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A CHARACTERIZATION OF SUPRATHRESHOLD STOCHASTIC RESONANCE IN AN ARRAY OF COMPARATORS BY CORRELATION COEFFICIENT

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Suprathreshold Stochastic Resonance (SSR), as described recently by Stocks, is a new form of Stochastic Resonance (SR) which occurs in arrays of nonlinear elements subject to aperiodic input signals and noise. These array elements can be threshold devices or FitzHugh-Nagumo neuron models for example. The distinguishing feature of SSR is that the output measure of interest is not maximized simply for nonzero values of input noise, but is maximized for nonzero values of the input noise to signal intensity ratio, and the effect occurs for signals of arbitrary magnitude and not just subthreshold signals. The original papers described SSR in terms of information theory. Previous work on SR has used correlation based measures to quantify SR for aperiodic input signals. Here, we argue the validity of correlation based measures and derive exact expressions for the cross-correlation coefficient in the same system as the original work, and show that the SSR effect also occurs in this alternative measure. If the output signal is thought of as a digital estimate of the input signal, then the output noise can be considered simply as quantization noise. We therefore derive an expression for the output signal to quantization noise ratio, and show that SSR also occurs in this measure.

Keywords: Stochastic resonance; suprathreshold stochastic resonance; correlation coefficient; neuron; quantization noise.

1. Introduction

Stochastic Resonance (SR) [1–5] occurs when the presence of noise in a nonlinear system can induce an optimal output from that system, and has been observed in a diverse range of physical and biological systems, including neurons and neuron models [6–12]. Since the original paper by Benzi *et al* [13], SR has usually been loosely defined as occurring when an increase in input noise leads to an increase in output signal-to-noise ratio (SNR), in a nonlinear system driven by a periodic force.

An important extension of SR was first addressed in detail in 1995 by Collins *et al* [14]; that of Aperiodic Stochastic Resonance (ASR). Until then, almost all studies on SR assumed periodic input signals. Collins *et al* instead considered an excitable system (a FitzHugh-Nagumo neuron model) subject to an aperiodic signal. They proposed a power-norm measure (a measure based on cross-correlation coefficient) to characterize ASR, instead of the signal to noise ratio (which is inappropriate for aperiodic signals). A more powerful measure is based on cross-correlation spectra [15] which is a frequency dependent generalization of the cross-correlation idea. However, for the sake of simplicity, we use only the cross-correlation coefficient in the present study.

Recently, a series of papers by Stocks has brought to light a new form of stochastic resonance, called Suprathreshold Stochastic Resonance (SSR) [16–20]. This occurs in an array of comparators (threshold devices) subject to the same input signal, but independent noise, where the output from each device is summed to give an overall output. It has been shown, using information theory, how stochastic resonance can occur in such a system, for signals of *arbitrary magnitude*. The measure used is *transmitted information*, which under certain conditions is maximized for nonzero values of the ratio of input noise to input signal intensity. Such a system is of interest, since it crosses the border between biology and engineering. The array resembles the summing of neuron outputs in the brain, as well as a DIMUS sonar array [21, 22], and a flash analogue to digital converter [23]. It is worth noting that SSR has also been shown to occur in more complex arrays of FitzHugh-Nagumo neuron models [20].

In this paper we analyze the same array of comparators as that in the papers on SSR, but from the perspective of cross-correlation coefficient. There are many SR papers that use cross-correlation coefficient [14, 24–29], but none that we are aware of that show the existence of SSR. We also derive a formula for the output signal to noise ratio, by considering the output signal to be a digital estimate of the input signal, so that the output noise is simply the quantization noise.

2. Why Use Correlation Coefficient?

A number of authors have argued against the use of correlation coefficient as a measure for studying SR. These kinds of criticisms can be avoided by the use of cross spectra [15]. However, when only the SNR based on the ratio of total signal power to total noise power independent of frequency is required, the correlation coefficient can correctly be used provided the system does not have a phase shift. Correlation coefficient is unity when two signals are linearly related to each other and thus can be thought of as a measure of linearity. It therefore may seem strange to use this measure on a thresholding channel, which is a nonlinear system to begin with. However, all this nonlinearity implies is that the correlation coefficient can never be unity; provided we interpret the results with this caveat in mind, correlation coefficient is a valid measure of how similar the output is to the input of our information channel.

For a linear system, both the transmitted information and correlation coefficient infers high information content. For a nonlinear system, we can no longer simply infer information from correlation coefficient. So provided our results are interpreted simply in terms of how closely the output matches the input signal, and we do not place any interpretation about information content, then the use of correlation coefficient is perfectly valid.

One advantage we find here for using correlation coefficient is that an exact analytical expression can be found for the correlation coefficient under more general conditions than any exact expression obtained for the transmitted information [18]. Exact expressions for the correlation coefficient in parallel arrays of devices have previously been derived using linear response theory, both theoretically [25], and specifically for a parallel array of threshold devices [26]. However in the latter case, the signal was always subthreshold, and the noise was not independent in each device.

3. Model Description

Consider an array of N comparators, subject to the same continuously valued input signal, x, as shown in Fig. 1. The i–th, $(i = 1, .., N)$ device is subject to independent continuously valued additive noise, η_i. The output from each comparator, y_i, is 1 if the sum of the input signal and the noise is greater than the threshold θ_i of that comparator and 0 otherwise. The outputs from the comparators are summed to give the output signal y. Hence y is a discrete signal taking on integer values from 0 to N and can be considered as the number of comparators that are currently "on."

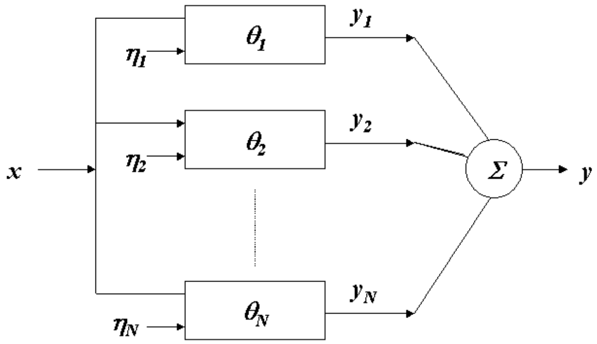

Fig. 1. Array of N summing comparators.

Thus, the output of comparator i is given by

$$y_i = \begin{cases} 1 & \text{if } x + \eta_i > \theta_i, \\ 0 & \text{otherwise,} \end{cases}$$

and the output of the array is $y = \sum_{i=1}^{N} y_i$.

If the output signal $y \in \{0, 1, .., N\}$ is normalized so that it takes on values between $-c$ and c, it becomes a digital approximation to the input signal. We will call this normalized signal $\hat{y}$ so that

$$\hat{y} = c\left(\frac{2y}{N} - 1\right) = \frac{c}{N}\sum_{i=1}^{N} \text{sign}(x + \eta_i)\,, \tag{1}$$

where

$$\text{sign}(x) = \begin{cases} 1 & \text{if } x > 0, \\ -1 & \text{if } x < 0. \end{cases}$$

We make a number of assumptions about the signal and noise distributions, and the threshold settings in the analysis that follows. These are that

- both the input signal, x, and the noise, η_i, are random variables that are both strictly stationary and ergodic;
- the noise in all N comparators is independent and identically distributed (iid);
- the signal and the noise have identical distributions with known variances;
- the signal and noise have even probability density functions with zero mean;
- all thresholds are set equal to the signal mean, that is, zero.

We denote the expected value, or mean, of a random variable as $\mathrm{E}[\cdot]$, and the variance of a random variable as $\mathrm{var}[\cdot]$. Hence, from its definition [30], the correlation coefficient of x and $\hat{y}$ is:

$$\rho_{x,\hat{y}} = \frac{\mathrm{cov}[x,\hat{y}]}{\sqrt{\mathrm{var}[x]\mathrm{var}[\hat{y}]}} = \frac{\mathrm{E}[x\hat{y}] - \mathrm{E}[x]\mathrm{E}[\hat{y}]}{\sqrt{\mathrm{var}[x]\mathrm{var}[\hat{y}]}}$$
$$= \frac{\mathrm{E}[x\hat{y}]}{\sqrt{\mathrm{var}[x]\mathrm{var}[\hat{y}]}} \tag{2}$$

since we assume $\mathrm{E}[x] = 0$.

Hence, to derive an expression for the correlation coefficient, we first need to obtain expressions for the cross-correlation of x and $\hat{y}$, $\mathrm{E}[x\hat{y}]$, and of the variance of $\hat{y}$, $\mathrm{var}[\hat{y}] = \mathrm{E}[\hat{y}^2] - \mathrm{E}[\hat{y}]^2$.

4. Expressions for Correlation Coefficient

We denote the probability that any given comparator is "on" (that is, the sum of the signal and noise is greater than zero), given knowledge of the signal, x, as $P_{1|x}$. Since we assume that the threshold of each comparator is zero and the noise distribution at each comparator is identical, $P_{1|x}$ is the same for each comparator. Hence (simplifying the notation by dropping the subscript from η) we can write

$$P_{1|x} = \text{Prob}(x + \eta \geq 0|x) = \int_{-x}^{\infty} R(\eta)d\eta, \tag{3}$$

where $R(\eta)$ is the probability density function of the noise.

Let $P(x)$ be the probability density function of the input signal, x. It is shown in the appendix that the mean square value of $\hat{y}$ is

$$\mathrm{E}[\hat{y}^2] = \frac{c^2}{N}\left(1 + (N-1)\left(4\mathrm{E}[P_{1|x}^2] - 1\right)\right), \tag{4}$$

and the cross-correlation of x and $\hat{y}$ is

$$\mathrm{E}[x\hat{y}] = 2c\int_{-\infty}^{\infty} xP(x)P_{1|x}dx\,. \tag{5}$$

Derivation of the correlation coefficient requires finding expressions for $\mathrm{E}[\hat{y}^2]$ and $\mathrm{E}[x\hat{y}]$. $\mathrm{E}[\hat{y}^2]$ is dependant on $\mathrm{E}[P_{1|x}^2]$, and $\mathrm{E}[x\hat{y}]$ is dependent on $P_{1|x}$. Given $R(\eta)$, $P_{1|x}$ can be derived from (3) and $\mathrm{E}[P_{1|x}^2]$ can be derived using $P_{1|x}$ and $P(x)$. Hence, knowledge of the probability density functions of the signal and noise are sufficient to allow us to derive expressions for $\rho_{x,\hat{y}}$.

We are able to obtain analytical expressions for $\rho_{x,\hat{y}}$ for both uniform and Gaussian distributions. It is convenient to express these formulas for $\rho_{x,\hat{y}}$ in terms of the ratio of the noise standard deviation to the signal standard deviation, which we denote as σ.

4.1. *Uniformly distributed signal and noise*

If the input signal, x is uniformly distributed between $-\sigma_p/2$ and $\sigma_p/2$, with zero mean, then

$$P(x) = \begin{cases} 1/\sigma_p & \text{for } -\sigma_p/2 \le x \le \sigma_p/2, \\ 0 & \text{otherwise.} \end{cases}$$

If the independent noise η in each device is uniformly distributed between $-\sigma_r/2$ and $\sigma_r/2$, with zero mean, then

$$R(\eta) = \begin{cases} 1/\sigma_r & \text{for } -\sigma_r/2 \le \eta \le \sigma_r/2, \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

The signal variance is $\sigma_p^2/12$ and the noise variance is $\sigma_r^2/12$. Hence $\sigma = \sigma_r/\sigma_p$. Substituting (6) into (3) gives

$$P_{1|x} = \begin{cases} 0 & \text{for } x < -\sigma_r/2, \\ x/\sigma_r + 1/2 & \text{for } -\sigma_r/2 \le x \le \sigma_r/2, \\ 1 & \text{for } x > \sigma_r/2. \end{cases} \tag{7}$$

From (7), $P_{1|x}$ is 0 for $x \le -\sigma_r/2$, a function of x for $-\sigma_r/2 \le x \le \sigma_r/2$ and 1 for $x \ge \sigma_r/2$, and $\rho_{x,\hat{y}}$ depends on whether σ_p is less than or greater than σ_r. Therefore we require separate derivations of $\rho_{x,\hat{y}}$ for σ less than 1 and greater than 1. When $\sigma = 1$, both cases give the same $\rho_{x,\hat{y}}$.

As given by (A.8), the correlation coefficient is

$$\rho_{x,\hat{y}} = \begin{cases} \frac{\sqrt{N}(3-\sigma^2)}{2\sqrt{\sigma(2-2N)+3N}} & (\sigma \le 1), \\ \frac{\sqrt{N}}{\sqrt{3\sigma^2+N-1}} & (\sigma \ge 1). \end{cases} \tag{8}$$

4.2. *Gaussian signal and noise*

If the input signal has a Gaussian distribution with zero mean and variance σ_p^2, then

$$P(x) = \frac{1}{\sqrt{2\pi\sigma_p^2}} \exp\left(-\frac{x^2}{2\sigma_p^2}\right).$$

If the independent noise in each device is Gaussian with zero mean and variance σ_r^2, then

$$R(\eta) = \frac{1}{\sqrt{2\pi\sigma_r^2}} \exp\left(-\frac{\eta^2}{2\sigma_r^2}\right). \tag{9}$$

Substituting (9) into (3) gives

$$P_{1|x} = \frac{1}{2} + \frac{1}{2}\mathrm{erf}\left(\frac{x}{\sqrt{2}\sigma_r}\right), \tag{10}$$

where erf is the error function [31].

As derived in the appendix, the input-output cross-correlation is given by

$$\mathrm{E}[x\hat{y}] = c\sigma_p\sqrt{\frac{2}{\pi(1+\sigma^2)}}, \tag{11}$$

and the output mean squared value (or auto-correlation) is

$$\mathrm{E}[\hat{y}^2] = \frac{c^2}{N}\left(1 + \frac{2(N-1)}{\pi}\arcsin\left(\frac{1}{\sigma^2+1}\right)\right). \tag{12}$$

As given by (A.11), the correlation coefficient is

$$\rho_{x,\hat{y}} = \sqrt{\frac{2N}{\pi(1+\sigma^2)}} \Bigg/ \sqrt{1 + \frac{2(N-1)}{\pi}\arcsin\left(\frac{1}{\sigma^2+1}\right)}. \tag{13}$$

5. Analysis of Correlation Coefficient

In all cases, the input-output cross-correlation, $\mathrm{E}[x\hat{y}]$ is independent of N (due to the normalization) and the correlation coefficient is independent of c. We present our results in the form of plots of the correlation coefficient against the input signal to noise ratio (SNR), in decibels ($\mathrm{SNR} = 10\log_{10}(\sigma_p^2/\sigma_r^2) = -20\log_{10}(\sigma)$). For $N > 1$, these plots clearly show the existence of a peak in the correlation coefficient at a non-zero noise intensity, hence indicating an SSR effect.

5.1. *Uniform signal and noise*

In the case of uniform signal and noise, for $\sigma \leq 1$ it is straightforward to show that $\rho_{x,\hat{y}}$ is maximized for $\sigma = (N - \sqrt{2N-1})/(N-1)$, and that for $\sigma \geq 1$, $\rho_{x,\hat{y}}$ is strictly decreasing. Hence there is a maximum in the correlation coefficient for a nonzero value of σ which approaches unity as N becomes large. When $\sigma = 0$, $\rho_{x,\hat{y}} = \sqrt{3}/2$. This behavior is shown in Fig. 2, where the correlation coefficient is plotted against the input SNR.

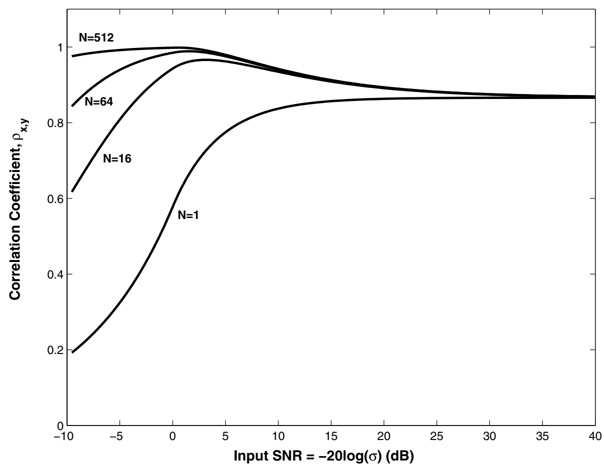

Fig. 2. Plot of Correlation Coefficient, $\rho_{x,\hat{y}}$ against the input SNR for various values of N and uniform signal and noise.

5.2. *Gaussian signal and noise*

Figure 3 shows how the correlation coefficient varies with the input SNR. Note that when $\sigma = 0$, $\rho_{x,\hat{y}} = \sqrt{2/\pi}$. For $N = 1$, $\rho_{x,\hat{y}}$ is strictly decreasing, and has a maximum value at $\sigma = 0$ (Input SNR $= \infty$) of $\sqrt{2/\pi}$. For $N > 1$, the maximum value of $\rho_{x,\hat{y}}$ occurs for a non zero value of noise, i.e. a finite input SNR. As N becomes very large, the maximum value of $\rho_{x,\hat{y}}$ approaches, but never reaches unity.

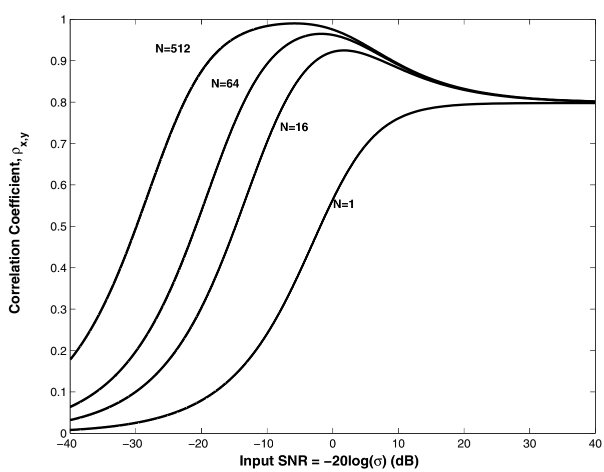

Fig. 3. Plot of Correlation Coefficient, $\rho_{x,\hat{y}}$ against the input SNR for various values of N and Gaussian signal and noise.

6. Signal to Noise Ratio

Our results relate closely to work done in the 1950s and 1960s by Baum [32] and Remley [33] related to DIMUS. Remley showed that the output mean square value of a DIMUS array (where the output values are between $-N$ and N) subject to Gaussian signal and noise is given by

$$R_o = N + \frac{2N(N-1)}{\pi} \arcsin\left(\frac{R_{\text{in}}}{1+R_{\text{in}}}\right) \tag{14}$$

where R_{in} is the input signal to noise ratio at a single input hydrophone. By letting $c = N$, and noting that $R_{\text{in}} = 1/\sigma^2$, then the expression derived in this paper for the output mean square value (Eq. (12)) becomes identical to (14). Remley derived his result in a different manner to that used here, from prior work by Van Vleck *et al* [34].

When there is no signal present, then $R_{\text{in}} = 0$ and from (14) $R_o = N$ and from (12), $\mathrm{E}[\hat{y}^2] = c^2/N$. If the output signal to noise ratio is defined as the increase in the output power owing to the arrival of a signal, divided by the output power when only noise is present [33], then the output signal to noise ratio is

$$\begin{aligned}\mathrm{SNR}_1 &= \frac{2(N-1)}{\pi} \arcsin\left(\frac{R_{\text{in}}}{1+R_{\text{in}}}\right)\\ &= \frac{2(N-1)}{\pi} \arcsin\left(\frac{1}{\sigma^2+1}\right).\end{aligned} \tag{15}$$

Note that when there is no signal, $\sigma = \infty$ and $\mathrm{SNR}_1 = 0$. When there is no noise, $\sigma = 0$ and $\mathrm{SNR}_1 = N-1$. When N is large and σ is large, then SNR_1 can be approximated as

$$\mathrm{SNR}_1 \simeq \frac{2N}{\pi\sigma^2}\,.$$

These formulae for SNR_1 are strictly decreasing functions in σ. Hence, although there is a maximum in the correlation coefficient for nonzero noise, the SNR_1 is maximized for no noise.

Stocks also derived a formula for the SNR [19]. The formula he obtained (using the notation used in this paper) can be expressed in terms of $P_{1|x}$. Rewriting his Eqs. (14) and (15) we get

$$\mathrm{SNR}_2 = \frac{N(\mathrm{E}[P_{1|x}^2] - \frac{1}{4})}{\frac{1}{2} - \mathrm{E}[P_{1|x}^2]}\,,$$

which for Gaussian signal and noise gives

$$\begin{aligned}\mathrm{SNR}_2 &= N \arcsin\left(\frac{1}{\sigma^2+1}\right)\left(\frac{\pi}{2} - \arcsin\left(\frac{1}{\sigma^2+1}\right)\right)\\ &= N \arcsin\left(\frac{1}{\sigma^2+1}\right)\Big/ \arccos\left(\frac{1}{\sigma^2+1}\right).\end{aligned} \tag{16}$$

Although this expression is also strictly decreasing, it is not in agreement with (15), in particular, in (16), if $\sigma = 0$, $\mathrm{SNR}_2 = \infty$. The difference is due to the definition used by Stocks, where he considered the output noise to approach zero as N approaches infinity. However, when N is large and σ is also large, then SNR_2 can be approximated from (16) as

$$\mathrm{SNR}_2 \simeq \frac{2N}{\pi\sigma^2}$$

which is identical to SNR_1 under these conditions.

In communication engineering, the signal to noise ratio of an Analog to Digital Converter (ADC) is usually defined as follows [35,36]. Given an analog input signal, x, and an output signal from an ADC, $\hat{y}$, the quantization error or noise is defined as $q_e = \hat{y} - x$. Hence, the power of the input signal is $\mathrm{E}[x^2]$ and the power of the quantization noise is $\mathrm{E}[q_e^2]$.

Therefore the Signal to Quantization Noise Ratio (SQNR) is

$$\begin{aligned} \mathrm{SQNR} &= \frac{\mathrm{E}[x^2]}{\mathrm{E}[q_e^2]} = \frac{\mathrm{E}[x^2]}{\mathrm{E}[(\hat{y}-x)^2]} \\ &= \frac{\mathrm{E}[x^2]}{\mathrm{E}[\hat{y}^2] - 2\mathrm{E}[x\hat{y}] + \mathrm{E}[x^2]}\,. \end{aligned} \tag{17}$$

For Gaussian signal and noise, we can combine (17) with (11) and (12) to get

$$\mathrm{SQNR} = \frac{N\sigma_p^2}{c^2\left(1 + \frac{2(N-1)}{\pi}\arcsin\left(\frac{1}{\sigma^2+1}\right)\right) - 2Nc\sigma_p\sqrt{\frac{2}{\pi(1+\sigma^2)}} + N\sigma_p^2}\,. \tag{18}$$

Since the input is a Gaussian random signal, if we let $c = \sigma_p$ then 68.27% of the input values fall between $\pm c$ and if $c = 3\sigma_p$, then 99.73% of input values fall between $\pm c$. Since we would ideally like the output signal to be a digital estimate of the input, one would expect that as large as possible a percentage of the input signal should be realized in the output. However, the resolution of the output is $2c/N$, and if c is made too large, the output signal becomes more distorted, unless N approaches infinity. Therefore, there is a tradeoff required between the output range and the output resolution. We would expect that the optimum value of output SQNR is obtained for an intermediate value of c. As N is increased, then c can be increased to obtain higher SQNR values.

Figure 4 shows the SQNR plotted against the input SNR for $c = 3\sigma_p$. It can be seen that the maximum value of SQNR occurs for a finite value of input SNR for all cases except $N = 1$, which strictly decreases from it's maximum at $\sigma = 0$.

7. Conclusions

In this paper we have shown that the Suprathreshold Stochastic Resonance effect present in an array of comparators subject to noise can be expressed in alternative measures to that of transmitted information. Firstly, we showed the existence of SSR in the input-output cross-correlation coefficient. Secondly, using the definition of SNR associated with an ADC, we have shown that an SSR effect occurs in the output SQNR. These results have the advantage of being exact, and show that the SSR effect is not dependent simply on the measure being used.

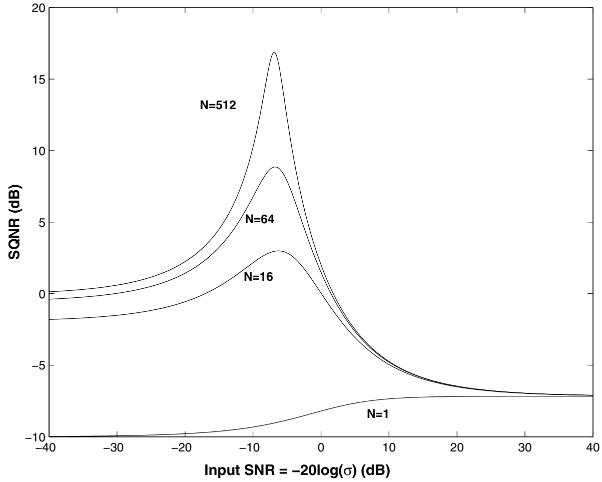

Fig. 4. Plot of SQNR in dB against the input SNR for various values of N and Gaussian signal and noise with $c = 3\sigma_p$.

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Appendix A. Derivation of Formulae

From (3), since $R(\eta)$ is even with mean zero we can write

$$P_{1|x} = \int_{-\infty}^{x} R(\eta)d\eta = \frac{1}{2} + \int_{0}^{x} R(\eta)d\eta\,.$$

Therefore, the expected value of $P_{1|x}$ over the signal distribution is

$$\begin{aligned} \mathrm{E}[P_{1|x}] &= \frac{1}{2} + \mathrm{E}\left[\int_0^x R(\eta)d\eta\right] \\ &= \frac{1}{2} + \int_{-\infty}^{\infty}\left(\int_0^x R(\eta)d\eta\right)P(x)dx\,. \end{aligned}$$

Since $R(\eta)$ is even, $\int_0^x R(\eta)d\eta$ is odd and therefore the integral above is zero, as $P(x)$ is even. Thus, $\mathrm{E}[P_{1|x}] = 0.5$.

From (1), the expected value of $\hat{y}$ given x is

$$\begin{aligned} \mathrm{E}[\hat{y}|x] &= \frac{c}{N}\mathrm{E}\left[\sum_{i=1}^{N} \mathrm{sign}(x+\eta_i) \mid x\right] \\ &= c\mathrm{E}[\mathrm{sign}(x+\eta)|x]\,, \end{aligned}$$

(since all η are independent and identically distributed)

$$= c(-1(1 - P_{1|x}) + 1P_{1|x})$$
$$= c(2P_{1|x} - 1)\,. \tag{A.1}$$

Therefore the expected value of $\hat{y}$ is

$$\mathrm{E}[\hat{y}] = \mathrm{E}[\mathrm{E}[\hat{y}|x]] = 2c\mathrm{E}[P_{1|x}] - c = 0\,.$$

The mean square value of $\hat{y}$ given x is

$$\begin{aligned}
\mathrm{E}[\hat{y}^2|x] &= \frac{c^2}{N^2}\mathrm{E}\left[\left(\sum_{i=1}^{N}\mathrm{sign}(x+\eta_i)\right)^2 \mid x\right] \\
&= \frac{c^2}{N^2}\Big(N\mathrm{E}[(\mathrm{sign}(x+\eta))^2 \mid x] + N(N-1)\mathrm{E}[\mathrm{sign}(x+\eta_i)\mathrm{sign}(x+\eta_j) \mid x]\Big) \\
&= \frac{c^2}{N^2}\Big(N + N(N-1)(-2P_{1|x}(1-P_{1|x}) + (1-P_{1|x})^2 + P_{1|x}^2)\Big) \\
&= \frac{c^2}{N^2}\left(N + N(N-1)(2P_{1|x}-1)^2)\right).
\end{aligned}$$

Therefore the mean square value of $\hat{y}$ is

$$\begin{aligned}
\mathrm{E}[\hat{y}^2] = \mathrm{E}[\mathrm{E}[\hat{y}^2|x]] &= \frac{c^2}{N} + \frac{c^2(N-1)}{N}\mathrm{E}[(2P_{1|x}-1)^2] \\
&= \frac{c^2}{N}\Big(1 + (N-1)\Big(4\mathrm{E}[P_{1|x}^2] - 4\mathrm{E}[P_{1|x}] + 1\Big)\Big) \qquad \text{(A.2)} \\
&= \frac{c^2}{N}\Big(1 + (N-1)\Big(4\mathrm{E}[P_{1|x}^2] - 1\Big)\Big).
\end{aligned}$$

The correlation of x and $\hat{y}$ is

$$\begin{aligned}
\mathrm{E}[x\hat{y}] = \mathrm{E}[\mathrm{E}[x\hat{y}|x]] &= \mathrm{E}[x\mathrm{E}[\hat{y}|x]] \\
&= \int_{-\infty}^{\infty} x\mathrm{E}[\hat{y}|x]P(x)dx \\
&= \int_{-\infty}^{\infty} xP(x)c(2P_{1|x}-1)dx \\
&= 2c\int_{-\infty}^{\infty} xP(x)P_{1|x}dx - c\int_{-\infty}^{\infty} xP(x)dx \qquad \text{(A.3)} \\
&= 2c\int_{-\infty}^{\infty} xP(x)P_{1|x}dx - c\mathrm{E}[x] \\
&= 2c\int_{-\infty}^{\infty} xP(x)P_{1|x}dx\,.
\end{aligned}$$

A.1. *Uniformly distributed signal and noise*

A.1.1. *Uniform signal and noise with $\sigma \leq 1$*

We have

$$\mathrm{E}[P_{1|x}^2] = \int_{-\infty}^{\infty} P_{1|x}^2 P(x) dx = \frac{1}{\sigma_p} \int_{-\sigma_r/2}^{\sigma_r/2} \left(\frac{1}{2} + \frac{x}{\sigma_r}\right)^2 dx + \frac{1}{\sigma_p} \int_{\sigma_r/2}^{\sigma_p/2} dx = \frac{1}{2} - \frac{\sigma}{6}.$$

Therefore from (A.2) we get

$$\mathrm{E}[\hat{y}^2] = c^2 \left[\frac{1}{N} + \frac{N-1}{N}\left(1 - \frac{2\sigma}{3}\right)\right], \tag{A.4}$$

and from (A.3)

$$\begin{aligned} \mathrm{E}[x\hat{y}] &= 2c \int_{-\infty}^{\infty} x P(x) P_{1|x} dx \\ &= 2c \int_{-\sigma_r/2}^{\sigma_r/2} x \left(\frac{1}{2} + \frac{x}{\sigma_r}\right) \frac{1}{\sigma_p} dx + 2c \int_{\sigma_r/2}^{\sigma_p/2} \frac{x}{\sigma_p} dx \\ &= \frac{c\sigma_p}{12}\left(3 - \sigma^2\right). \end{aligned} \tag{A.5}$$

A.1.2. *Uniform signal and noise with $\sigma \geq 1$*

We have

$$\mathrm{E}[P_{1|x}^2] = \int_{-\infty}^{\infty} P_{1|x}^2 P(x) dx = \frac{1}{\sigma_p} \int_{-\sigma_p/2}^{\sigma_p/2} \left(\frac{1}{2} + \frac{x}{\sigma_r}\right)^2 dx = \frac{1}{4} + \frac{1}{12\sigma^2}.$$

Therefore from (A.2) we get

$$\mathrm{E}[\hat{y}^2] = c^2 \left[\frac{1}{N} + \frac{N-1}{N}\left(\frac{1}{3\sigma^2}\right)\right], \tag{A.6}$$

and from (A.3)

$$\begin{aligned} \mathrm{E}[x\hat{y}] &= 2c \int_{-\infty}^{\infty} x P(x) P_{1|x} dx \\ &= 2c \int_{-\sigma_p/2}^{\sigma_p/2} x \left(\frac{1}{2} + \frac{x}{\sigma_r}\right) \frac{1}{\sigma_p} dx \\ &= \frac{c\sigma_p}{6\sigma}. \end{aligned} \tag{A.7}$$

A.2. *Correlation coefficient for uniform signal and noise*

From (2),(A.4), (A.5), (A.6) and (A.7) we get

$$\rho_{x,\hat{y}} = \begin{cases} \frac{\sqrt{N}(3-\sigma^2)}{2\sqrt{\sigma(2-2N)+3N}} & (\sigma \leq 1), \\ \frac{\sqrt{N}}{\sqrt{3\sigma^2+N-1}} & (\sigma \geq 1). \end{cases} \tag{A.8}$$

A.3. *Gaussian signal and noise*

We have

$$
\begin{aligned}
\mathrm{E}[P_{1|x}^2] &= \int_{-\infty}^{\infty} P_{1|x}^2 P(x)dx \\
&= \int_{-\infty}^{\infty} \left(\frac{1}{2} + \frac{1}{2}\mathrm{erf}\left(\frac{x}{\sqrt{2}\sigma_r}\right)\right)^2 P(x)dx \\
&= \int_{-\infty}^{\infty} \left(\frac{1}{4} + \mathrm{erf}\left(\frac{x}{\sqrt{2}\sigma_r}\right) + \frac{1}{4}\mathrm{erf}\left(\frac{x}{\sqrt{2}\sigma_r}\right)^2\right) P(x)dx \\
&= \frac{1}{4}\int_{-\infty}^{\infty} P(x)dx + \int_{-\infty}^{\infty} \mathrm{erf}\left(\frac{x}{\sqrt{2}\sigma_r}\right) P(x)dx \\
&\quad + \frac{1}{4}\int_{-\infty}^{\infty} \mathrm{erf}^2\left(\frac{x}{\sqrt{2}\sigma_r}\right) P(x)dx \\
&= \frac{1}{4} + 0 + \frac{1}{4}\int_{-\infty}^{\infty} \mathrm{erf}^2\left(\frac{x}{\sqrt{2}\sigma_r}\right) P(x)dx\,,
\end{aligned}
$$

since $P(x)$ is even and $\mathrm{erf}(x)$ is odd and therefore the second term above is zero. We make use of the following result [37]:

$$
\int_{-\infty}^{\infty} \exp(-a^2x^2)\mathrm{erf}^2(x)dx = \frac{2}{a\sqrt{\pi}} \arctan \frac{1}{a\sqrt{a^2+2}}
$$

which gives

$$
\begin{aligned}
\mathrm{E}[P_{1|x}^2] &= \frac{1}{4} + \frac{1}{4\sqrt{2\pi\sigma_p^2}} \int_{-\infty}^{\infty} \mathrm{erf}^2\left(\frac{x}{\sqrt{2}\sigma_r}\right) \exp\left(-\frac{x^2}{2\sigma_p^2}\right) dx \\
&= \frac{1}{4} + \frac{\sigma}{4\sqrt{\pi}} \int_{-\infty}^{\infty} \mathrm{erf}^2(\tau) \exp(-\sigma^2\tau^2)d\tau \\
&= \frac{1}{4} + \frac{1}{2\pi} \arctan\left(\frac{1}{\sigma\sqrt{\sigma^2+2}}\right) \\
&= \frac{1}{4} + \frac{1}{2\pi} \arcsin\left(\frac{1}{\sigma^2+1}\right),
\end{aligned}
$$

since $\sigma \geq 1$. Therefore from (A.2)

$$
\mathrm{E}[\hat{y}^2] = \frac{c^2}{N}\left(1 + \frac{2(N-1)}{\pi} \arcsin\left(\frac{1}{\sigma^2+1}\right)\right), \tag{A.9}
$$

and from (A.3)

$$\begin{aligned}
\mathrm{E}[x\hat{y}] &= 2c \int_{-\infty}^{\infty} xP(x)P_{1|x}dx \\
&= 2c \int_{-\infty}^{\infty} xP(x) \left(\int_{-\infty}^{x} R(\eta)d\eta \right) dx \\
&= 2c \int_{-\infty}^{\infty} R(\eta) \left(\int_{\eta}^{\infty} xP(x)dx \right) d\eta \\
&= 2c \int_{-\infty}^{\infty} R(\eta) \left(\int_{\eta}^{\infty} \frac{x}{\sqrt{2\pi}\sigma_p} \exp\left(-\frac{x^2}{2\sigma_p^2} \right) dx \right) d\eta \\
&= \frac{2c\sigma_p}{\sqrt{2\pi}} \int_{-\infty}^{\infty} R(\eta) \exp\left(-\frac{\eta^2}{2\sigma_p^2} \right) d\eta \\
&= \frac{2c\sigma_p}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_r} \exp\left(-\frac{\eta^2}{2\sigma_r^2} \right) \exp\left(-\frac{\eta^2}{2\sigma_p^2} \right) d\eta \\
&= \frac{c}{\pi\sigma} \int_{-\infty}^{\infty} \exp\left(-\frac{\eta^2}{2} \left(\frac{1+\sigma^2}{\sigma_r^2} \right) \right) d\eta \,.
\end{aligned}$$

The final integrand is a Gaussian density function, with variance $\sigma_r^2/(1+\sigma^2)$. Hence the integral from negative to positive infinity is 1 times the normalizing factor, i.e.

$$\mathrm{E}[x\hat{y}] = \frac{c}{\pi\sigma}\sqrt{2\pi\left(\frac{\sigma_r^2}{1+\sigma^2}\right)} = c\sigma_p\sqrt{\frac{2}{\pi(1+\sigma^2)}}\,. \tag{A.10}$$

A.4. *Correlation coefficient for Gaussian signal and noise*

From (2), (A.9) and (A.10) we get

$$\rho_{x,\hat{y}} = \sqrt{\frac{2N}{\pi(1+\sigma^2)}} \Bigg/ \sqrt{1+\frac{2(N-1)}{\pi}\arcsin\left(\frac{1}{\sigma^2+1}\right)}. \tag{A.11}$$

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