma^r_{tt} = \frac{c^2\mu(D - 2\mu)}{D'D^3},\ \Gamma^r_{rt} = \frac{\mu D'}{D(D - 2\mu)}. \tag{B.3}$$

Using the geodesic equation, they give the radial acceleration

$$\frac{\mathrm{d}^2 r}{\mathrm{d}\tau^2} = -\left(\frac{c^2\mu(1 - 2\mu/D)}{D'D^2}\right)\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 - \left(\frac{D''}{D'} - \frac{\mu D'}{D^2(1 - 2\mu/D)}\right)\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2. \tag{B.4}$$

This then recovers the Newtonian acceleration $\frac{\mathrm{d}^2 r}{\mathrm{d}\tau^2} = -\frac{GM}{r^2}$ in the weak field limit independent of D.

Appendix C. Velocity dependence of tidal force

Up to this point, apart from equation (11), we have only dealt with coordinate dependent effects. However in gravity there is a frame in which the effect cannot be transformed away by coordinate boosts. These are the frames that experience tidal accelerations, which in general relativity characterize gravity and the curvature of spacetime, via geodesic deviation. We give only an elementary investigation of such effects here, with the aim of showing that under gravity, the individual coordinate dependent effects do affect coordinate independent tidal accelerations.This section is also based on the work of authors such as Munoz that show the gravitational field has solutions consistent with uniform acceleration.

Following on from section 1.1.1 we consider a group of $PG1_n$ evenly separated, and each occupying a point in flat spacetime. Together these form a simple inertial frame in which all clocks can be synchronized and agree on simultaneity. Therefore if a point is accelerated past this frame, then the accelerated point would encounter $PG1_1$, $PG1_2$,…, $PG1_n$. Hence if $PG1_n$ measures the acceleration from equation (9), then all clocks in this inertial frame can agree exactly when the particle passed the point $PG1_n$. If, for example, the accelerating point passes $PG1_a$ before PG_b then $PG1_b$ will measure a slightly smaller acceleration for the point on the rocket compared to $PG1_a$. Now, invoking the principle of equivalence for each individual particle, the $PG1_n$ individually follow geodesics that when close together form an approximate inertial reference frame consistent with the laws of special relativity. Therefore to a good approximation the $PG1_n$ are still able to agree on their respective clock synchronizations, and hence on observations they make of their respective observed accelerations and individually they will observe acceleration dependence on velocity.

Since the $PG1_a$, $PG1_b$ are at slightly different positions in the field and therefore at slightly different potentials, they will accelerate at slightly different rates, and hence experience geodesic deviation. Therefore while individually the $PG1_n$ are coordinate dependent, together they are not. This leads to a natural investigation of how the velocity dependencies of each individual $PG1_n$, affect the geodesic deviation between them.

The Newtonian equation for tidal accelerations, based on the gravitational acceleration, around a mass M, of $a = -\frac{GM}{r^2}$, is

$$\frac{\mathrm{d}a}{\mathrm{d}r} = \frac{2GM}{r^3}. \tag{C.1}$$

This implies that for objects at a radial separation $\mathrm{d}r$ in a gravitational field experience a differential acceleration $\mathrm{d}a$.

In the domain of special relativity, we can treat equation (9) as a relativistic correction to an acceleration akin to Newtonian gravity, implying a tidal acceleration

$$\frac{\mathrm{d}a}{\mathrm{d}r} = \frac{2GM}{\gamma^3 r^3}\left[1 - \frac{3}{4\gamma}\frac{2GM}{rc^2}\right] \approx \frac{2GM}{\gamma^3 r^3}, \tag{C.2}$$

where we have neglected the effect of the second term in brackets. This is justified as in the weak field we can assume $r \gg \frac{2GM}{c^2} = r_s$, where r_s is the Schwarzschild radius and also for relativistic velocities $\frac{1}{\gamma} \to 0$. Therefore, special relativity adds to the Newtonian result a velocity dependent term, which implies that tidal forces will appear to reduce for higher velocities. Also of interest is that the Schwarzschild radius r_s arises naturally within special relativity with a Newtonian acceleration profile.

In the low velocity limit weak field limit, this approximates to

$$\frac{\mathrm{d}a}{\mathrm{d}r} \approx \frac{2GM}{r^3}\left(1 - \frac{3v^2}{2c^2}\right), \tag{C.3}$$

showing the velocity dependence compared with the Newtonian result in equation (C.1).

As we have solved the geodesic equation in equation (24), which gives the one-dimensional radial acceleration as a function of r, we can find the relative acceleration with respect to any fixed point in the field by differentiation with respect to the radius r, giving

$$\frac{\mathrm{d}a}{\mathrm{d}r} = \frac{2GM}{r^3}\left(\frac{3m^2c^4}{E_0^2}\left(1 - \frac{r_s}{r}\right)\left(1 - \frac{2r_s}{r}\right) - \left(2 - \frac{3r_s}{r}\right)\right). \tag{C.4}$$

Substituting for the total particle energy E_0 in terms of velocity in equation (23), we find the effective tidal acceleration

$$\frac{\mathrm{d}a}{\mathrm{d}r} = \frac{2GM}{r^3}\left(1 - \frac{3r_s}{r} - \frac{3v^2}{c^2}\frac{\left(1 - \frac{2r_s}{r}\right)}{\left(1 - \frac{r_s}{r}\right)^2}\right), \tag{C.5}$$

where $v = \frac{\mathrm{d}r}{\mathrm{d}t}$.

For the weak field case where $r \gg r_s$, we have the effective tidal acceleration

$$\frac{\mathrm{d}a}{\mathrm{d}r} \approx \frac{2GM}{r^3}\left(1 - \frac{3v^2}{c^2}\right). \tag{C.6}$$

Hence the velocity dependence in Schwarzschild coordinates, in the weak field limit, is twice that obtained previously using special relativity as shown in equation (C.3). Hence, the special relativistic approach gives qualitatively the same result as general relativity, with the commonly recurring factor of two discrepancy. This is therefore an alternative approach to gain insight into geodesic deviation, based on velocity dependence within special relativity.

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