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THE PHYSICAL BASIS FOR PARRONDO'S GAMES

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It has been reported that the original inspiration for Parrondo's games was a physical system called a "flashing Brownian ratchet." The relationship seems to be intuitively plausible but has not previously established with rigor. This is the problem that we address in this paper.

The dynamics of a Brownian particle in a flashing Brownian ratchet are the result of diffusion and of externally applied forces. The probability density, of finding the particle at a certain place and time, can be mathematically modelled using a Partial Differential Equation (PDE) namely the Fokker-Planck Equation. In this paper, we apply standard finite-difference methods of numerical analysis to the Fokker Planck Equation. We derive a set of finite difference equations and show that they have the same form as Parrondo's games which justifies the claim that Parrondo's games are a discrete-time, discrete-space version of a flashing Brownian ratchet. We claim that Parrondo's games are effectively a particular way of sampling a Fokker-Planck Equation. Our difference equations are a natural and physically motivated generalization of Parrondo's games. We refer to some well established theorems of numerical analysis to suggest conditions under which the solutions to the difference equations and partial differential equations would converge to the same solution.

The diffusion operator, implicitly assumed in Parrondo's original games, reduces to a pre-existing numerical method called "The Schmidt formula." There is actually an infinite continuum of possible diffusion operators and the Schmidt formula is at one extreme of the feasible range. We point out that an operator in the middle of the feasible range, with half-period binomial weightings, would be a better representation of the underlying physics. We present a modified form of Parrondo's games based on the central diffusion operator.

We suggest that the finite difference method presented here will be useful in the simulation and design of real physical flashing Brownian ratchets.

Keywords: Brownian ratchet; Parrondos games; Fokker-Planck Equation.

1. Introduction

Methods for mapping between discrete and continuous systems are well known [1–3] and have been carefully studied since the times of Newton and Cotes. Recent authors have examined the relationship between discrete and continuous ratchets [4–6]. In this paper we concentrate on the specific case of Parrondo's games. We achieve the mapping between the continuous and discrete systems using finite difference techniques. We show that Parrondo's games represent a ratchet with unnatural diffusion and moreover we show how to modify Parrondo's original games to reflect a more natural diffusion process.

A Brownian ratchet is a transport mechanism that relies on diffusion and on the modulation of an external field to produce a steady flow of particles. Brownian ratchets are believed to occur in nature and artificial physical Brownian ratchets have been constructed and have worked [7–11]. Flashing Brownian ratchets have been analyzed mathematically [12, 13] and have inspired a number of simplified games of chance, such as Parrondo's games [14–17]. Parrondo's games are discrete and are simple enough to be solved completely, in closed form.

It is tempting to infer that properties of the solutions of Parrondo's games tell us something about real physical Brownian ratchets. This kind of inference, "by analogy," is not valid unless the relationship between the discrete games and the continuous ratchets is established with rigor. In this paper we show a method that can be used to establish a one to one, reversible mapping between continuous ratchets and discrete games of chance.

2. The Fokker-Planck Equation

We seek a macroscopic statistical description for the diffusion of a very small particle in a uniform fluid, under the influence of external forces. There are two common approaches to this type of problem; to use a Stochastic Differential Equation (SDE) [19], such as the Langevin equation, or to use a Partial Differential Equation (PDE), such as the Fokker-Planck Equation [18]. Each approach has its advantages and disadvantages; its partisans and detractors. Kurtz [20, 21] has shown that is possible to directly approximate the solution to an SDE using a Markov chain. This is certainly a valid approach to the problem and is worthy of future consideration. We have found it more convenient to work with the Fokker-Planck PDE. Further discussion of the relative merits of the two approaches, SDE versus PDE, can be found in Risken [18].

We denote the probability of finding a Brownian particle at a certain point on space, z, and time, t, by $p = p(z,t)$. The time-evolution of $p(z,t)$ is governed by a partial differential equation called the Fokker-Planck Equation:

$$\frac{\partial^2}{\partial z^2}\left(D(z,t)\,p(z,t)\right) - \frac{\partial}{\partial z}\left(\alpha(z,t)\,p(z,t)\right) - \frac{\partial}{\partial t}p(z,t) = 0\ . \tag{1}$$

The functions $\alpha(z,t)$ and $D(z,t)$ are referred to as the infinitesimal first and second moments of diffusion. In practice, the infinitesimal second moment does sometimes depend on concentration of the solute, $p(z,t)$, but is usually regarded as constant and is called the "Fick's law constant." A typical value (for a hydrated sodium ion in water) would be of the order $D \approx 1.3\times10^{-9}\ \mathrm{m^2 s^{-1}}$. The infinitesimal

first moment depends on the magnitude of externally imposed forces and on the mobility of the Brownian particle which is given by

$$u = \frac{Z_e}{6\pi\eta a} \tag{2}$$

where Z_e is the electrical charge on the particle, η is the kinematic viscosity of the solvent and a is the effective radius of the particle. A typical value for the mobility (of a hydrated sodium ion in water) would be $u \approx 51.9 \times 10^{-9}\ \mathrm{m^2 s^{-1} volt^{-1}}$. Further descriptions and numerical data may be found in books on physical chemistry and statistical physics [22–25]. If we apply an electrical potential, or voltage, of $V(z,t)$ then the infinitesimal first moment is given by

$$\alpha(z,t) = -u\frac{\partial}{\partial z}V(z,t)\ . \tag{3}$$

The theory behind Eqs. (2) and (3) is due to Stokes, Kirchhoff and Einstein [26]. More information about the methods of solution and the applications of the Fokker-Planck Equation can be found in Risken [18].

When we take into account the functional forms of D and α then we can rewrite the Fokker-Planck Equation as:

$$D\frac{\partial^2 p}{\partial z^2} - \frac{\partial \alpha}{\partial z}p - \alpha\frac{\partial p}{\partial z} - \frac{\partial p}{\partial t} = 0\ . \tag{4}$$

This is the form of the Fokker-Planck Equation which we will sample at regular intervals in time and space, to yield finite difference equations.

3. Finite Difference Approximation

Many Partial Differential Equations, or PDEs, including Eqn. (4), can be very difficult to solve analytically. One well established approach to this problem is to sample possible solutions to a PDE at regular intervals, called mesh points [27]. The true solution is approximated locally by a collocating polynomial. The values of the derivatives of the true solution are approximated by the corresponding derivatives of the collocating polynomial.

We can define local coordinates, expanded locally about a point (z_0, t_0) we can map points between a real space (z,t) and an integer or discrete space (i,j). Time, t, and position, z, are modelled by real numbers, $t, z \in \mathcal{R}$ and the corresponding sampled position, i, and sampled time, j, are modelled by integers $i, j \in \mathcal{Z}$. We sample the space using a simple linear relationship

$$(z,t) = (z_0 + i\lambda, t_0 + j\tau) \tag{5}$$

where λ is the sampling length and τ is the sampling time.

In order to map Eq. (4) into discrete space, we need to make suitable finite difference approximations to the partial derivatives. The notation is greatly simplified if we define a family of difference operators:

$$\Delta_{i,j} = p(z_0 + i\lambda, t_0 + j\tau) - p(z_0, t_0)\,. \tag{6}$$

In principle, this is a doubly infinite family of operators but in practice we only use a small finite subset of these operators. This is determined by our choice of sampling points. This choice is not unique and is not trivial. The set of sampling points is called a "computational molecule [27]" or "computational template [1]." Some choices lead to over-determined sets of equations with no solution. Some other choices lead to under-determined sets of equations with infinitely many solutions. We chose a computational molecule called "Explicit" computation with the following sample points: $(i, j) \in \{(0,0), (-1,-1), (0,-1), (+1,-1)\}$.

We also need to make a choice regarding the form of the local collocating polynomial. This is not unique and inappropriate choices do not lead to unique solutions. A polynomial which is quadratic in z and linear in t is the simplest feasible choice:

$$p(z,t) = p(z_0,t_0) + A_1 \cdot (z - z_0) + A_2 \cdot (z - z_0)^2 + B_1 \cdot (t - t_0) \tag{7}$$

where A_1, A_2 and B_1 are the real coefficients of the polynomial.

More complicated polynomials would be possible and they would, presumably, lead to other different sets of Parrondian games. It would be possible to use other classes of approximating functions, such as the sinc function, $\mathrm{sinc}(x) = \sin(\pi x)/(\pi x)$. We know, from Taylor's theorem, that we can always approximate other classes of functions and higher order polynomials, to any desired degree of accuracy, using the simple polynomials suggested in this paper. We only have to choose sufficiently small sample distances and times. So, in the limit, all games should reduce to the same game. Different functions may approach the limit at different rates and be better, or worse, approximations. This is an interesting numerical issue and should be the topic of further research. The ultimate test of whether a set of games is physically "valid" or not should be decided on the basis of whether the solutions to the Partial Difference Equations, and the corresponding Partial Differential Equations, converge to the same solution as we reduce the sampling distance and time to zero.

Equations (5), (6) and (7) imply a simple system of linear equations that can be expressed in matrix form:

$$\begin{bmatrix} -\lambda & +\lambda^2 & -\tau \\ 0 & 0 & -\tau \\ +\lambda & +\lambda^2 & -\tau \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ B_1 \end{bmatrix} = \begin{bmatrix} \Delta_{-1,-1} \\ \Delta_{0,-1} \\ \Delta_{+1,-1} \end{bmatrix} . \tag{8}$$

These can be solved algebraically, using Cramer's method to obtain expressions for A_1, A_2 and B_1:

$$A_1 = \frac{p(z_0 + \lambda, t_0 - \tau) - p(z_0 - \lambda, t_0 - \tau)}{2\lambda} \tag{9}$$

and

$$A_2 = \frac{p(z_0 - \lambda, t_0 - \tau) - 2p(z_0, t_0 - \tau) + p(z_0 + \lambda, t_0 - \tau)}{2\lambda^2} \tag{10}$$

and

$$B_1 = \frac{p(z_0, t_0) - p(z_0, t_0 - \tau)}{\tau} . \tag{11}$$

These are all intuitively reasonable approximations but their choice is not arbitrary. Equations (9), (10) and (11) form a complete and consistent set. We could not change

one without adjusting the others. We can evaluate the derivatives of Eq. (7) to obtain a complete and consistent set of finite difference approximations for the partial derivatives:

$$\frac{\partial p}{\partial z} = A_1 = \frac{p\,(z_0+\lambda, t_0-\tau) - p\,(z_0-\lambda, t_0-\tau)}{2\lambda}\ , \tag{12}$$

$$\frac{\partial^2 p}{\partial z^2} = 2A_2 = \frac{p\,(z_0-\lambda, t_0-\tau) - 2p\,(z_0, t_0-\tau) + p\,(z_0+\lambda, t_0-\tau)}{\lambda^2} \tag{13}$$

and

$$\frac{\partial p}{\partial t} = B_1 = \frac{p\,(z_0, t_0) - p\,(z_0, t_0-\tau)}{\tau}\ . \tag{14}$$

We can apply the same procedure to $\alpha\,(z, t)$ to obtain

$$\frac{\partial \alpha}{\partial z} = A_1 = \frac{\alpha\,(z_0+\lambda, t_0-\tau) - \alpha\,(z_0-\lambda, t_0-\tau)}{2\lambda}\ . \tag{15}$$

Equations (12), (13), (14) and (15) can be substituted into Eq. (4) to yield the required finite partial difference equation:

$$p\,(z_0, t_0) = a_{-1}\cdot p\,(z_0-\lambda, t_0-\tau) + a_0\cdot p\,(z_0, t_0-\tau) + a_{+1}\cdot p\,(z_0+\lambda, t_0-\tau) \tag{16}$$

where

$$a_{-1} = \frac{\frac{D\tau}{\lambda^2} + \frac{\alpha(z_0,t_0)\tau}{2\lambda}}{\frac{\alpha(z_0+\lambda,t_0-\tau)-\alpha(z_0-\lambda,t_0-\tau)}{2\lambda}\tau + 1}\ , \tag{17}$$

$$a_0 = \frac{-2\frac{D\tau}{\lambda^2} + 1}{\frac{\alpha(z_0+\lambda,t_0-\tau)-\alpha(z_0-\lambda,t_0-\tau)}{2\lambda}\tau + 1} \tag{18}$$

and

$$a_{+1} = \frac{\frac{D\tau}{\lambda^2} - \frac{\alpha(z_0,t_0)\tau}{2\lambda}}{\frac{\alpha(z_0+\lambda,t_0-\tau)-\alpha(z_0-\lambda,t_0-\tau)}{2\lambda}\tau + 1}\ . \tag{19}$$

We can overload the arguments of p and write them in terms of the discrete space (i, j) using the mapping defined in Eq. (5),

$$p_{i,j} = a_{-1}\cdot p_{i-1,j-1} + a_0\cdot p_{i,j-1} + a_{+1}\cdot p_{i+1,j-1}\ . \tag{20}$$

The meaning of the arguments should be clear from the context and from the use of subscript notation, $p_{i,j}$, rather than function notation, $p(z,\, t)$. Equation (20) is precisely the form required for Parrondo's games.

4. Parrondo's Games

In the original formulation, the conditional probabilities of winning or losing depend on the state, i, of capital but not on any other information about the past history of the games:

- Game A is a toss of a biased coin:

$$p_{\text{win}} = \frac{1}{2} - \epsilon \tag{21}$$

where ϵ is an adverse external bias that the game has to "overcome". This bias, ϵ, is typically a small number such as $\epsilon = 1/200$, for example [14, 16].

- Game B depends on the capital, i:

If $(i \bmod 3) = 0$, then the odds are unfavorable.

$$p_{\text{win}} = \frac{1}{10} - \epsilon \tag{22}$$

If $(i \bmod 3) \neq 0$, then the odds are favorable.

$$p_{\text{win}} = \frac{3}{4} - \epsilon \ . \tag{23}$$

It is straightforward to simulate a randomized sequence of these games on a computer using a very simple algorithm [17].

4.1. *Game A as a partial difference equation*

We can write the requirements for game A in the form of Eq. (20).

$$p_{i,j} = \left(\frac{1}{2} - \epsilon\right) \cdot p_{i-1,j-1} + 0 \cdot p_{i,j-1} + \left(\frac{1}{2} + \epsilon\right) \cdot p_{i+1,j-1} \ . \tag{24}$$

This implies a constraint that $a_0 = 0$ which implies that $D\tau/\lambda^2 = 1/2$ which defines the relative scales of λ and τ so we can give it a special symbol:

$$\beta = \frac{D\tau}{\lambda^2} \tag{25}$$

with $\beta = 1/2$ in this case. The scaling relationship in Eq. (25) is closely related to the Einstein-Smoluchowski equation [24].

The constraints on a_{-1} and a_{+1} imply a value for Parrondo's "ϵ" parameter:

$$\epsilon = \left\{\frac{\lambda}{4D}\right\} \alpha\left(z_0, t_0\right) \tag{26}$$

which can be related back to an externally imposed electric field, $E = -\partial V/\partial z$ using Eqs. (2) and (3):

$$\epsilon = \left(\frac{\lambda}{4D}\right)\left(\frac{Z_e}{6\pi\eta a}\right)\left(-\frac{\partial V}{\partial z}\right) \ . \tag{27}$$

The small bias, ϵ, is proportional to the applied external field which justifies Parrondo's original intuition.

4.2. *Game B as a partial difference equation*

There is still zero probability of remaining in the same state that implies a constraint that $a_0 = 0$ implying that we still have the same scale, $\beta = \frac{1}{2}$. If we are in state i then we can denote the probability of winning by
$q_i = P\,(\text{win}|\text{initial position is } i)$. We can write the difference equations for game B in the form:

$$p_{i,j} = q_{i-1} \cdot p_{i-1,j-1} + 0 \cdot p_{i,j-1} + (1 - q_{i+1}) \cdot p_{i+1,j-1} \ . \tag{28}$$

This, together with Eqs. (17), (18) and (19), gives

$$\frac{q_{i-1}}{1 - q_{i+1}} = \frac{a_{-1}}{a_{+1}} = \frac{1 + \frac{\lambda}{2D\tau}\alpha_{i,j}}{1 - \frac{\lambda}{2D\tau}\alpha_{i,j}} \ , \tag{29}$$

which implies that

$$\alpha_{i,j} = 2\lambda\beta \frac{q_{i-1} - (1 - q_{i+1})}{q_{i-1} + (1 - q_{i+1})} \ . \tag{30}$$

This can be combined with Eq. (3) and then directly integrated to calculate the required voltage profile. We can approximate the integral with a Riemann sum:

$$V_i = -\frac{2\beta}{u} \sum_{k=0}^{i} \frac{1 - \left(\frac{1-q_{k+1}}{q_{k-1}}\right)}{1 + \left(\frac{1-q_{k+1}}{q_{k-1}}\right)} \tag{31}$$

so we can construct the required voltage profile for the ratchet, which means that given the values of q_i, it is possible to construct a physical Brownian ratchet that has a finite difference approximation that is identical with Parrondo's games. We can conclude that Parrondo's games are literally a finite difference model of a flashing Brownian ratchet.

We note that game B, as defined here, is quite general and actually includes game A as a special case.

4.3. *Conditions for convergence of the solution*

We would like to think that as long as $\beta = D\tau/\lambda^2$ is preserved then the solution to the finite partial difference equation (20) would converge to the true solution of the PDE in Eq. (4), as the mesh size, λ goes to zero. Fortunately, there is a theorem due to O'Brien, Hyman and Kaplan [28] which establishes that the numerical integration of a parabolic PDE, in explicit form, will converge to the correct solution as $\lambda \to 0$ and $\tau \to 0$ provided $0 \leq \beta \leq \frac{1}{2}$. Similar results may also be found in standard texts on numerical analysis [1–3, 27].

We see that Parrondo's choice of diffusion operator, with $\beta = \frac{1}{2}$ is at the very edge of the stable region.

4.4. *An appropriate choice of scale*

There is a possible range of values for β. As $\beta \to 0$ we require the time step $\tau \to 0$ which means that the number of time steps required to simulate a given time

interval, T, increases without bound $N_{\text{steps}} = T/\tau \to \infty$. It is computationally infeasible to perform simulations with very small values of β. On the other hand, the value of $\beta = 1/2$ implied in Parrondo's original games is at the very limit of stability. In fact, the presence of small roundoff errors in the arithmetic could cause the the discrete simulation to diverge significantly from the continuous solution. This is a weakness in the original formulation of Parrondo's games. We suggest that this weakness can be fixed by choosing $\beta = 1/4$, in the middle of the feasible range. If we consider the case of pure diffusion, with $\alpha = 0$, then Eq. (20) reduces to

$$p_{i,j} = \beta \cdot p_{i-1,j-1} + (1 - 2\beta) \cdot p_{i,j-1} + \beta \cdot p_{i+1,j-1} \ . \tag{32}$$

If we choose $\beta = \frac{1}{2}$ then we get the Schmidt formula [27], $p_{i,j} = \frac{1}{2} \cdot p_{i-1,j-1} + \frac{1}{2} \cdot p_{i+1,j-1}$, which is at the very limit of numerical stability. If we choose $\beta = \frac{1}{4}$ then Eq. (32) reduces to

$$p_{i,j} = \frac{1 \cdot p_{i-1,j-1} + 2 \cdot p_{i,j-1} + 1 \cdot p_{i+1,j-1}}{4} \ , \tag{33}$$

which is the same as Pascal's triangle with every second row removed. The solution to the case where the initial condition is a Kronecker delta function, $p_{i,0} = \delta_{i,0}$ is relatively easy to calculate:

$$p_{i,j} = 2^{-2j} \cdot \binom{2j}{j+i} = 2^{-2j} \cdot \frac{(2j)!}{(j+i)!\,(j-i)!} \ , \tag{34}$$

which is a half period, or double frequency, binomial and is shown in Fig. 1. We can invoke the Laplace and De Moivre form of the Central Limit Theorem which establishes a correspondence between Binomial (or Bernoulli) distribution and the Gaussian distribution to obtain

$$p_{i,j} \approx \frac{1}{\sqrt{2\pi \left(\frac{j}{2}\right)}} \exp\left(\frac{-i^2}{j}\right) \ . \tag{35}$$

This expression is only approximate but is true in the limiting case as $j \to \infty$.

In the case where $\alpha = 0$; the Fokker-Planck Eq. (4) reduces to a diffusion equation:

$$D\frac{\partial^2 p}{\partial z^2} - \frac{\partial p}{\partial t} = 0 \ . \tag{36}$$

Einstein's solution to the diffusion equation is a Gaussian probability density function:

$$p\,(z,t) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{-z^2}{2\sigma^2}\right) \tag{37}$$

where the variance, σ^2, is a linear function of time:

$$\sigma^2 = 2Dt \ . \tag{38}$$

This last expression is known as the "Einstein-Smoluchowski" equation [24]. It is possible to verify that Eq. (37) is a solution to Eq. (36) by direct substitution:

$$D\frac{\partial^2 p}{\partial z^2} = \frac{\partial p}{\partial t} = \left(\frac{1}{2t}\right) \cdot \left(\left(\frac{z}{\sigma}\right)^2 - 1\right) \cdot p\,(z,t) \ . \tag{39}$$

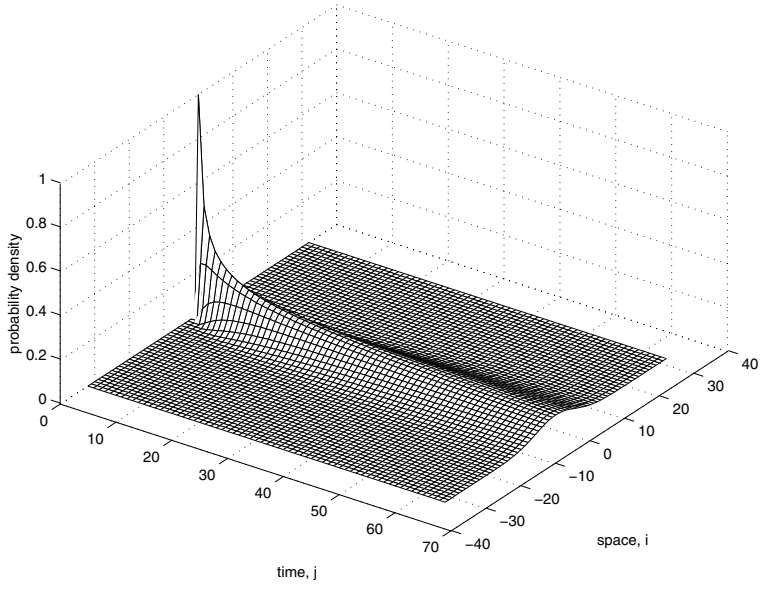

Fig. 1. The numerical solution to Eq. (32) is shown above. The discrete time axis, j, is on the lower left. The discrete space axis, i, is on the lower right. The vertical axis is probability density, approximated by point probability in this discrete simulation. The sum of probabilities, over all values of i, has the constant value of one, regardless of the time, j.

If we sample the solution in Eq. (37) using the mapping in Eq. (5) then we obtain Eq. (35) again. This is an exact result. We conclude that the choice of $\beta = 1/4$ is very appropriate for the solution to the diffusion equation. This new operator is shown graphically in Fig. 2 (b). The Schmidt operator is shown in Fig. 2 (a) for comparison. It is important to note that the new operator includes the possibility of particles remaining where they are. This is very natural and intuitive from a physical point of view. We suggest that the half-period binomial weightings would also be appropriate for the Fokker-Planck Equation, in the case where α is "small." The appropriate choice of β, given arbitrarily large, or rapidly varying, α is still an unsolved problem. In general, we would expect that much smaller values, $\beta \to 0$, would be needed to accommodate more extreme choices of α.

4.5. *An example of a simulation*

We simulated a physically reasonable ratchet with a moderately large modulo value, $M = 8$. (The value for the original Parrondo's games was $M = 3$.) We used the value of $\beta = 1/4$. The simulation was based on a direct implementation of Eq. (20) in Matlab. We chose a sampling time of $\tau = 12$ μs and a sampling distance of $\lambda \approx 0.25$ μm. The result is shown in Fig. 3, where we indicate how the expected position of a particle can move within a Brownian flashing ratchet during four cycles of the modulating field. We can see a steady drift of the mean position of

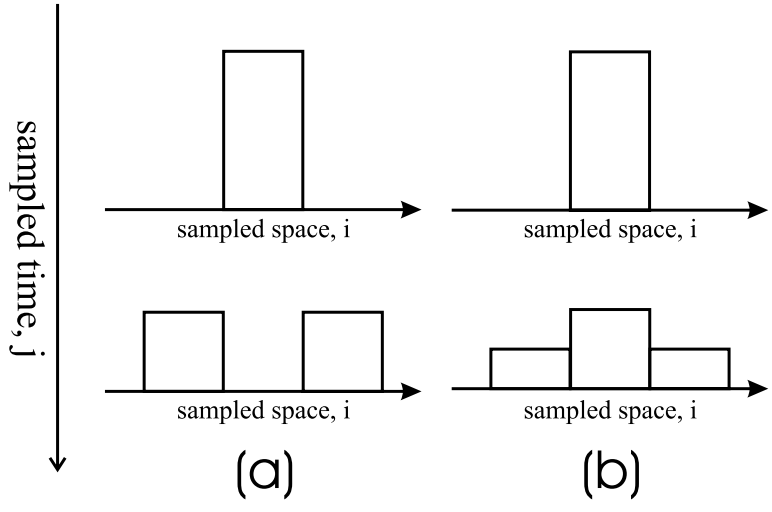

Fig. 2. A graphical representation of the diffusion operators: In part (a) we see the Schmidt operator, implied by Parrondo's games, with $\beta = 1/2$. In part (b) we see the half period binomial weightings suggested in this paper, with $\beta = 1/4$.

the particle in response to the ratchet action. This simulation includes a total of 500 time samples. Note that the average rate of transport quickly settles down to a steady value.

4.6. *Parrondo's games with natural diffusion*

We consider the effect of a change in the diffusion operator on the structure of the corresponding discrete games.

The model for diffusion used in Parrondo's games is equivalent to the Schmidt formula of numerical analysis but also has another precedent in physics, the Ehrenfest model for diffusion through a membrane [29, 30]. In this model, we consider a large number of molecules, M, distributed in two containers, or "urns," labelled A and B. At each trial, or time "tick," a molecule is chosen at random and moved from its container to the other. We can consider the state of the system at discrete time, j to be the number of molecules in container A, denoted by i_j. The time series $\{i_j\}$ is a temporally homogeneous Markov chain. The structure of the state transitions is shown in Fig. 4(a). It is possible to take the limiting case as the number of molecules becomes very large, $M \to \infty$ and using an initial condition for the state $i = M/2$. Note that in the limit it does not matter if M is odd or even, we can choose M to be even so that i will be an integer. The limiting case leads to the state transition diagram shown in Fig. 4(b) which is equivalent to Parrondo's original model. Finally, in order to include "natural" diffusion into Parrondo's games we must include "do nothing" or "no-change" transitions into the state transition diagram. This is shown in Fig. 4(c). If we choose $\beta = 1/4$ then we get half-period binomial weightings for the state transitions, which are also shown in Fig. 4(c).

If we use a periodic potential function, $V(z, t)$, for the ratchet and we use pre-

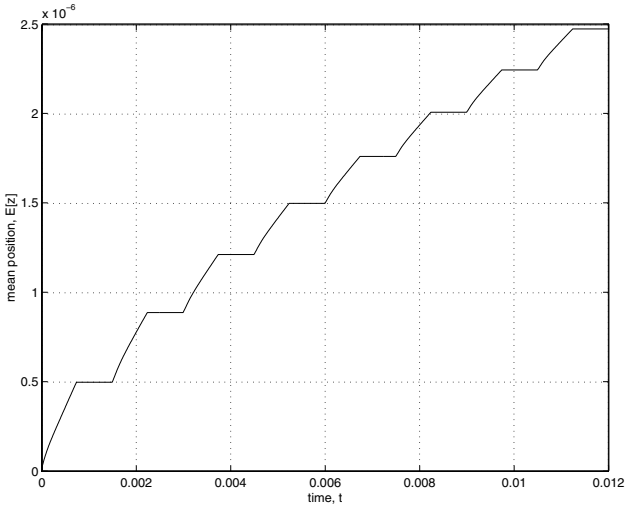

Fig. 3. Time-evolution of the mean of the distribution $p(z,t)$, $E[z]$. When the field is asserted, the mean position of the particles moves in the "upward" direction. When the field is turned off, the mean remains constant although diffusion causes the field to spread. The total shift in mean position of this ratchet is very modest compared with the spacing between the teeth of the ratchet. Part of the motivation of this work is to optimise the transport effect of the Brownian ratchet, subject to typical constraints such as degree of asymmetry, physical size, maximum electric field or frequency of operation.

cisely three samples per spatial period then the state transition diagrams are reduced modulo three. These are shown in Fig. 5. The state transition diagram in Fig. 5(a) is the natural development of the diagram in Fig. 4(b), if we reduce the number of states by periodically sampling the ratchet, modulo three. This is the original structure for Parrondo's games, where $\{a_{+1,0}, a_{+1,1}, a_{+1,2}\}$ are the probabilities of a "win" for events in each of the three equivalence classes. The state transition diagram in Fig. 5(b) is the corresponding development of the state transition diagram in Fig. 4(c). This is the structure that must be used in order to to fully represent natural diffusion. The coefficients, $\{a_{-1,k}, a_{0,k}, a_{+1,k}\}$ are the probabilities of "loss," "no change" and "win," respectively for events in equivalence class $k = i \pmod 3$, and i is the discrete, sampled, spatial coordinate. These quantities, $\{a_{-1,k}, a_{0,k}, a_{+1,k}\}$ can be evaluated using Eqs. (17), (18) and (19). For the purpose of simulation, it is convenient to reduce these new forms of Parrondo's games to a non-deterministic decision tree as is shown in Fig. 6. It is easy to code the algorithm implied by this decision tree. Note that this tree is general and covers all cases including Parrondo's "Game A" and "Game B."

5. Conclusions

We conclude that Parrondo's games *are* a valid finite-difference simulation of a flashing Brownian ratchet, which justifies Parrondo's original intuition. We have established that Parrondo's "ϵ" parameter is a reasonable way to simulate a gradual

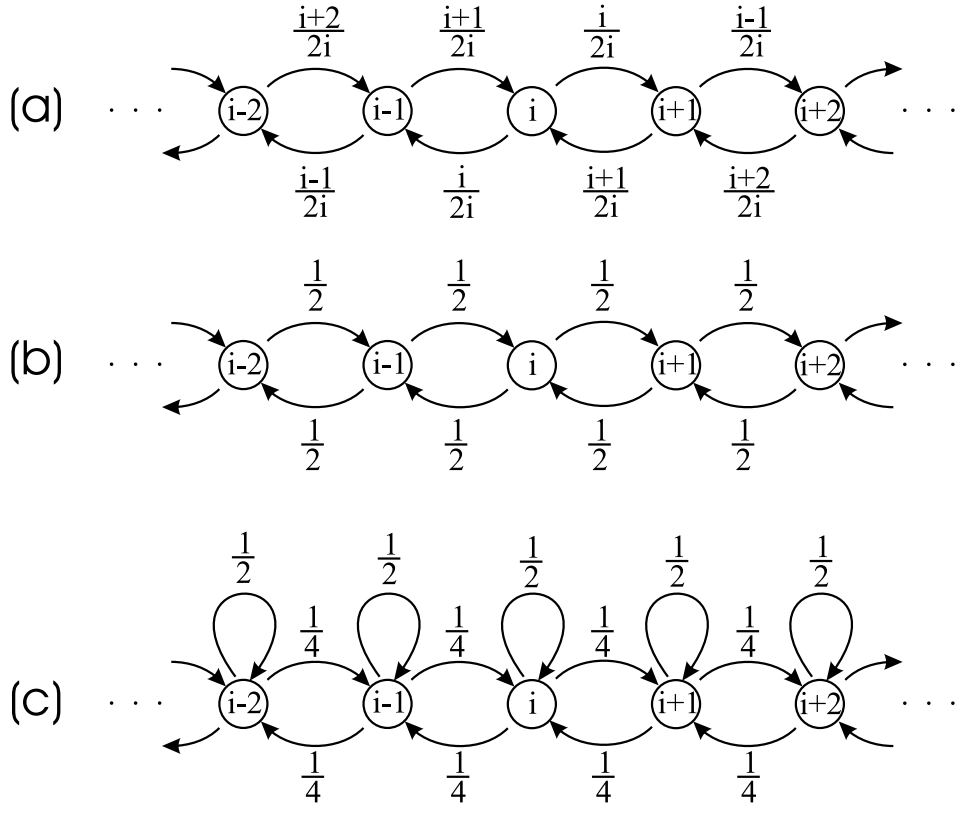

Fig. 4. State transition diagrams for various models of diffusion: In part (a) we see the state transition diagram for the Ehrenfest model. In part (b) we see the unbounded case, as the number of molecules, $M \to \infty$ and initial condition $i = M/2$. This is the model of diffusion assumed in Parrondo's original games. In part (c) we see the half-period binomial model proposed in this paper.

externally imposed electric field, or voltage gradient. We have established that Parrondo's implied choice of the β parameter does lead to a stable simulation but we suggest that the choice of $\beta = 1/4$ is more appropriate from mathematical and physical points of view. Finally, we have generalized Parrondo's games, in the form of a set of finite difference equations and we have shown that these can be implemented on a computer and do exhibit a Brownian ratchet transport effect.

Each new development always raises more questions than it solves and there are a number of open questions that seem to demand attention. Given the scheme proposed in this paper, precisely how should we adapt the choice of β to correspond to different driving functions $\alpha(z,t)$? Can we use an SDE approach rather than a PDE approach? Perhaps we could approximate the solutions to the SDE using Markov chains? Finally, we would like to have general criteria for the choice of collocating functions and computational templates. Under what circumstances are

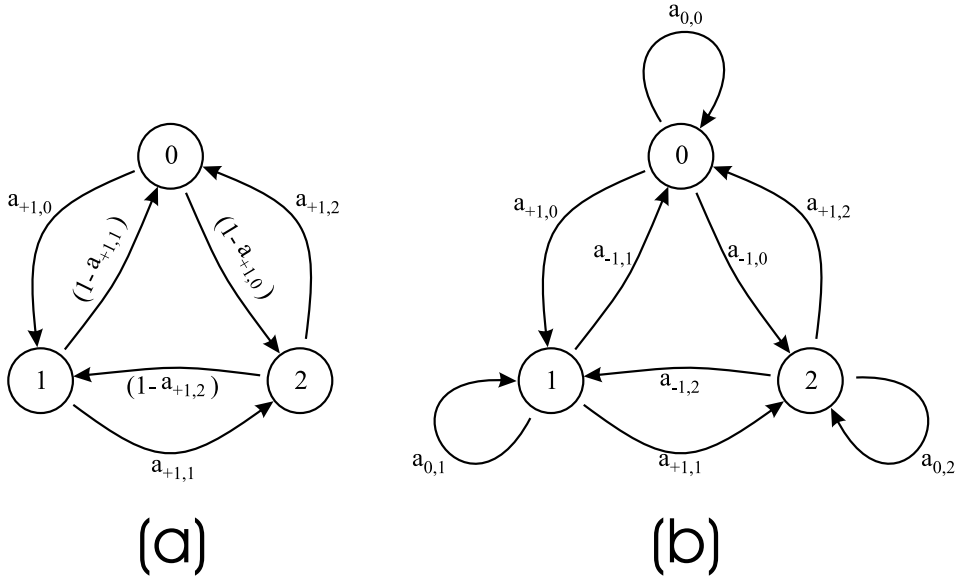

Fig. 5. State transition diagrams for Parrondo's games: In part (a) we see the state transitions for Parrondo's original games. In part (b) we see the transitions which are necessary to model natural diffusion in Parrondo's games, keeping the same number of states.

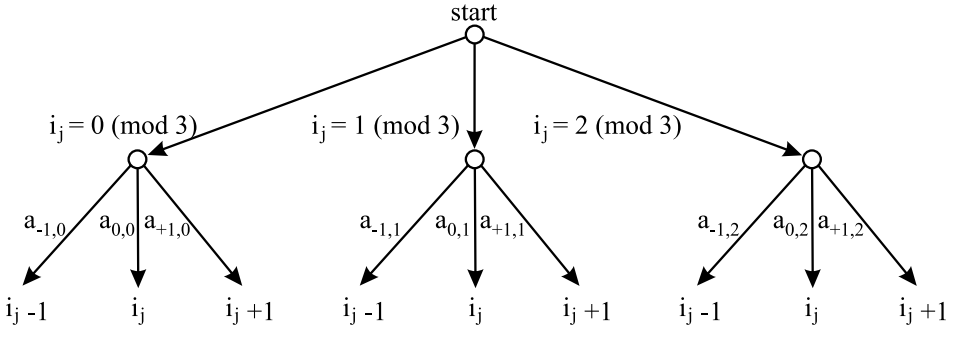

Fig. 6. This non-deterministic decision tree shows the decisions that must be made in order to update the value of the state i_j from discrete time j to discrete time $j+1$. Note the presence of "do-nothing" branches, with weights $a_{0,k}$. These weights correspond to conditional probabilities of random events. The numbers at the bottom of the tree are the new values of i at time $j+1$, denoted by i_{j+1}. The sequence of calculation is that the equivalence class is evaluated, $k = i \pmod 3$. The probabilities, $a_{m,k}$, are looked up or calculated; a random (or pseudo-random) event is generated and a branch of the tree is selected at random, using the appropriate weights. Finally, The values of j and i_j are updated.

we free to choose these aspects of the model without affecting the quality of the solution? Is there an optimum trade-off between accuracy and computational effort?

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