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CAN A MINORITY GAME FOLLOW REAL MARKET DYNAMICS?

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It is commonly known in economics that markets follow both positive and/or negative trends, crashes and bubble effects. In general a strong positive trend is followed by a crash. Famous examples of these effects were seen in the recent crash on the NASDAQ (April 2000) and prior to the crash on the Hong Kong market, which was associated with the Asian crisis in the early 1994. In this paper we use real market data input into a minority game with a variable payoff function and a nonlinear super exponential model for bubbles, to explore financial bubbles. By changing the payoff function in the minority game we study how one can get the price function to follow the dynamics of a real market.

Keywords: Minoirity game; $-Game; agent models; bubbles; stock market dynamics.

1. Introduction

Before the seminal paper on the minority game [1], there were a great number of physicists already exploring various economic related issues. Around the late 90's a number of groups proposed multi-agent models for the stock market [2–5] — these were important studies that showed that interacting agent models could produce realistic price histories, with crashes, clustered volatility, chronic bubbles and depression. However a limitation of these models is that the relevant features of the interaction are buried under so many parameters that a systematic understanding becomes unclear. This is mainly because the market mechanisms are intrinsically nonlinear, which means small variations in any of the parameters can lead to dramatic changes masking the cause of each price movement.

In order to begin to address this problem one has to adopt a completely different approach. In physics the usual procedure in constructing models is to start from the simplest model, capturing the essential features in question, and to then progressively add complexities to it. A famous example of this is the Ising model [7, 8], used to describe the magnetization in materials.

It is in this spirit of simplicity that led to the creation of a model known as the minority game. This model is aimed at having a simple yet rich platform to examine various phenomena, including those arising from financial markets.

At this point we have a minority game as it was originally defined by Challet and Zhang [1]. The general idea of the minority game is as follows: at any given time some people have two choices, they make their decisions simultaneously without any kind of communications between them, and those who happen to be in the minority win. In this context it is not in the interest of any agent to behave in the same way as the rest of the agents.

In this paper we use the direction of real price movements (not their magnitudes) to drive the Minority Game of Challet and Zhang [1], to see how well the outputs of the game follow real market dynamics. This exploration shows that a modified form of this game, called the \$-Game [23], is in fact much better at reproducing real market dynamics. We finally discuss how this improvement relates to the modified payoff function in the \$-Game.

2. Minority Game

2.1. *The original minority game model*

The dynamics of the Minority Game (MG)[a] are defined in terms of the dynamical variables $U_{s,i}(t)$ in discrete time $t \in \mathcal{N}^+$. These are the scores that each agent $i = \{1, \ldots, N\}$ attaches to each other of his possible choices $s = \{1, \ldots, S\}$. Each agent makes a decision $s_i(t)$ with probability

$$\mathrm{Prob}\{s_i(t) = s\} = \frac{\exp[\Gamma_i U_{s,i}(t)]}{\sum_{s'} \exp[\Gamma_i U_{s',i}(t)]}, \tag{1}$$

where $\Gamma_i > 0$ appears as an "individual inverse temperature". The original MG corresponds to $\Gamma_i = \infty$ [1] and was later generalized to $\Gamma_i \equiv \Gamma < \infty$ [9].

The public information variable $\mu(t)$ is given to all agents, it belongs to the set of integers $(1, \ldots, P)$ and can either be the binary encoding of the last M winning choices [1] or drawn randomly from a uniform distribution [10].

The action $a^{\mu(t)}_{s_i(t),i}$ of each agent depends on choices $s_i(t)$ and on $\mu(t)$. The coefficients $a^{\mu}_{s_i,i}$, which are either $+1$ or -1, are called strategies and play the role of quenched disorder. These are randomly drawn with probability of a $1/2$ for each i, s and μ. They can also be thought of as agents buying (when $+1$) or selling (when -1) an asset.

[a] In this paper we consider the minority game defined by Challet and Zhang [1]. When we refer to the Minority Game written in capital letters we are specifically talking about the dynamics defined in [1] and when we refer to the minority games in general we use lower case letters.

On the basis of the outcome

$$A(t) = \sum_{i=1}^{N} a^{\mu(t)}_{s_i(t),i}, \tag{2}$$

each agent updates his scores according to

$$U_{s,i}(t+1) = U_{s,i}(t) - a^{\mu(t)}_{s_i(t),i} \frac{A(t)}{P}, \tag{3}$$

where $P = 2^M$ is the total number of predictions. The idea of this equation is that agents reward $[U_{s,i}(t+1) > U_{s,i}(t)]$ those strategies that would have predicted the minority sign, i.e., $A(t)/|A(t)|$.

Similar results may be obtained when one considers the case when there is a nonlinear dependence on $A(t)$, i.e., with the dynamics

$$U_{s,i}(t+1) = U_{s,i}(t) - a^{\mu(t)}_{s_i(t),i} \mathrm{sgn}[A(t)], \tag{4}$$

where the sgn function is the sign function also known as the signum function, and is defined as

$$\mathrm{sgn}(A(t)) = \begin{cases} +1 & \text{if } A(t) > 0, \\ -1 & \text{if } A(t) < 0, \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

This leads to qualitatively similar results. A more lengthy discussion may be found elsewhere [11–14].

The source of randomness is in the choice of $\mu(t)$ and $s_i(t)$. These are fast fluctuating degrees of freedom. As a consequence $U_{s,i}(t)$ is also fast fluctuating and hence the probability with which the agents choose $s_i(t)$ are subject to stochastic fluctuations.

The key parameters is the ratio $\alpha = P/N$ and the two relevant quantities are

$$\sigma^2 = \langle A^2(t) \rangle, \quad \text{and} \quad H = \frac{1}{P} \sum_{\mu=1}^{P} \langle A|\mu \rangle^2, \tag{6}$$

which measure, respectively, the fluctuations of attendance $A(t)$, i.e., the smaller σ^2 is, the larger a typical minority group is — in other words σ^2 is a reciprocal of the global efficiency of the system and the predictability[b]; here $\langle \cdots \rangle$ denotes the temporal average over time.

One of the striking properties of this model is the fact that agents cooperate measured by σ^2. Agents taking random decisions would produce fluctuations equal to N so that agents cooperate if they manage to produce fluctuations lower than N. In Fig. 1 we show the graph of the global efficiency σ^2/N and the predictibility H/N versus the critical parameter $\alpha = 2^M/N$ for a sequence of number of agents

[b]In this work we follow the same terminology for the term *predictibility* and *global efficiency* as in [1, 11, 14].

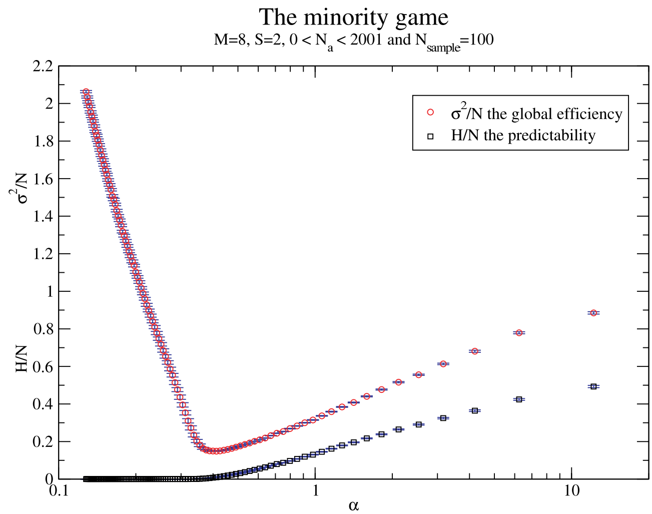

Fig. 1. The global efficiency σ^2/N and the predictibility H/N versus the critical parameter $\alpha = 2^M/N$ for a sequence of number of agents varying from 1 to 2001 when $M = 8$ and $S = 2$, in each simulation with $(N)_i$ number of agents it has been ensemble averaged over 100 samples ($N_{\text{sample}} = 100$). In this graph we can clearly see the three different regions, the first one fluctuations rapidly increase beyond the random agents and the game enters what has been called crowded region. At intermediate α the agent are at best coordination with each other, and finally at large α the game is more or less in a random mode.

varying from 1 to 2001 when $M = 8$ and $S = 2$, in each simulation with $(N)_i$ number of agents and with ensemble averaging over 100 samples ($N_{\text{sample}} = 100$).

In Fig. 2 we show the graph of the same quantities but this time plotted for two different values of S, that is, $S = 2, 4$ and 6. This time the graph is a log-log plot so that we can get a good view of the behavior of both the global efficiency and the predictability as S varies. In Figs. 3 and 4 the global efficiency and the predictability are graphed, respectively. It was initially pointed out [1] that one could observe three different regions in this graph. The first one is found when α is small. In that case there is a large number of agents. In that region fluctuations rapidly increase beyond the level of random agents and the game enters what has been called crowded region since it is reached by keeping M constant and N increasing. In other words the agents display a herding behavior and produce non-Gaussian fluctuations $\sigma^2 \sim N^2$ [1, 11, 14].

At intermediate α, as N decreases that is, when the game enters into a regime where agents cooperate to reduce fluctuations. In other words, that is when maximal cooperation is achieved.

Now if we go to the region where α is large, which means that N is small, then the outcome is more or less random. That is cooperation slowly disappears and the variance of the outcome tends to the value that would be produced by agents making random decisions. The reason for this is that the information, which

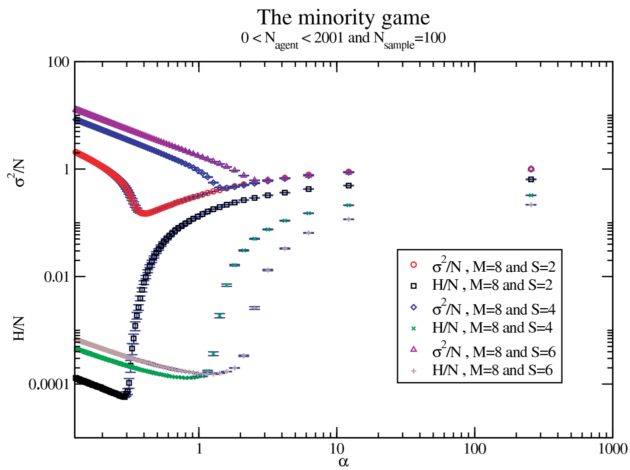

Fig. 2. The global efficiency σ^2/N and the predictibility H/N versus the critical parameter $\alpha = 2^M/N$ for a sequence of number of agents varying from 1 to 2001 when $M = 8$ and $S = 2, 4$ and 6, in each simulation with $(N)_i$ number of agents it has been ensemble averaged over 100 samples ($N_{\text{sample}} = 100$). This is the same graph as Fig. 1, but for different scenarios.

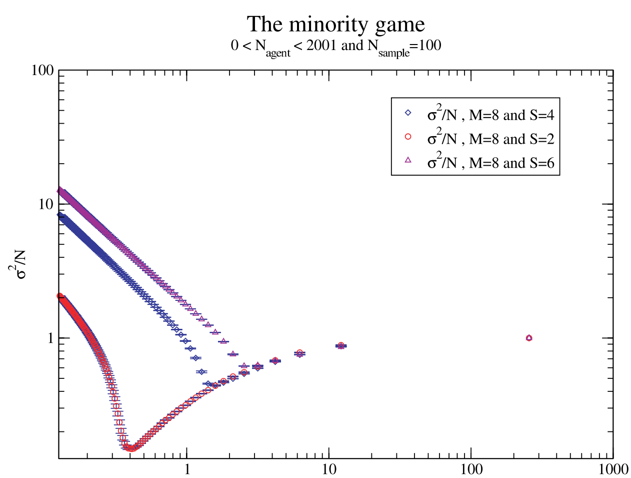

Fig. 3. The global efficiency σ^2/N versus the critical parameter $\alpha = 2^M/N$ for a sequence of number of agents varying from 1 to 2001 when $M = 8$ and $S = 2, 4$ and 6, in each simulation with $(N)_i$ number of agents it has been ensemble averaged over 100 samples ($N_{\text{sample}} = 100$) for the Minority Game.

agents receive about the past history, is too complex and their behavior over-fits the fluctuations of past attendance.

When S is varied the crowded region moves to the right, whereas σ^2/N for $N \ll 2^M$ seems to collapse on roughly the same curve. The measure of σ^2/N is less and less pronounced when S is larger, as shown in Fig. 3.

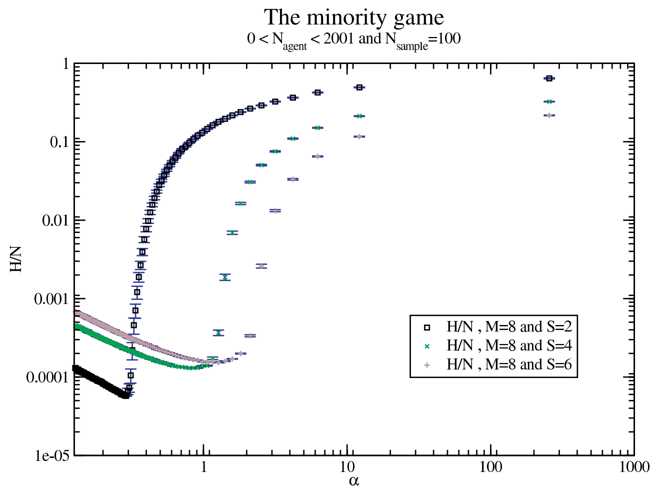

Fig. 4. The predictibility H/N versus the critical parameter $\alpha = 2^M/N$, for a sequence of number of agents varying from 1 to 2001 when $M = 8$ and $S = 2, 4$ and 6, in each simulation with $(N)_i$, agents has been ensemble averaged over 100 samples ($N_{\text{sample}} = 100$).

Also shown in Figs. 1, 2 and 4 is the predictability, which is another quantity of interest in the Minority Game. The predictability is a major issue in finance. It is commonly believed that markets are not efficient markets, violating the Efficient Market Hypothesis (EMH) [15]. Even in their weakest form (that is all public information on past prices and volumes affects the current price at every time), empirical studies [16, 17] show that there are systematic correlations in most financial markets.

In the case of the Minority Game there are different pieces of information such as the histories, which are common pieces of public information encoding the previous M last minority choices. Another aspect is the memory of the game in Eq. (3) with a given payoff function, in the case of the Minority Game it is given by

$$g_i(t) = -a^{\mu(t)}_{s_i(t),i} A(t), \tag{7}$$

with $A(t)$ defined as in Eq. (2) for the agents.

The scores, given by Eq. (3), contain information about the game. The normalized predictability in the Minority Game is calculated from $A(t)$, that is,

$$H = \frac{1}{2^M} \sum_{i=1}^{P} \langle A(t)|\mu(t)\rangle^2. \tag{8}$$

At the point where H starts to differ from 0 (at around $\alpha_c \approx 0.34$ for $S = 2$) and starts to increase, the system becomes predictable. In statistical physics this is commonly known as a phase transition with symmetry breaking as α varies. For $S = 2$, where $\alpha_c \approx 0.34$ when $\alpha > \alpha_c$, we then have an asymmetric phase. This is when the outcome becomes probabilistically predictable.

In Figs. 1 and 4 we see a graph of the predictability for $S = 2$, and $M = 8$, and for the number of agents varying from 1 to 2001.

2.2. *The price function in the minority game*

To connect the Minority Game with the financial market, one needs to examine the price dynamics. Here we shall focus on a market for a single asset and call $P(t)$, its price function at a time t. Let us assume that the price is driven by the difference between the number of shares being bought and sold, called the *excess demand*. This is how the connection is made with the Minority Game. In the Minority Game we assume that the behavior of agents is restricted to the two possible actions, that is buy (i.e., $a_i(t) = 1$) and sell (i.e., $a_i(t) = -1$). The $A(t) = \sum_i a_i(t)$, Eq. (2), is simply the difference between demand and supply, i.e., the excess demand.

Several price formulation rules can be found in the literature, which link the excess demand $A(t)$ to the price return. The simplest one is to suppose that the price return $r(t)$ depends linearly on $A(t)$ [18],

$$r(t) = \ln\left(\frac{P(t)}{P(t-1)}\right) = \frac{A(t)}{\lambda}, \tag{9}$$

where λ is sometimes called the liquidity or the market depth [19]. This relationship is implicit in many early works, which refer to σ^2 as price volatility, but a plot of $\ln(P(t)) = \sum_{t' \leq t}^{t} A(t'/\lambda)$ was not shown until the paper by [20]. Equation (9) can be justified in limit order markets, that is markets where people can submit limit orders [1, 11, 14], which are requests to buy or sell a given quantity of the asset

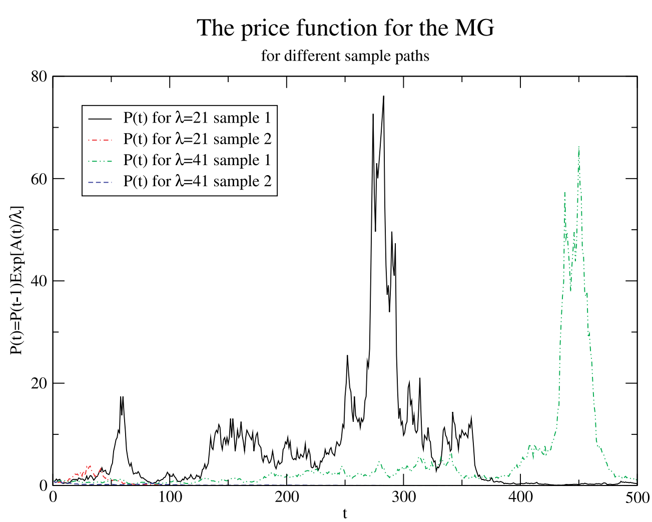

Fig. 5. The price function $P(t)$, Eq. (9), for two different liquidity values $\lambda = N = 21$ and 41 for two different samples for t up to five hundred ticks. This is for simulated data within the Minority Game.

at a given price. Each of these orders can only be executed if there is an opposite matching request. In this way, the quantity and the price of the transaction are fixed and the time when the limit order will be executed is left undetermined. Orders waiting to be executed are stored in the order book. In Fig. 5 we show the price function for two different values of the liquidity $\lambda = N = 21$ and 41 for two different samples for t up to five hundred ticks. Now supposing that at time $t - \epsilon, 0 < \epsilon \ll 1, N$ market orders of size 1 arrive simultaneously on the market. Assuming that $(N + A)/2$ are buy orders and $(N - A)/2$ are sell orders, it is then possible to match $(N - |A|)/2$ buy and sell orders and to execute them at the current price. This leaves unexecuted $|A|$ orders of one kind. If $A > 0$ they will be buy orders, else sell orders. There orders will be matched with the best limit orders of the opposite type present in the order book.

Now assuming that there is a uniform density λ of limit orders, that is λ orders per tick (ticks are evenly spaced), the price will be displaced by a quantity A/λ, as all the orders between $P(t-1)$ and $P(t) \equiv P(t-1) + A/\lambda$ will be executed. This is what Eq. (9) postulates. This process can go on assuming that there are new limit orders that fill the gap between $P(t-1)$ and $P(t)$, restoring a uniform distribution of limit orders. [22] have shown that the assumption of uniform order density of the order book, which is responsible for the linear relationship between A and r is a very rough approximation.

An alternative definition for the price function, under the same assumption as the Minority Game for each agent, is specified as follows. Supposing that $a_i(t) = +1$ means that agent i invests \$1 in order to buy the asset at time t, whereas $a_i(t) = -1$ means that he/she sells $1/P(t-1)$ units of assets, where $P(t-1)$ is the price of the last transaction. Then the total demand is $(N + A)/2$ and the total supply is $(N - A(t))/2P(t-1)$ units of asset where $A(t) = \sum_i a_i(t)$. Then the price $P(t)$ is fixed in such a way that the demand matches the supply, that is

$$P(t) = P(t-1)\frac{N + A(t)}{N - A(t)}. \tag{10}$$

If $A(t) \ll N$, taking the logarithm of both sides and keeping the leading order terms leads to an expression that is very similar to Eq. (9) with $\lambda = N/2$.

Using these two definitions we compare the price time series in the Minority Game. This is shown in Fig. 6. The graph of the price function in the Minority Game for the two definitions of $P(t)$ given by Eqs. (9) and (10) for the full range of Minority Game time ticks. Here $S = 2, N = 121 \equiv \lambda$ and $M = 8$, for the full time time series in the Minority Game when each of the 121 agents have 2 strategies and when the memory is of the order of 8, i.e., $M = 8$ and in Fig. 7 for the first 2500 time ticks of the time series. In these two figures the number of agents is $N = 121$ and was used for the liquidity λ.

Now looking at the two trajectories we can see that Eq. (10) gives a higher estimate than Eq. (9) while giving very similar trajectories. These two definitions

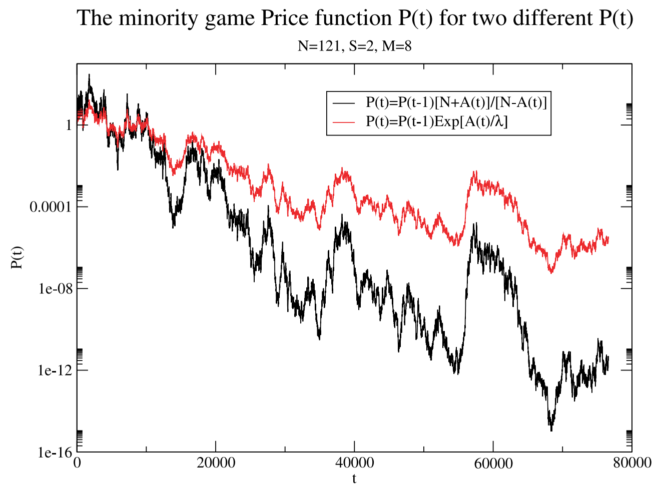

Fig. 6. The simulated price function in the Minority Game for the two definition of $P(t)$ given by Eqs. (9) and (10) for the full Minority Game time ticks. Here $S = 2$, $N = 121 \equiv \lambda$ and $M = 8$. Ignoring the scaling issue we can see that the price function remains stable for large value of time.

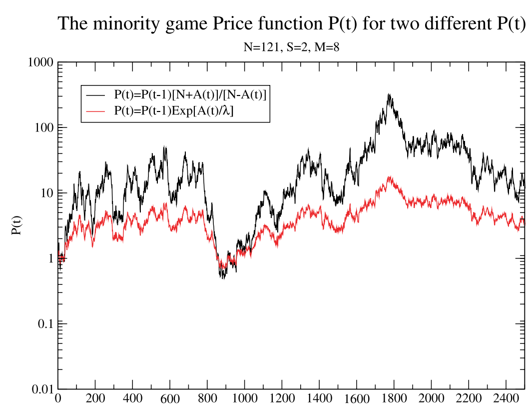

Fig. 7. The price function in the Minority Game for the two definition of $P(t)$ given by Eqs. (9) and (10) for the first 2500 time ticks. Here $S = 2$, $N = 121 \equiv \lambda$ and $M = 8$. This is the same as Fig. 6 but this time on a smaller range.

may be compared better when real data is used in the Minority Game with a different payoff, see Sec. 2.3 for later discussion.

As previously mentioned, Fig. 5 shows the price time series evolution for Eq. (9) for the first 500 time ticks when $\lambda = 21$ and 41, $S = 2$ and $M = 8$ in the Minority Game for 2 different initial configurations or samples. In this figure we can see some sharp peaks. Here at this level the time series either diverges to infinity or converges to 0. This is because we have taken $\lambda = N$ when λ should not be taken

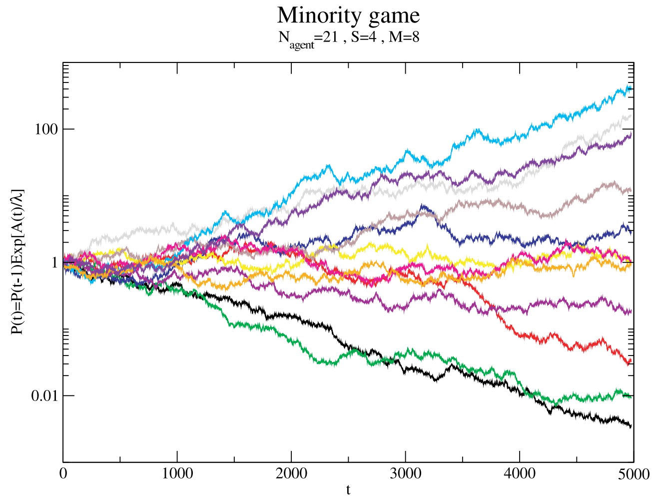

Fig. 8. The price function in the Minority Game for $P(t)$ given by Eq. (9) for the first 5000 time ticks. Here $S = 2, \lambda \propto N = 21$ and $M = 8$ taken over many different initial configuration on a smaller range of time. From this graph we can see how the different paths are not biased.

as a constant and there is also a time scale associated with it. Furthermore, if λ is taken as the market depth it is commonly accepted that the market depth is also a time series and varies in time thus taking λ to be a constant is partially incorrect.

Finally to make sure that we have not any bias in the time series we have repeated the experiment a number of times and plotted Eq. (9) for many different configurations, this is shown in Fig. 8. From this graph we can clearly see that each paths are clearly distinct and it therefore shows no bias overall.

2.3. *The Dollar Game*

We now consider the \$-Game [23], and point out the small difference between the Minority Game and the \$-Game.

The Minority Game is a repeated game where N agents, have to choose one out two possible alternatives at each step. Each agent, i, has a memory of the past. At each time step t every agent decide whether to buy or sell an asset. The agent takes an action $a_i(t) = \pm 1$ where 1 is when buying an asset as opposed to -1 when selling. The Excess demand $A(t)$ at time t is then given by Eq. (2), that is $A(t) = \sum_{i=1}^{N} a^{\mu(t)}_{s_i(t),i}$. The payoff of agent i in the Minority Game is given by Eq. (7).

In order to model financial markets, some authors have used the following definition for the return $r(t)$ using the price time series $P(t)$ [18, 19]

$$r(t) \equiv \ln[P(t)] - \ln[P(t-1)] = \frac{A(t)}{\lambda}, \tag{11}$$

which means that price time series is defined by

$$P(t) = P(t-1)\exp\left[\frac{A(t)}{\lambda}\right]. \tag{12}$$

Here the liquidity λ is proportional to the number of agents N. In the Minority Game the agents predicts the price movements only over the next time step. However, [23] have shown that in order to know when the price reaches its next local extreme-mum and optimize their gain the agents need to predict the price movement over the next two time steps ahead (t and $t+1$) and they therefore have postulated the correct payoff function to be given by

$$g_i^{\$}(t+1) = a_i(t)A(t+1). \tag{13}$$

This small difference in the payoff function is what defines the $-Game. From now on when we are refering to the $-Game, we really mean the Minority Game with the payoff function defined by Eq. (13). In this case we define the game as the $-Game.

3. Financial Bubbles

It is well known in economics that markets follow both positive and/or negative trends, crashes, and bubble effects. In general a strong positive trend is followed by a crash, famous examples of these effects were seen in the recent crash on the NASDAQ (April 2000) and prior to the crash in the Hong Kong market, which was associated with the Asian crisis in the early 1994.

A strong positive trend in economics is commonly called a *bubble*. Bubbles can occur in all sorts of different sectors, for example in the technology sector, resources sector, housing sector, the music industry or the pharmaceutical sector. So a bubble is really when investors follow the same trend or strategies for a given time (e.g., buying or selling) for a while until the demand decreases, which may be due to economic slowdown or change of perspectives in economical strategies. At that time the trend usually takes an opposite direction (either a positive trend corresponding to buying, then once the bubble has matured everyone starts selling, or vice versa).

A common approach to viewing the market is carried out by assuming that these are complex evolutionary systems that are adaptive and that they are populated by rational agents interacting with each other. These sorts of models are researched at the Santa Fe Institute in New Mexico [18, 24] as well as other institutions worldwide [1, 25–27].

One of the main problems in most of the models is that they do not capture the characteristic structure of bubbles. However if such effects are actually present in markets (which is commonly accepted that they are) they probably constitute one of the most important facts in explaining and detecting market behavior with their associated consequences such as large potential losses during crashes and recession following these bubbles.

Since the earlier works on Rational Expectation (RE) bubbles [18, 25] the size of the literature on the subject has been growing with theoretical improvements of the original concept and on the empirical detectability of RE bubbles in financial data [28, 29]. At the same time, empirical research has largely concentrated on testing for explosive exponential trends in the time series of asset price and foreign rates [30, 31].

Many RE bubbles produce curves that are not always consistent with economic facts, a major problem is that the appearance of bubbles can be reinterpreted in terms of market fundamentals that are not observed by the researcher. Another suggestion is that if stock prices are not more explosive than dividends then it can be concluded that rational bubbles are not present, since bubbles are taken to generate an explosive component of stock prices [32]. However periodically collapsing bubbles are not detectable by using standard tests to determine whether stock prices are more explosive or less stationary than dividends [30]. So in short, the present evidence for an ability to speculate on bubbles remains an unsolved problem.

3.1. *Positive feedback model with multiplicative noise*

In this section, a model to generate the bubble price $B(t)$ is described. This model has been developed by [32] and has been used in previous studies [32, 33]. Here we use the same notation and interpretation as in these references. Readers interested in how the model is derived may see these last two references for further details.

The bubble price model is an hyperbolic stochastic finite-time singularity formula, which transforms a Wiener process into a time series containing no correlation of returns [34] long range correlation of volatility [35], fat–tail of returns distribution [36–38], apparent multifractality [39, 40], sharp peak through flat pattern of price peaks [41], as well as accelerated speculative bubbles preceding crashes [42].

One of the key aspects of this model is that bubbles are growing superexponentially, that is, self growing in time, this leads to power law acceleration, which eventually leads to a singularity as opposed to other bubble models, which are based on exponential growth.

The formulation of the bubble price $B(t)$ is initially constructed from the Black–Scholes–Merton option pricing model [43], $dB(t) = \mu B(t)dt + \sigma B(t)dW_t$ with μ the instantaneous return rate and σ the volatility. The Gaussian noise W_t is the standard Wiener process.

The bubble model is generalized as

$$dB(t) = \mu(B(t))B(t)dt + \sigma(B(t))B(t)dW_t - \kappa(t)B(t)dj, \tag{14}$$

where $B(t)$ is the price of the bubble, μ the abnormal return rate above the fundamental return, σ is the volatility of the bubble and the jump term dj describes a correction or a crash that may occur with amplitude κ. The crash amplitude can be a stochastic variable taken from an arbitrary distribution.

Straight after the last crash, which becomes the new origin of time 0, dj is reset to 0 and will eventually jump to 1 with hazard rate $h(t)$ with probability $P(t < t_{\text{crash}} < t + dt) = h(t)dt$. The discontinuous process for the jump dj is defined as in Cox *et al.* [44, 45] to define the jump as a discontinuous process. Its average $\langle dj \rangle$ is given by

$$\langle dj \rangle = 1 \times h(t)dt + 0 \times (1 - h(t)dt) = h(t)dt. \tag{15}$$

Using this definition of the hazard rate $h(t)dt = \langle dj \rangle$ we can see for Eq. (14) that the expectation, over all possible outcomes since the last crash, leads to

$$\mu(B(t))B(t) - \langle \kappa \rangle B(t)h(t) = 0, \tag{16}$$

which gives an expression for the hazard rate

$$h(t) = \frac{\mu(B(t))}{\langle \kappa \rangle}. \tag{17}$$

It is possible to generalize Eq. (14) by allowing some nonlinearity in $\mu(B(t))$ and $\sigma(B(t))$, as shown in [32] and in [33]:

$$\mu(B(t))B(t) = \frac{m}{2B(t)}[B(t)\sigma(B(t))]^2 + \mu_0 \left[\frac{B(t)}{B_0}\right]^m, \tag{18}$$

$$\sigma(B(t))B(t) = \sigma_0 \left[\frac{B(t)}{B_0}\right]^m. \tag{19}$$

Here $B_0, \mu_0, m > 0$, and σ_0, are, respectively, four parameters of the model that are a reference scale, an effective drift, the strength of nonlinearity and the magnitude of stochastic component which sets the scale of the volatility (i.e., the nonlinear positive feedback). The first term in Eq. (18) was added for convenience to simplify the Îto calculation of the stochastic differential equation.

Herding is perhaps the most obvious reason that leads to positive nonlinear feedback of $\mu(B(t))$ and $\sigma(B(t))B(t)$ on stock prices.

The solution of Eq. (14) with Eqs. (18) and (19) is derived in references such as [32] and in [33] and is given by

$$B(t) = \alpha^\alpha \frac{1}{\left(\mu_0 \left[t - t_c\right] - \frac{\sigma_0}{B_0^m} W(t)\right)^\alpha}, \tag{20}$$

where $\alpha \equiv 1/m - 1$ and with $t_c = y_0/(m-1)\mu_0$. The critical time t_c is a finite time singularity that is determined by initial conditions with $y_0 = 1/[B^{(m-1)}(t = 0)]$, see Appendix in [32]. In Fig. 9, the graph of the time series for the bubble defined in Eq. (20) versus the time $t, 0 \leq t \leq 2500$ with fixed parameters $m = 3, \mu_0 = 0.01, B_0 = y_0 = 1, \delta t = 0.0003$ and the critical time $t_c = 1$ for two distinct sample path of the Wiener process.

That is the graph of Eq. (20) versus the time $t, 0 \leq t \leq 2500$ with fixed parameters $m = 3, \mu_0 = 0.01, B_0 = y_0 = 1, \delta t = 0.0003$ and the critical time $t_c = 1$ for

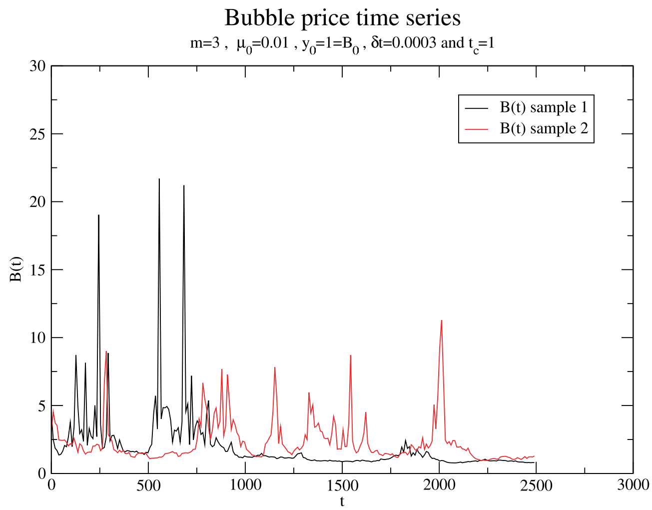

Fig. 9. The time series for the bubble defined in Eq. (20) versus the time $t, 0 \leq t \leq 2500$ with fixed parameters $m = 3, \mu_0 = 0.01, B_0 = y_0 = 1, \delta t = 0.0003$ and the critical time $t_c = 1$ for two distinct sample path of the Wiener process.

two distinct sample path is shown. In both cases the graphs show some very sharp but finite peaks after a certain time of normal activity.

Note that Eq. (20) is correct as long as a crash $dj = 1$ has not occurred, which may happen at any time according to the crash hazard rate $h(t)$, given by Eq. (17) determined from nonarbitrage conditions. Here $\langle\kappa\rangle$ is the average amplitude calculated over some predetermined distribution of κ. In the deterministic case $\sigma_0 = 0$ reduces to $B(t) \propto 1/[t_c - t]^{1/m-1}$, that is the bubble follows a hyperbolic growth path which would diverge in finite time if not checked by crashes according to Eq. (17).

One must note that this hyperbolic growth is a sign of the positive feedback characterized by $m > 1$ of the price $B(t)$ on the return rate μ.

On the other hand if $\sigma \neq 0$ we see that the crash hazard rate grows even further than the bubble price we then do not obtain a singularity. In the limit $1/\alpha \to 0 (m \to 1)$ in Eq. (20)

$$B(t) = \exp\left[\mu_0 t + \sigma_0 W(t)\right], \tag{21}$$

one recovers the standard Black–Scholes–Merton solution.

4. Minority Game and Dollar Game Price Function with Real Data

In this section, we combine the results from the previous sections to monitor the price function when real data is inserted into the Minority Game with the

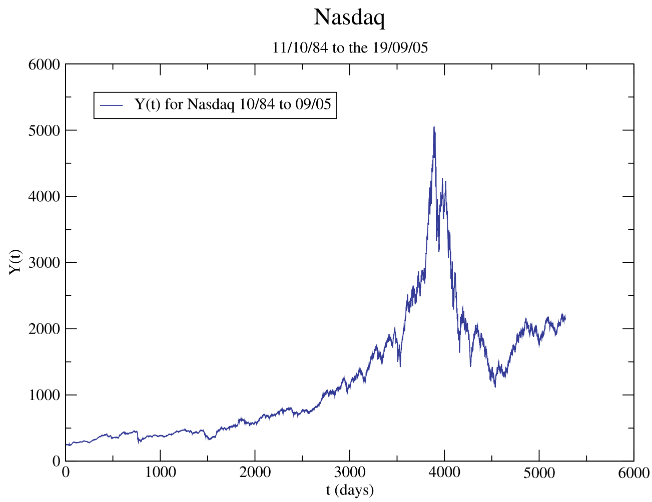

Fig. 10. The graph of the NASDAQ versus the time $t, 1 \leq t \leq 5283$ over the period of 11/09/84 to 19/09/05, showing clearly the signs of a bubble over the time.

dollar game payoff. The idea is to see if the agent model does follow the real data trajectories.

From past historical data we can see where bubbles have occurred in the past, and use this information to see how an agent model — such as the Minority Game — will track the real data.

Here we will use the historical price time series of the NASDAQ over a period of about twenty years, that is from October 1984 to late September 2005, Fig. 10.

Over this period we can clearly see the bubbles due to the technological sector from the mid eighties until the bubble burst in the early 2000. Large growth was then followed by a big crash, where billions of dollars were wiped off the market.

The other set of data that will be considered will be from the S&P500 from the late nineties to the present day, see Fig. 11, that is over the period of January 1998 to September 2005.

We now use this data to insert it into the Minority Game to see how the game behaves and evolves as a function of time t with two different payoff, that of two different dynamical processes. Here the payoff function is updated differently as in the standard Minority Game described in the earlier section. We introduce an extra parameter that looks over a certain time in the past, we call it T. It can be understood as a window parameter that can be attributed a certain length. In this setting we update the scores, defined in equation as in Eq. (3)

$$\Delta U_{s,i}(t) = \sum_{\kappa}^{t} a^{\mu(j)}_{s_i(t),i} \frac{A(j)}{P}, \quad \text{with } \kappa = \begin{cases} j = t - T + 1 & \text{if } t - T + 1 > 0, \\ j = 1 & \text{if } t - T + 1 \leq 0. \end{cases} \tag{22}$$

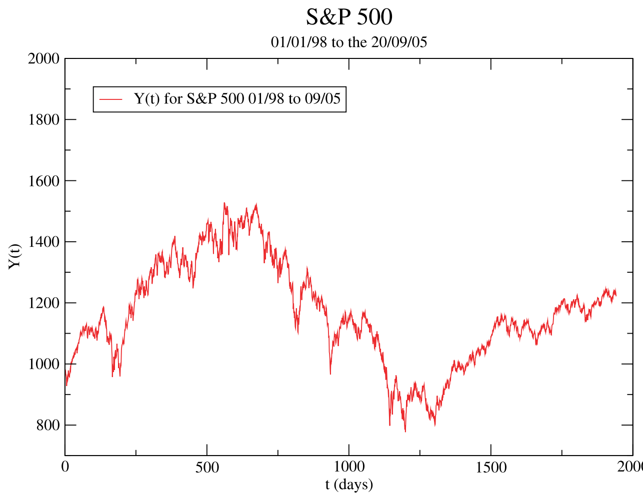

Fig. 11. The graph of the S&P500 versus the time $t, 1 \leq t \leq 1941$ over the period of 01/01/98 to 20/09/05, showing clearly the signs of a bubble over the time.

where $A(t)$ is from the Minority Game strategy selection as described in Sec. 2.1. The scores are then updated such as

$$U_{s,i}(t+1) = \Delta U_{s,i}(t). \tag{23}$$

The real data is inserted via the action $a^{\mu(t)}_{s_i(t),i}$. In the simulated case the action $a^{\mu(t)}_{s_i(t),i}$ is generated randomly and take the value of $+1$ or -1. Using the real data we can generate the evolution of the action functional as we evolve through the real data. Supposing that we denote the real data set by $S(t)$. We set $a^{\mu(t)}_{s_i(t),i} = 1$ when the value goes up, in other words when $S(t+1) > S(t)$ and set $a^{\mu(t)}_{s_i(t),i} = -1$ when the value goes down, that is when $S(t+1) < S(t)$. When the value stays unchanged, $S(t+1) = S(t)$, we flip a coin with equal probability.

Using this method we can compare the dynamics of both games. In Fig. 12, we show the graph of the price function as a time series for the \$-Game versus the Minority Game in the Minority Game as a function of time t on a linear scale, $1 \leq t \leq 5283$. This is compared to the real data from the NASDAQ over the period of 11/09/84 to 19/09/05, showing clearly the signs of a bubble over the time. Here the number of agents $N = 41$ and each agent have $S = 2$ strategies to choose from with a memory of 8, $M = 8$ and with a window size of $T = 100$.

Ignoring the scale factor problem between the games and the real data — something that will need to be resolved later on — we can see that in Fig. 12 the \$-Game and the real data follow very similar trajectories as opposed to the Minority Game, which is not sensitive to the existence of a bubble. So in this figure we can see that the \$-Game is significantly more sensitive to the bubble showing clear evidences of lumps and troughs displayed in the real data. There is also clear evidence that there

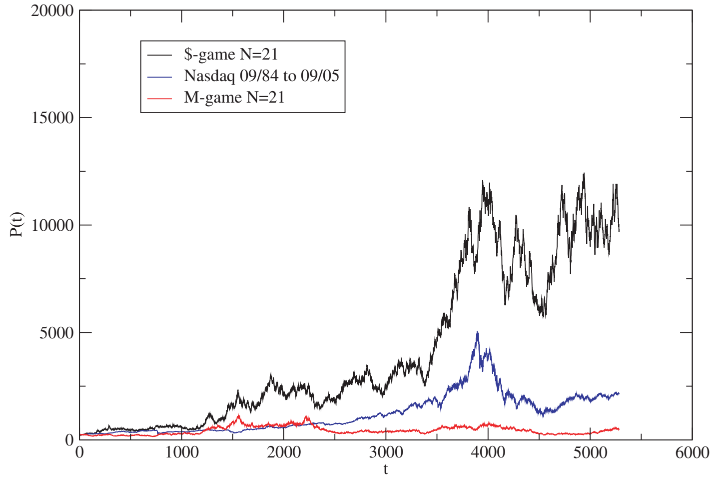

Fig. 12. The graph of the NASDAQ versus the time $t, 1 \leq t \leq 5283$ over the period of 11/09/84 to 19/09/05, showing clearly the signs of a bubble over the time. The is compared with the plot of both the Minority Game and the \$-Game. We can clearly see that the Minority Game does not sensitive to the existence of the bubble, but the \$-Game does.

is a scaling problem. This comes from the fact that the liquidity is approximated to be $\lambda \sim N$. The liquidity is however, usually affected, as the market depth is. The market is not constant right through and should be taken as a time series. In Fig. 13 we show the graph of the price function as a time series for the \$-Game versus the Minority Game in the Minority Game as a function of time t on a logarithmic scale, $1 \leq t \leq 5283$. This is compared to the real data from the NASDAQ over the period of 11/09/84 to 19/09/05. Here the number of agents $N = 41$ and each agent have $S = 4$ strategies to choose from with a memory of 4, $M = 4$. The window size is $T = 10$.

If we look at Fig. 13 when the number of strategies is set to $S = 4$ with a memory of $M = 4$, where we have set λ to two different values, we can easily see the dependence of the liquidity over time therefore showing clearly the sign of non constant liquidity over the time series evolution. In this figure the black curve is when the liquidity $\lambda = 10N$ while the red curve is when $\lambda = 15N$. On the other hand one should note that increasing the factor in front of the liquidity does not always bring the curve closer to the real data, sometimes it is the opposite.

We can now compare the dynamics of both payoffs for a different number of agents N and liquidity λ. In Fig. 14 we show on a logarithmic scale the graph of the price function as a time series for the \$-Game versus the Minority Game as a function of time $t, 1 \leq t \leq 5283$. This is compared to the real data from the NASDAQ (the blue curve) over the period of 11/09/84 to 19/09/05. Here the number of agents $N = 21$, 41 and 61 in each games agents have $S = 4$ strategies to choose from with a memory of 4, $M = 4$. The window size is $T = 10$. In Fig. 14 we

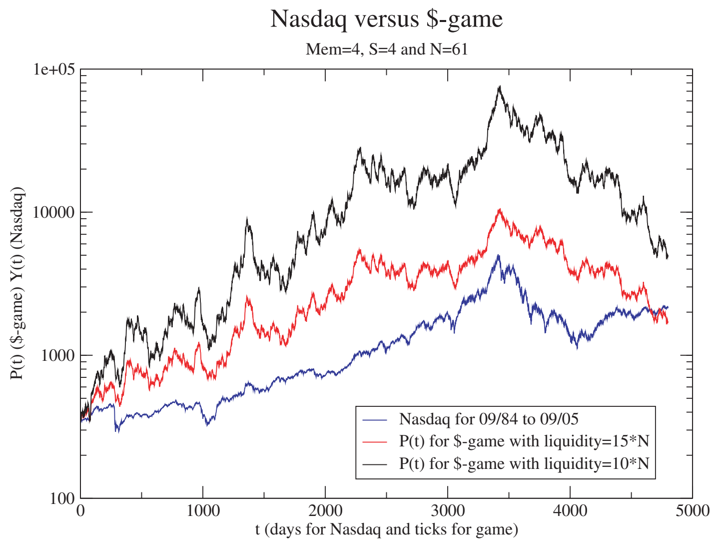

Fig. 13. The price function as a time series for the $-Game in the Minority Game on a logarithmic scale, $1 \le t \le 5283$. This is compared to the real data from the NASDAQ over the period of 11/09/84 to 19/09/05, showing clearly the signs of a bubble over the time. Here the number of agents $N = 41$ and each agent has $S = 4$ strategies to choose from with a memory of 4, $M = 4$. The window size is $T = 10$.

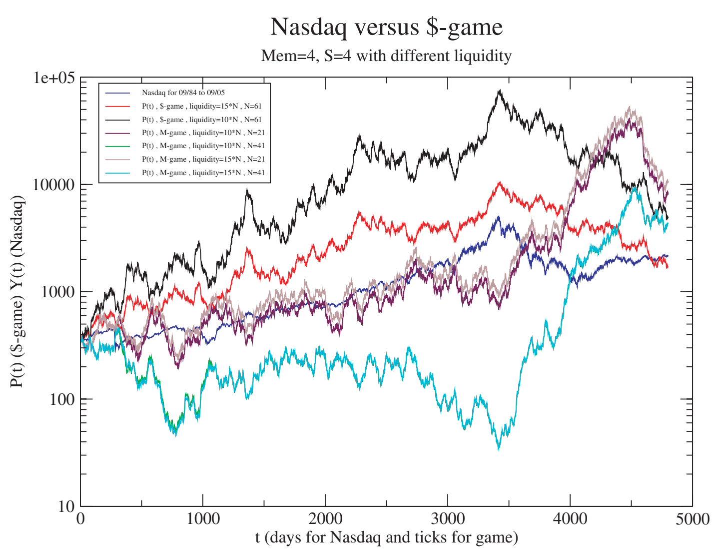

Fig. 14. The price function for the $-Game versus the Minority Game in the Minority Game as a function of time t on a logarithmic scale, $1 \le t \le 5283$. This is compared to the real data from the NASDAQ over the period of 11/09/84 to 19/09/05, showing clearly the signs of a bubble over the time. Here the number of agents $N = 21$, 41 and 61 in each games agents have $S = 4$ strategies to choose from with a memory of 4, $M = 4$. The window size is $T = 10$.

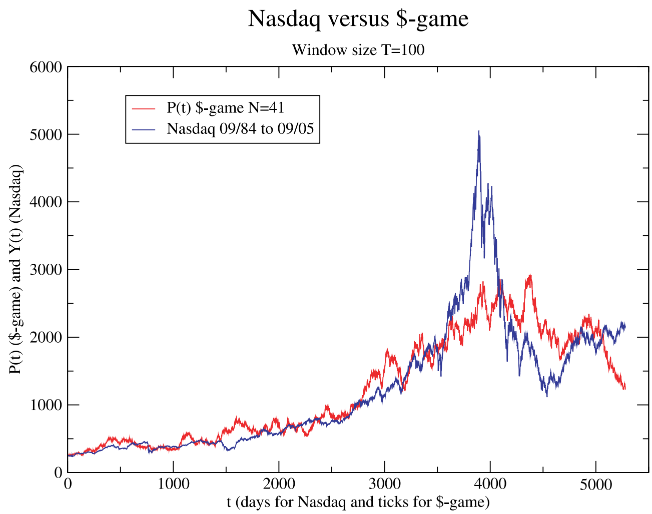

Fig. 15. The time series for the NASDAQ versus the \$-Game price function in the Minority Game as a function of time $t, 1 \leq t \leq 5283$ over the period of 11/09/84 to 19/09/05, showing clearly the signs of a bubble over the time. Here the number of strategies is $S = 2$ and the memory is $M = 8$.

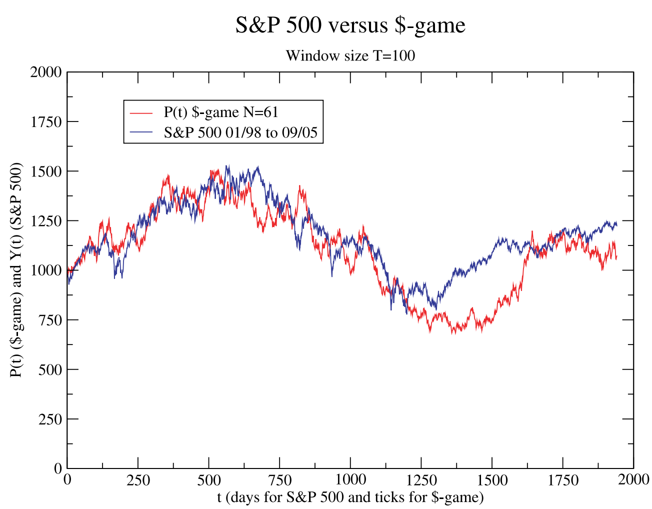

Fig. 16. The graph of the S&P 500 versus the \$-Game price function in the Minority Game as a function of time $t, 1 \leq t \leq 1941$ over the period of 11/01/98 to 20/09/05, showing clearly the signs of a bubble over the time. Here the number of strategies is $S = 2$ and the memory is $M = 8$.

can see that in all cases the dynamics of the Minority Game does not quite follow those of the real data, contrarily to the \$-Game.

As a final test we turn off the dynamics of both games by setting $\Delta U_{s,i}(t) = 0$ which means that the scores do not get updated, and seeing how each game performs

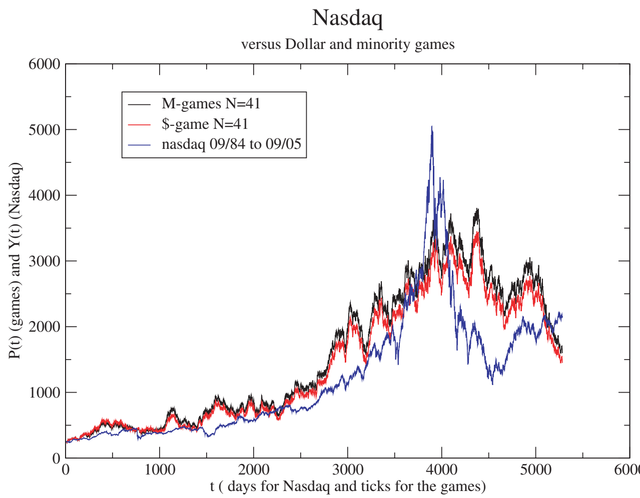

Fig. 17. The price function for the \$-Game versus the Minority Game when the payoff function are set to 0 in the Minority Game as a function of time t, $1 \leq t \leq 5283$. This is compared to the real data from the NASDAQ over the period of 11/09/84 to 19/09/05, showing clearly the signs of a bubble over the time.

on real data, namely on the NASDAQ and S&P500. This is shown in Figs. 15 and Fig. 16, where we can see that both curves follow each other quite well.

Finally comparing the outcome of both the \$-Game and the Minority Game when the payoff $\Delta U_{s,i}(t) = 0$ gives trajectories that overlap on top of another, see Fig. 17, as one would expect.

5. Conclusion

We used the Minority Game, which is a special class of agent models, to simulate the evolution of the price function using real data. It is well established that the NASDAQ has undergone a major bubble effect, which started during the late 90s and bursting in the early years of this millennium, as shown in Fig. 10. This is commonly known as the "tech bubble".

Bubble detection remains an unsolved problem in economics, attempts like the one mentioned in Sec. 3.1 can be used to model these phenomena, but from Fig. 9 we remark that these models still remain unstable. However by using an agent model such as the \$-Game it is possible to mimic the dynamics of the bubbles. It is also clear that the Minority Game does not really follow the dynamics of the real data and that it is not sensitive to the presence of the bubble, as shown in Fig. 12, but the \$-Game is a more suitable way to explore the dynamics. The flow in the Minority Game is that the updating of the scores is carried out at the wrong time, and this is what has been corrected by [32] by the introduction of the \$-Game.

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