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EVALUATION OF BISTABLE SYSTEMS VERSUS MATCHED FILTERS IN DETECTING BIPOLAR PULSE SIGNALS

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This paper presents a thorough evaluation of a bistable system versus a matched filter in detecting bipolar pulse signals. The detectability of the bistable system can be optimized by adding noise, i.e. the stochastic resonance (SR) phenomenon. This SR effect is also demonstrated by approximate statistical detection theory of the bistable system and corresponding numerical simulations. Furthermore, the performance comparison results between the bistable system and the matched filter show that (a) the bistable system is more robust than the matched filter in detecting signals with disturbed pulse rates, and (b) the bistable system approaches the performance of the matched filter in detecting unknown arrival times of received signals, with an especially better computational efficiency. These significant results verify the potential applicability of the bistable system in signal detection field.

Keywords: Bistable system; matched filter; signal detection statistics; robust; computational efficiency.

1. Introduction

Stochastic resonance (SR) phenomena involve optimizing a system performance measure via adding noise, and have received considerable attention in the past few decades [1–22]. The prototype SR model is an overdamped bistable system, which was used to describe the earth's climatic change [1–3]. Since then, this generic system is a frequently used model for characterizing SR phenomena in diverse scientific fields [4–14]. Moreover, many experimental verifications of SR effects have been demonstrated for a bistable characteristic, such as in a Schmitt Trigger [15], bistable ring laser [16], paramagnetically driven bistable buckling ribbon [17], bistable electron paramagnetic resonance systems [18], bistable superconducting quantum interference devices [19], vertical cavity surface emitting lasers [20], etc. In addition, new SR-type phenomena were also observed in bistable systems subjected to suprathreshold input signals [21, 22].

Recent research has focused on potential SR applications in signal transmission [20, 23–27], estimation [28, 29] and detection [30–48]. Signal detection theory offers a powerful tool for analyzing linear systems, such as the matched filter with a Gaussian white noise background and its impulse response matched to the input signal maximizing the filter output signal-to-noise ratio [49]. These nonlinear systems, thereby, are potentially rich in signal detection applications, though generally more difficult to theoretically tackle [5, 7, 34, 35]. A variety of appealing nonlinear systems or models exhibiting SR effects have been explored as detection devices, such as the threshold detector [35, 37], electronic circuits [5, 40–42], biological systems [5–7, 10, 13], neuron models [45, 46, 48], bistable systems [30–32, 65], etc. Although many researchers stressed that SR can not do wonders in signal detection, in the sense that nonlinear systems exploiting SR can never do better than the corresponding linear systems [30–32, 46, 50–52, 61] in the presence of Gaussian white noise, some promising results have been obtained for non-Gaussian noise conditions [34–36, 43, 44], the computational and design efficiency of nonlinear systems [32, 53], noise-floor limited systems [54, 55], psychophysical experiments [23] or models [56], and parallel arrays of nonlinear devices via the suprathreshold SR effect [57, 58, 62–64].

In line with this, we investigate further the performance of bistable systems versus matched filters in detecting bipolar pulse signals. Some new merits of bistable systems will be emphasized in the comparison of results. Section 2 develops a theoretical non-stationary probability density model of the bistable system. This approximate theory describes the SR phenomenon in the bistable system well, and agrees with the numerical simulation results. The comparison of the pertinent signal detection statistics, between bistable systems and matched filters, demonstrates the fact that the bistable system is a suboptimal detector against the matched filter [30–32, 57, 58]. Beyond this fact, we evaluate in detail the detectability of bistable systems, versus matched filters, in detecting pulse signals with disturbed pulse rates and unknown arrival times in Sec. 3. From the power spectrum analysis of bistable system outputs, the different pulse rates of the disturbed signal can be clearly discriminated. The matched filter, however, is more like a lowpass filter, wherein too much noise at low frequency bands weakens its resolution. In the case of detecting the unknown arrival time of received signals, the bistable system, with lower computational overhead, is an appropriate substitute for the matched filter. The corresponding numerical simulation results also indicate that the bistable system is a more competitive detector than the matched filter, with respect to its robustness and computational efficiency. This may be of importance to bistable electronic or optical devices utilized in signal detection. We argue that the potential applicability of the bistable system deserves further study in other signal processing problems.

2. Signal Detection Statistics of the Bistable System Versus the Matched Filter

Here, we consider the detection problem of a bipolar pulse signal $s(t)$ and introduce two hypotheses

$$\begin{aligned} \boldsymbol{H}_0 &: s(t) = -A,\ [(n-1)T_p,\ nT_p], \\ \boldsymbol{H}_1 &: s(t) = +A,\ [(n-1)T_p,\ nT_p], \end{aligned} \tag{1}$$

where $n = 1, 2, \ldots$. The input pulse signal $s(t)$, for $(n-1)T_p \le t \le nT_p$, has the pulse duration time T_p and pulse amplitudes $\pm A$ (see an example of Fig. 1). The background noise $\eta(t)$ is additive Gaussian white noise with autocorrelation $\langle \eta(t)\eta(0)\rangle = 2D\delta(t)$ and zero-mean. Here, D denotes the noise intensity. The mixture of $s(t)$ plus $\eta(t)$ is then applied to a detector, such as the matched filter or the bistable system introduced below.

2.1. *Non-stationary probability density model of the bistable system*

The bistable system under study is described as

$$\tau_a \frac{dx(t)}{dt} = x(t) - \frac{x^3(t)}{X_b^2} + s(t) + \eta(t), \tag{2}$$

with real system parameters τ_a and X_b [25]. τ_a is related to the system relaxation time. The dynamics of Eq. (2) are derived from the symmetrical double-well potential $V_0(x) = -x^2/2 + x^4/(4X_b^2)$, having the two minima $V_0(\pm X_b) = -X_b^2/4$. Parameters τ_a and X_b have the units of time and signal amplitude respectively, and define natural scales associated to the process of Eq. (2).

In each pulse duration T_p, the system of Eq. (2) is subjected to the constant amplitude $+A$ or $-A$, with an additive input Gaussian white noise $\eta(t)$. Under these conditions, the statistically equivalent description for the corresponding probability density $\rho(x,t)$ is governed by the Fokker–Planck equation

$$\tau_a \frac{\partial \rho(x,t)}{\partial t} = \left[\frac{\partial}{\partial x} V'(x) + \frac{D}{\tau_a}\frac{\partial^2}{\partial x^2}\right]\rho(x,t), \tag{3}$$

where $V'(x) = -x + x^3/X_b^2 \mp A$ and the Fokker–Planck operator is $L_{FP} = \frac{\partial}{\partial x}V'(x) + \frac{D}{\tau_a}\frac{\partial^2}{\partial x^2}$ [66]. Here, $\rho(x,t)$ obeys the natural boundary condition that it vanishes at large x for any t. The steady-state solution of Eq. (3), for a permanent input at $+A$ or $-A$, is given by

$$\rho(x) = \lim_{t\to\infty} \rho(x,t) = C \exp\left[-\frac{\tau_a V(x)}{D}\right], \tag{4}$$

where C is the normalization constant [66]. This solution of Eq. (4) is for global equilibrium in the double-well potential.

Now, we will seek the non-stationary solution $\rho(x,t)$ of the Fokker–Planck equation, Eq. (3), in the case of an input transition from $s(t) = -A$ to $s(t) = +A$, or vice versa. This calculation is carried out in Appendix A. We show in Appendix A that the transition from the stationary density corresponding to $s(t) = -A$ to the stationary density corresponding to $s(t) = +A$, or vice versa, is dominated by an exponential temporal relaxation with a time constant $1/\lambda_1$. This allows us to deduce a response time $T_r = \tau_a/\lambda_1$ for the bistable system, which is a measure of the time taken by the system to switch from one potential well to the other, when the input changes from $\pm A$ to $\mp A$, in the presence of noise. In Appendix A, the non-stationary solution $\rho(x, t)$ for an input transition from $s(t) = -A$ to $s(t) = +A$ (or vice versa) is approximated with the two first terms from its asymptotic representation of Eq. (A.11), as

$$\rho(x, t|s(t) = \pm A) \simeq \rho(x|s(t) = \pm A) + [\rho(x|s(t) = \mp A) - \rho(x|s(t) = \pm A)] \exp(-t/T_r), \quad (5)$$

where $\rho(x|s(t) = \pm A)$ are the steady-state solutions of Eq. (4). In Eq. (5), when $t = 0$, the term $\exp(-t/T_r) = 1$ and $\rho(x, t|s(t) = \pm A)$ starts with the initial condition of $\rho(x|s(t) = \mp A)$. As $t \to +\infty$, the term $\exp(-t/T_r) = 0$, and $\rho(x, t|s(t) = \pm A)$ tends to the stationary condition of $\rho(x|s(t) = \pm A)$. This theoretical model, although approximate, nicely captures the double role played by the noise, both in accelerating the switching dynamics between wells and in enhancing the fluctuations inside the wells. With this non-stationary solution of Eq. (5), as we will see, the nonmonotonic evolution of the signal detection statistics of the bistable system can be analyzed theoretically.

2.2. *Signal detection statistics of the bistable system and the matched filter*

We are interested in detecting the input pulse signal $s(t)$ from the observation of the outputs of bistable system or matched filter. The input pulse signal $s(t)$ is emitted at a rate of one waveform every T_p and lasts over a pulse duration T_p. In this detection problem, we assume that the interval T_p is both known at the emitter and the detector. But, the waveform amplitudes $+A$ and $-A$ are not known at the detector. In order to detect $s(t) = +A$ or $s(t) = -A$, we sample the system output signals at equispaced times $t_j = jT_P$ for $j = 1, 2, \ldots$. The probability density $\rho(x, T_p|s(t) = \pm A)$ of Eq. (5) is then easily obtained for the bistable system. For the input bipolar pulse signals $s(t)$ mixed with additional Gaussian white noise $\eta(t)$, the impulse response of matched filter is $h(t) = s(T_p - t)$, where $h(t)$ is a scaled, time-reversed and shifted version of the pulse signal $s(t)$ $(0 < t \leq T_p)$. If the output of the matched filter is sampled at jT_p that maximizes the output signal-to-noise ratio, the probability density $\rho(x, T_p|s(t) = \pm A)$ has a Gaussian distribution

$$\rho(x, T_p|s(t) = \pm A) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(x \mp A)^2}{2\sigma^2}\right], \quad (6)$$

with $\sigma = \sqrt{A^2 T_P/2D}$ [49].

In this paper, the signal detection statistics of the bistable system and the matched filter are measured by the false alarm probability

$$P_{FA} = \int_{\ell}^{+\infty} \rho(x, T_p|s(t) = -A) dx, \quad (7)$$

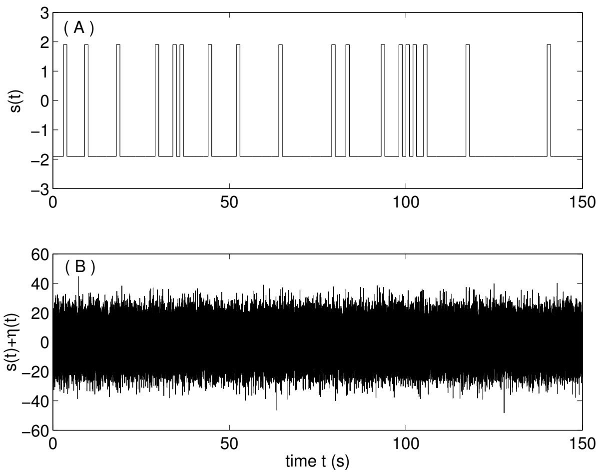

Fig. 1. Time evolution of the signals for the system of Eq. (2) with $\tau_a = 0.1$ s and $X_b = 5$ V. The sampling time step $\Delta t = 0.001$ s. (A) The input pulse signal $s(t)$ is with $A = 1.9$ V and $T_p = 1$ s. $s(t) = +A$ appears randomly in successive time intervals $[(n-1)T_p,\ nT_p]$ $(n = 1, 2, \ldots)$; (B) The mixture of signal plus noise ($D = 0.1$ V^2/Hz).

and the detection probability

$$P_D = \int_{\ell}^{+\infty} \rho(x, T_p | s(t) = +A) dx, \tag{8}$$

where ℓ is the detection threshold, and the positive pulse signal $s(t) = +A$, shown in Fig. 1, is considered as the information carrying part of the signal in this detection problem.

Figure 2 (A) compares the signal detection performance of the bistable system and the matched filter. As the noise intensity D increases from 0.1 V^2/Hz to 2.0 V^2/Hz, the signal detection statistics of the bistable system displays a non-monotonic behavior, i.e. the SR phenomenon in signal detection. The detectability of the bistable system is optimized at $D = 0.8$ V^2/Hz. The discrete points, illustrated in Fig. 2 (A), represent the corresponding numerical simulation results of the bistable system. Here, we numerically integrate the stochastic differential equation of Eq. (2) using a Euler-Maruyama discretization method with a small sampling time step $\Delta t \ll \tau_a$ [67]. The theoretical model of Eq. (5), although approximate, clearly describes the SR effect in the bistable system, and it agrees with the corresponding numerical results. Furthermore, the output signals of the bistable system and the matched filter are also given in Figs. 2 (B), (C), (D) and (E) for different noise intensities. The SR-type effect in the bistable system is also visible, as shown in Figs. 2 (B), (C) and (D).

In Fig. 2, the bistable system, even optimized at $D = 0.8$ V^2/Hz via the SR effect, can not outperform the matched filter for the case of Gaussian white noise. This fact has been pointed repeatedly by many researchers in a variety of investigations [30–32, 46, 50–52, 61]. However, we have assumed complete knowledge of the probability density functions under hypotheses $\boldsymbol{H}_0$ and $\boldsymbol{H}_1$, from which the con-

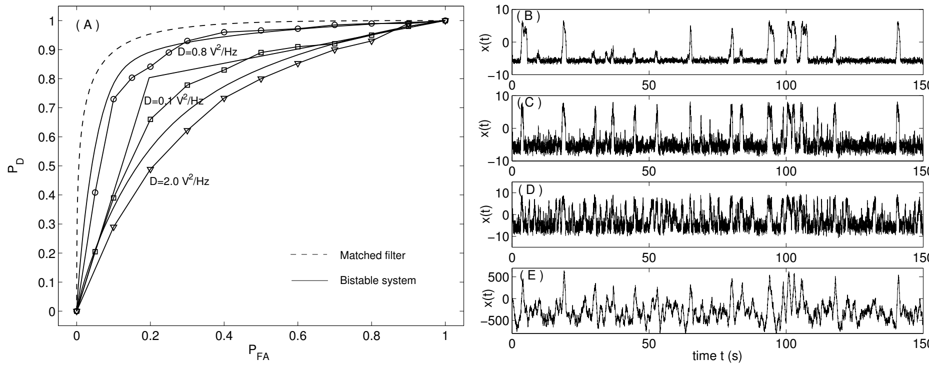

Fig. 2. (A) The detection statistics of the bistable system (solid lines) with $\tau_a = 0.1$ s and $X_b = 5$ V at $D = 0.8\ \mathrm{V}^2/\mathrm{Hz}$, $0.1\ \mathrm{V}^2/\mathrm{Hz}$ and $2.0\ \mathrm{V}^2/\mathrm{Hz}$ from the top down, and the matched filter (dashed line) at $D = 0.8\ \mathrm{V}^2/\mathrm{Hz}$. Numerical results of the bistable system are also presented at $D = 0.8\ \mathrm{V}^2/\mathrm{Hz}$ (circles), $D = 0.1\ \mathrm{V}^2/\mathrm{Hz}$ (squares) and $D = 2.0\ \mathrm{V}^2/\mathrm{Hz}$ (down triangles), respectively. The output signals $x(t)$ of the bistable system at (B) $D = 0.1\ \mathrm{V}^2/\mathrm{Hz}$, (C) $D = 0.8\ \mathrm{V}^2/\mathrm{Hz}$ and (D) $D = 2.0\ \mathrm{V}^2/\mathrm{Hz}$; (E) The output signal $x(t)$ of the matched filter at $D = 0.8\ \mathrm{V}^2/\mathrm{Hz}$. $\Delta t = 0.001$ s, $A = 1.9$ V and $T_p = 1$ s.

clusion of the bistable system being suboptimal by comparison with the matched filter is deduced. In next section, we will turn to more realistic problems in which some parameters of the signal are not completely known. New merits of the bistable system will be observed.

3. Evaluating the Bistable System Versus the Matched Filter in Unknown Detection Problems

In this section, we mainly study two signal detection problems: the first is detecting the pulse signal with disturbed pulse rates $1/T_p$; the second is a situation that the return pulse signal is delayed by the propagation time of the signal through the medium, resulting in the unknown arrival time t_0. Both the bistable system and the matched filter are employed as detectors in the two signal detection problems. The performances of the bistable system and the matched filter, with different measurements in different detection tasks, are compared thoroughly.

3.1. *The performance of the bistable system and the matched filter in detecting the pulse signal with disturbed pulse rates*

Consider the heuristic problem of detecting the received pulse signals with changed pulse rates (i.e. multi-rate pulse signals), for example, which may occur due to the desynchronization of devices during the detection of moving objects. For a given noisy environment, we can deduce the response time of the bistable system in terms of the signal amplitude A, system parameters τ_a and X_b (ref. Appendix A). If the disturbed pulse duration T_p is not too large (or small) with respect to the system response time T_r, it will be seen in the following that the bistable system can detect different pulse rates from a frequency spectrum analysis. It is well known that intersymbol errors will appear in this situation with a matched filter [49]. The

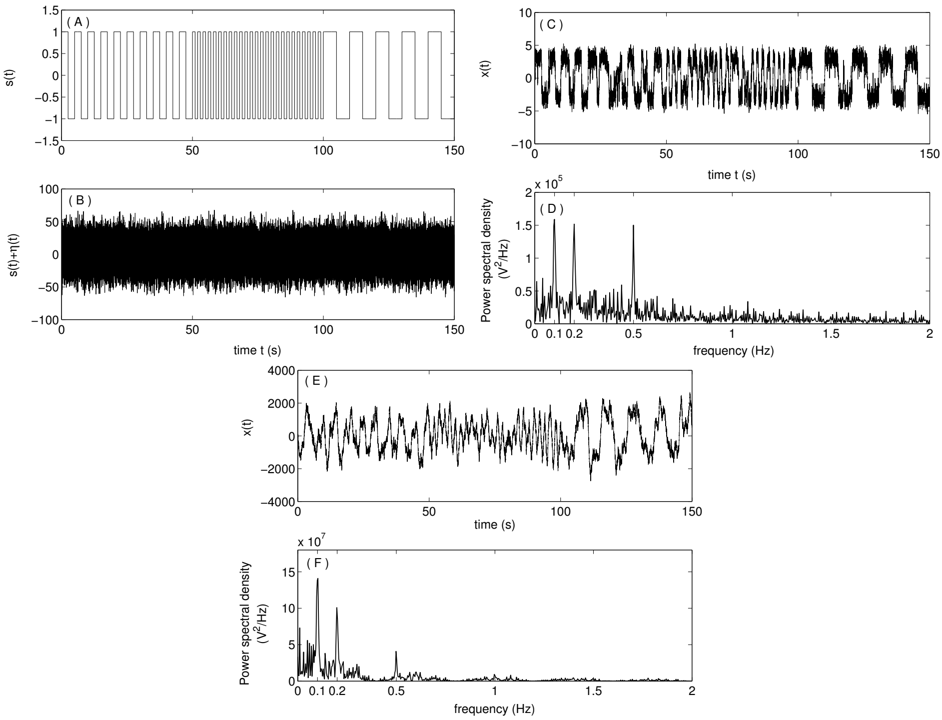

Fig. 3. (A) Input pulse signal $s(t) = \pm 1$ V with multi-rates $1/T_p = 0.2$ Hz, 0.5 Hz and 0.1 Hz; (B) The mixture of input $s(t)$ plus noise $\eta(t)$ ($D = 0.15$ V^2/Hz); (C) The output signal $x(t)$ of the bistable system with $\tau_a = 0.1$ s and $X_b = 2.8$ V; (D) The power spectrum of bistable system outputs; (E) The output signal $x(t)$ of the matched filter. The impulse response of $h(t)$ is selected in terms of the reference pulse rate of 0.2 Hz; (F) The power spectrum of matched filter outputs. $\Delta t = 0.001$ s.

frequency spectrum analysis of the matched filter output also shows that too many low frequency components degrade the matched filter's detectability.

In numerical simulations, the reference pulse rate is set as $1/T_p = 0.2$ Hz. Then, for the matched filter, the impulse response should be the "flipped around" version of the signal, i.e. $h(t) = s(5 - t)$, $t \in [0,\ 5]$ s. We assume that the received pulse rates of signal $s(t)$ is distributed as 0.5 Hz and 0.1 Hz. Figure 3 (A) shows the disturbed pulse signal $s(t)$ with three rates of 0.2 Hz, 0.5 Hz and 0.1 Hz. From the output signals of the bistable system and the matched filter, i.e. Figs. 3 (C) and (E), the corresponding power spectra are computed and plotted in Figs. 3 (D) and (F) respectively. The power spectrum of the bistable system output, as seen in Fig. 3 (D), clearly illustrates three pulse rates of input signal $s(t)$ at $1/T_p = 0.1$ Hz, 0.2 Hz and 0.5 Hz. However, the frequency component at 0.5 Hz, shown in the power spectrum of the matched filter output, is weaker than the spectrum lines in the low frequency domain $1/T_p < 0.1$ Hz, as seen in Fig. 3 (F). This indicates that the bistable system seems to suppress the noise at all frequency bands, but the matched filter, more like a lowpass characteristic, allows too much noise at low frequencies to interfere with its frequency discrimination.

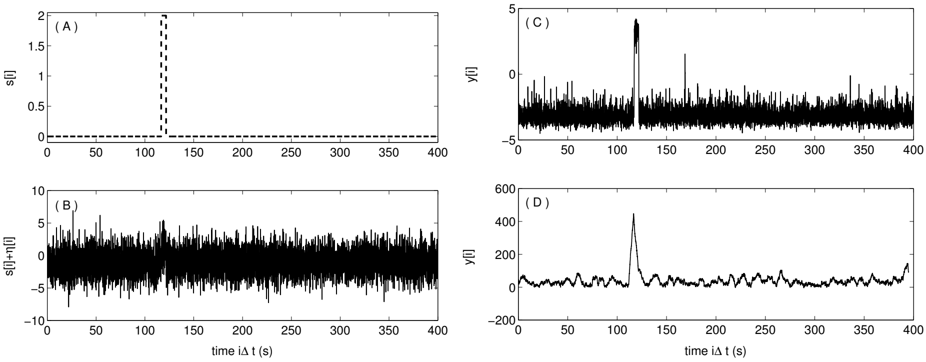

Fig. 4. (A) Input signal $s[i]$ is one pulse with the amplitude $2A = 2$ V and the pulse duration $T_p = 5$ s; (B) The mixture of the signal $s[i]$ plus noise $\eta[i]$ ($D = 0.08$ V^2/Hz); (C) The output signal of the subthreshold bistable system with $X_b = 2.8$ V and $\tau_a = 0.1$ s; (D) The convolution output signals computed by the matched filter, $y[i] = \sum_{i=\hat{i}_0}^{\hat{i}_0+M-1} h[i]x[i]$. $\Delta t = 0.05$ s.

In terms of the above comparisons, we argue that the bistable system is not so sensitive to the spread of the pulse rate values as the matched filter. This robust feature of the bistable system is not trivial in signal detection. Note that there exists an optimal pulse rate matching the bistable system for fixed signal amplitude and noise intensity [1, 2, 5], while the interest here is the system's robust property.

3.2. *The performance of the bistable system versus the matched filter in detecting the pulse signal with unknown arrival time*

In this subsection, we consider the detection problem of the received pulse signal with an unknown arrival time t_0. The hypotheses can be represented as

$$\begin{aligned} \boldsymbol{H}_0 &: x[i] = \eta[i], \\ \boldsymbol{H}_1 &: x[i] = s[i - i_0] + \eta[i], \end{aligned} \tag{9}$$

with $i = 0, 1, \ldots, N-1$. Here, the observation time interval is $[0, T]$ and $N = T/\Delta t$. The received pulse signal $s[i]$ is with only one pulse, as shown in Fig. 4 (A). The pulse amplitude is $2A$ over the interval $[i_0, i_0+M-1]$ ($M = T_p/\Delta t$), but the arrival time $t_0 = i_0\Delta t$ is unknown.

For the matched filter, the test statistic can be written as

$$H(y) = \max_{i\in[0,\, N-M+1]} \sum_{i=\hat{i}_0}^{\hat{i}_0+M-1} h[i]x[i] \geq \gamma, \tag{10}$$

where $h[i] = s[M-1-i]$ for $i \in [0,\, M-1]$ and the decision threshold $\gamma \leq \varepsilon = \sum_{i=0}^{M-1} s^2[i]$ [49]. The arrival time of input pulses is then $\hat{t}_0 = \hat{i}_0\Delta t + T_p(\varepsilon-\gamma)/\varepsilon$. In order to find the arrival time $\hat{t}_0$, the matched filter will compute $T/\Delta t$ convolutions in this detection problem. Figure 4 (D) gives an example of the convolution output signals of the matched filter.

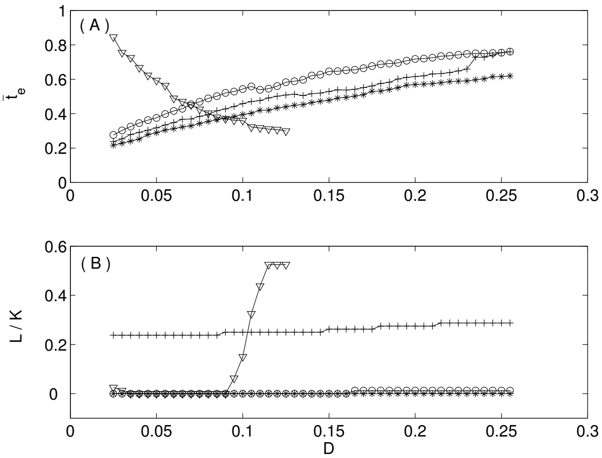

Fig. 5. (A) Average arrival time $\bar{t}_e$ versus D; (B) Loss ratio L/K versus D. For different values of D, each subject is simulated with $K = 80$ different arrival times t_0. $A = 1$ V, $T_p = 5$ s, $T = 400$ s and $\Delta t = 0.05$ s. The circle symbols represent the performance of the suprathreshold bistable system with $X_b = 1\ V < \sqrt{27}A/2$ and $\tau_a = 1$ s, while the down triangle symbols correspond to the subthreshold bistable system with $X_b = 2.8\ V > \sqrt{27}A/2$ and $\tau_a = 0.1$ s. The performance curves of the matched filter are plotted for the decision levels $\gamma = 200$ V^2 ($*$) and 400 V^2 (+), with $h(i) = s[99 - i]$, $i \in [0,\ 99]$ and $\varepsilon = 400$ V^2.

Considering the symmetric characteristics of bistable systems, a bias level $-A$ is added to the input $x[i]$, i.e., $x[i] = s[i - i_0] + n[i] - A$ or $x[i] = n[i] - A$. After the bistable system with output $y[i]$, illustrated in Fig. 4 (C), the test statistic can be expressed as

$$H(y) = y[\hat{i}_0] \geq x_+,\ \hat{i}_0 \in [0,\ N - M + 1], \tag{11}$$

where x_+ is the positive root of equation $x - x^3/X_b^2 + A = 0$ [22]. Here, we assume that the bistable system needs a time interval of T_r, i.e. the system response time introduced in Appendix A, to reach the corresponding steady-state x_+. The arrival time of the pulse signal is then approximated as $\hat{t}_0 = \hat{i}_0\Delta t - T_r$.

There are no general theories for evaluating performance of the bistable system and the matched filter in this detection problem [49]. In numerical simulations, the performance of both detectors is quantified by the average error time

$$\bar{t}_e = \sum_{j=1}^{K} \frac{|\ \hat{t}_0[j] - t_0\ |}{K}, \tag{12}$$

where K is the simulation time and $j = 1, 2, \ldots, K$. Here, $\hat{t}_0[j]$ is the j_{th} arrival time of the signal given by the detector. The matched filter or the bistable system, however, might detect a totally wrong arrival time $\hat{t}_0$, which is measured by the loss ratio L/K and the loss time L. If we take $|\ \hat{t}_0[m] - t_0\ | > T_p$ ($m = 1, 2, \cdots, L$) as one loss time, the average error time $\bar{t}_e$ can be rewritten as

$$\bar{t}_e = \sum_{j=1}^{K-L} \frac{|\ \hat{t}_0[j] - t_0\ |}{K - L}. \tag{13}$$

Moreover, we know the computation time $T_c[j]$ of the bistable system and matched filter in the $j_{\rm th}$ numerical simulation. Thus, the computational efficiency of both detectors can be quantified by the average computation time

$$\bar{T}_c = \sum_{j=1}^{K} \frac{T_c[j]}{K}. \tag{14}$$

The plots of Figs. 5 (A) and (B) show the performance of bistable systems and the matched filter in detecting the unknown arrival time of input signals, this is, the average error time $\bar{t}_e$ and the loss ratio L/K. It is seen that the matched filter's performance depends on selecting an appropriate decision threshold γ [49]. If the decision threshold $\gamma = 400$ V^2, it will present a higher loss ratio L/K than in bistable systems. The decision threshold $\gamma = 200$ V^2 makes the matched filter optimized. Two kinds of bistable systems are employed in this detection problem: one is subthreshold ($X_b > \sqrt{27}A/2$) and the other is suprathreshold ($X_b < \sqrt{27}A/2$) [7,22,68]. In Fig. 5, the average error time $\bar{t}_e$ and the loss ratio L/K of the suprathreshold bistable system is a monotonic increasing function of the noise intensity D. This suprathreshold bistable system is suboptimal compared to the matched filter with $\gamma = 200$ V^2, but outperforms the matched filter with $\gamma = 400$ V^2. For the subthreshold bistable system shown in Fig. 5, the average error time $\bar{t}_e$ decreases as the noise intensity D increases, and almost approaches the performance of the matched filter ($\gamma = 200$ V^2) before the loss ratio L/K begins to increase. This non-monotonic influence of the noise on the performance of the subthreshold bistable system is the signature of the SR phenomenon.

Moreover, an important merit of the bistable system is its computational efficiency. The average computation time of the bistable system, i.e. Eq. (14), is about $\bar{T}_c = 0.3361$ s for processing the data observed in the time interval $[0,\ T]$ ($T = 400$ s). The matched filter will use about $\bar{T}_c = 0.9048$ s to deal with the same data. This computational characteristic of the bistable system maybe meaningful in the optimization of speed and efficacy of information processing with limited resources for data handling, storage and energy supply [29, 32, 53].

4. Conclusion

We thoroughly evaluated the performance of the nonlinear bistable system versus the linear matched filter in pulse signal detection problems. Based on an approximate theory of the probability density model, the signal detection statistics of the bistable system were deduced theoretically. The detection performance of the nonlinear bistable system can be improved by adding an optimal amount of noise, i.e. the SR phenomenon, which was also demonstrated by numerical experiments. The theoretical and numerical results here lead to the fact, emphasized by many researchers [30–32, 46, 50–52, 61], that the bistable system is suboptimal to the matched filter for detecting the signal with complete known parameters. However, the bistable system is superior in detecting a pulse signal with unknown parameters. The power spectrum analysis demonstrated that the bistable system is more robust than the matched filter for discriminating the disturbed pulse rate. In the case of detecting the unknown arrival time of the pulse signal, the bistable system is also comparable to the matched filter. Furthermore, the detection scheme of the bistable system is more efficient with computational saving. These significant results verified the potential applicability of nonlinear bistable systems in the signal detection field, which might deserve further studies, for example, in exploiting other nonlinear systems exhibiting SR effects in practical detection problems.

Acknowledgments

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Appendix A. System Response Time and Non-stationary Probability Density Model

A.1. *System response time*

In Eq. (3), the Fokker–Planck operator $L_{FP} = \frac{\partial}{\partial x}V'(x) + \frac{D}{\tau_a}\frac{\partial^2}{\partial x^2}$ is not a Hermitian operator [66]. We rescale the variables as

$$\bar{X}_b = X_b/\sqrt{D/\tau_a},\ \bar{A} = A/\sqrt{D/\tau_a},\ \tau = t/\tau_a,\ y = x/\sqrt{D/\tau_a}, \tag{A.1}$$

Eq. (3) becomes

$$\frac{\partial \rho(y,\tau)}{\partial \tau} = \left[\frac{\partial}{\partial y}V'(y) + \frac{\partial^2}{\partial y^2}\right]\rho(y,\tau), \tag{A.2}$$

where $V'(y) = -y + y^3/\bar{X}_b^2 \mp \bar{A}$. The steady-state solution of Eq. (A.2) is given by

$$\rho(y) = \lim_{\tau\to\infty}\rho(y,\tau) = C\exp[-V(y)], \tag{A.3}$$

where C is the normalization constant. A separation ansatz for $\rho(y,\tau)$ [66],

$$\rho(y,\tau) = u(y)\exp\left[-\frac{V(y)}{2}\right]\exp(-\lambda\tau), \tag{A.4}$$

leads to

$$Lu = -\lambda u, \tag{A.5}$$

with a Hermitian operator $L = (\partial^2/\partial y^2) - [\frac{1}{4}V'^2(y) - \frac{1}{2}V''(y)]$. The functions $u(y)$ are eigenfunctions of the operator L with the eigenvalues λ. Multiplying both sides of Eq. (A.5) by $u(y)$ and integrating it, yields

$$\lambda = \frac{\int_{-\infty}^{+\infty}\{u'^2(y) + u^2(y)[\frac{1}{4}V'^2(y) - \frac{1}{2}V''(y)]\}dy}{\int_{-\infty}^{+\infty}u^2(y)dy}, \tag{A.6}$$

where eigenfunctions $u(y)$ satisfy the boundary conditions of $\lim_{y\to\pm\infty}u(y) = 0$ and $\lim_{y\to\pm\infty}u'(y) = 0$. The eigenvalue problem of Eq. (A.5) is then equivalent to the variational problem consisting in finding the extremal values of the right side of Eq. (A.6). The minimum of this expression is then the lowest eigenvalue $\lambda_0 = 0$, corresponding to the steady state solution of Eq. (A.3) [66]. Here, we adopt eigenfunctions $u(y) = p(y)\exp[-V(y)/2]$ and $p(y) \neq 0$, Eq. (A.6) then becomes

$$\lambda = \frac{\int_{-\infty}^{+\infty}\{p'^2(y) + \frac{1}{2}p^2(y)V'^2(y) - \frac{1}{2}[V'(y)p^2(y)]'\}\exp[-V(y)]dy}{\int_{-\infty}^{+\infty}p^2(y)\exp[-V(y)]dy}. \tag{A.7}$$

Since

$$\int_{-\infty}^{+\infty} [V'(y)p^2(y)]' \exp[-V(y)]dy = V'(y)p^2(y)\exp[-V(y)]|_{-\infty}^{+\infty} + \int_{-\infty}^{+\infty} p^2(y)V'^2(y)\exp[-V(y)]dy = \int_{-\infty}^{+\infty} p^2(y)V'^2(y)\exp[-V(y)]dy,$$

Eq. (A.7) can be rewritten as

$$\lambda = \frac{\int_{-\infty}^{+\infty} p'^2(y)\exp[-V(y)]dy}{\int_{-\infty}^{+\infty} p^2(y)\exp[-V(y)]dy}. \tag{A.8}$$

Assume $p(y) = d_0 + d_1 y + \cdots + d_n y^n$ and the order n is an integer, we obtain

$$([K] - \lambda[M])\{d\} = 0, \tag{A.9}$$

with eigenvectors $\{d^i\} = [d_0^i, d_1^i, \ldots, d_n^i]$ corresponding to eigenvalues λ_i for $i = 0, 1, \ldots, n$. The elements of matrices $[M]$ and $[K]$ are

$$m_{ij} = \int_{-\infty}^{+\infty} y^{i+j}\exp[-V(y)]dy > 0,$$

$$k_{ij} = \int_{-\infty}^{+\infty} ijy^{i+j-2}\exp[-V(y)]dy \geq 0,$$

where i, $j = 0, 1, \ldots, n$. The matrix $[M]$ is positive definite and the matrix $[K]$ is semi-positive definite. The minimal eigenvalue λ_0 is zero. The inverse of minimal positive eigenvalue λ_1 describes the main time of the system tending to the steady state solution of Eq. (A.3), what we call the system response time. Note the time scale transformation in Eq. (A.1), the minimal positive eigenvalue should be λ_1/τ_a and the real system response time is $T_r = \tau_a/\lambda_1$.

A.2. *Non-stationary probability density model*

From Eq. (A.9), we can obtain the eigenfunctions $u_i(y) = p_i(y)\exp[-V(y)/2]$ corresponding to the eigenvalue λ_i for $i = 0, 1, \ldots, n$, where $p_i(y) = d_0^i + d_1^i y + \cdots + d_n^i y^n$. The eigenvectors $\{d^i\} = [d_0^i, d_1^i, \ldots, d_n^i]$ are normalized. Because L is a Hermitian operator, eigenfunctions $u_i(y)$ and $u_j(y)$ are orthogonal

$$\int_{-\infty}^{+\infty} u_i(y)u_j(y)dy = \delta_{ij}, \tag{A.10}$$

where $i, j = 0, 1, \ldots, n$. $\rho(y, \tau)$ can be expanded, according to eigenfunctions $u_i(y)$ and eigenvalues λ_i, as

$$\rho(y, \tau) = \sum_{i=0}^{n} C_i u_i(y)\exp\left[-\frac{V(y)}{2}\right]\exp[-\lambda_i \tau], \tag{A.11}$$

where C_i are normalization constants. In this paper, we take an approximate expression of $\rho(y, \tau) \simeq \sum_{i=0}^{1} C_i u_i(y)\exp[-V(y)/2]\exp[-\lambda_i \tau]$ instead of Eq. (A.11).

Then, if the preceding input signal is $s(t) = \mp A$ and the next one is $s(t) = \pm A$, a simple non-stationary probability density model is derived as

$$\rho(x,t|s(t) = \pm A) \simeq \rho(x|s(t) = \pm A)+[\rho(x|s(t) = \mp A)-\rho(x|s(t) = \pm A)]\exp(-t/T_r), \tag{A.12}$$

with the initial condition $\rho(x, t = 0|s(t) = \pm A) = \rho(x|s(t) = \mp A)$ and the stationary condition $\rho(x, t = +\infty|s(t) = \pm A) = \rho(x|s(t) = \pm A)$, respectively. Here, $\rho(x|s(t) = \pm A)$ are the steady-state solutions given in Eq. (4), and $V(x) = -x^2/2+x^4/(4X_b^2)\mp Ax$ correspond to the constant inputs $s(t) = \pm A$ respectively. This kind of non-stationary solution of $\rho(x, t|s(t) = \pm A)$ can be further developed into the case of $\rho(y, \tau) \simeq \sum_{i=0}^{n} C_i u_i(y) \exp[-V(y)/2] \exp[-\lambda_i \tau]$ for $n \geq 2$.

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