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INVESTIGATION OF CHAOTIC SWITCHING STRATEGIES IN PARRONDO'S GAMES

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An analysis of Parrondo's games with different chaotic switching strategies is carried out. We generalize a fair way to compare between different switching strategies. The performance of Parrondo's games with chaotic switching strategies is compared to random and periodic switching strategies. The rate of winning of Parrondo's games with chaotic switching strategies depends on coefficient(s) defining the chaotic generator, initial conditions of the chaotic sequence and the proportion of Game A played. Maximum rate of winning can be obtained with all the above mentioned factors properly set, and this occurs when chaotic switching strategy approaches periodic-like behavior.

Keywords: Parrondo's paradox; chaos; chaotic switching.

1. Introduction

1.1. *Parrondo's games*

Parrondo's games were devised by the Spanish physicist Juan M. R. Parrondo in 1996 and they were presented in unpublished form at a workshop in Torino, Italy [1]. After about three years, in 1999, Harmer and Abbott published the seminal paper on Parrondo's games [2]. The games are named after their creator and the counterintuitive behavior is called "Parrondo's paradox" [3].

The main idea of Parrondo's paradox is that two individually losing games can be combined to win via deterministic or non-deterministic mixing of the games [4]. There has been a lot of research on Parrondo's games after the first published paper, giving birth to new games such as history dependent games [5] (instead of capital dependent) and cooperative games [6, 7] (multi-player games instead of one player). Parrondo's games are closely related to Brownian ratchets [8, 9]. However, in this paper the original Parrondo's games will be used for analyzing the differences between chaotic, random and periodic switching strategies. The motivation for exploring chaotic strategies is inspired by the fact that they have been shown to be superior for a number of different types of optimization problems [10–14]. The seminal papers that considered chaotic switching in Parrondo's games were by Arena *et al.* [15] and Bucolo *et al.* [10]. These papers tried to show that a chaotic switching strategy is better than a random switching strategy in Parrondo's games. However, it is not fair to compare between random and chaotic switching strategies with arbitrarily chosen parameters. In this paper, we generalize a fair way to compare random and chaotic Parrondo's games. The original Parrondo's games are defined as below [3, 4, 16, 17], where C is the current capital at discrete-time step n.

Game A consists of a biased coin that has a probability p of winning,

Game B consists of 2 games, the condition of choosing either one of the games is given as below:

If C mod $M = 0$, play a biased coin that has probability p_1 of winning,

If C mod $M \neq 0$, play a biased coin that has probability p_2 of winning.

For the original Parrondo's games, the parameters are set as follows: $M = 3$, $p = 1/2 - \epsilon$, $p_1 = 1/10 - \epsilon$ and $p_2 = 3/4 - \epsilon$. To control the three probabilities p, p_1 and p_2, a biasing parameter, ϵ is utilized, where in this paper ϵ is chosen to be 0.005.

Game A is a fair game if ϵ is chosen to be zero. To make Game A losing, ϵ has to be positive ($\epsilon > 0$). Similarly, Game B is fair if ϵ is zero, otherwise if ϵ is positive, Game B is losing. Game B is made up of two coin tossing games, where one consists of a good coin and the other consists of a bad coin. If the good coin of Game B can be played more often than when Game B is played individually, by feedback from mixing of Game A and B, the consequent payoff is more than sufficient to cover the loss from Game A. In this situation, Parrondo's paradox is said to exist.

1.2. *Chaotic maps*

Chaos is used to describe fundamental disorder generated by simple deterministic systems with only a few elements [18]. The irregularities of chaotic and random sequences in the time domain are often quite similar. As an illustration, the random and Logistic sequences (the definition of a Logistic sequence is explained below) are plotted as shown in Fig. 1(a) and Fig. 1(b) where it is difficult to observe any difference. However, by plotting the phase space of random and Logistic sequences, as shown in Fig. 1(c) and Fig. 1(d), the chaotic sequence can be easily identified because there is a regular pattern in its phase space plot. Consecutive points of a chaotic sequence are highly correlated, but not for the case of a pure random sequence. A chaotic sequence, X is usually generated by nested iteration of some functions. This is shown as below, where n is sample number and $f_n(\cdot)$ is nth iteration of function $f(\cdot)$,

$$x_n = f_n(f_{n-1}(...f_2(f_1(x_0))...)). \quad (1)$$

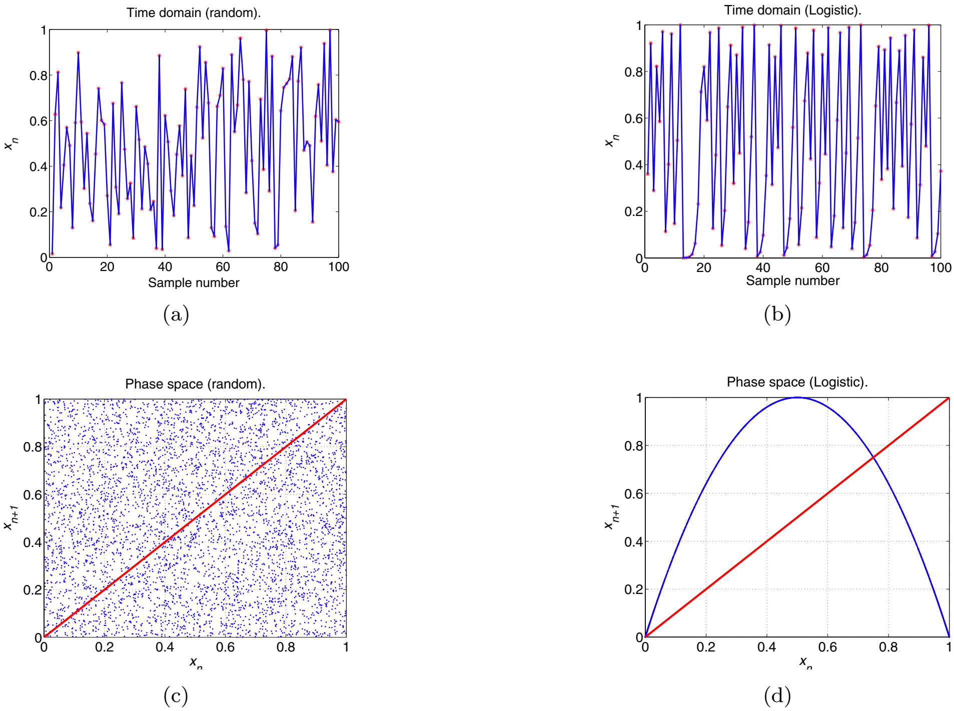


Fig 1. This figure illustrates that random and chaotic sequences are markedly different when plotted in phase space, but not easily distinguishable in the time domain. (a) A random sequence consisting of 100 samples was generated using the `rand` MATLAB function. The unpredictable pattern of the sequence is observed. (b) The irregularities of a Logistic sequence with $a = 4$ is observed. There is no *prima facie* difference between a chaotic sequence and a random sequence in the time domain. This sequence of 100 samples was generated using initial condition = 0.1. (c) Phase space plot of a random sequence with 5,000 samples. The $x_{n+1} = x_n$ line is drawn to divide the phase space plot into two equal halves. There is no distinct pattern observed in the phase space plot of a random sequence. (d) Phase space plot of a Logistic sequence with $a = 4$. It is observed that the phase space plot gives rise to a symmetric shape. By varying the coefficient a, the phase space plot shifts in shape. This phase space plot is generated with initial condition, $x_0 = 0.1$ using 5,000 samples.

For simplicity, one-dimensional and two-dimensional chaotic maps are used. More analysis on the Logistic map is carried out in this paper because it is one of the oldest and typical chaotic maps.

1. One-dimensional chaotic generators:

Logistic map [19]

$$x_{n+1} = ax_n(1 - x_n) \tag{2}$$

Tent map [18]

$$x_{n+1} = \begin{cases} ax_n & \text{if } x_n \leq 0.5 \\ a(1 - x_n) & \text{otherwise} \end{cases} \tag{3}$$

Sinusoidal map [15]

$$x_{n+1} = a x_n^2 \sin(\pi x_n) \tag{4}$$

Gaussian map [10]

$$x_{n+1} = \begin{cases} 0 & \text{if } x_n = 0 \\ \frac{1}{x_n} \bmod 1 & \text{if } x_n \neq 0. \end{cases} \tag{5}$$

2. Two-dimensional chaotic generators:

Henon map [18]

$$\begin{cases} x_{n+1} = y_n + 1 - a{x_n}^2 \\ y_{n+1} = b x_n \end{cases} \tag{6}$$

Lozi map [10]

$$\begin{cases} x_{n+1} = y_n + 1 - a|x_n| \\ y_{n+1} = b x_n. \end{cases} \tag{7}$$

1.3. *Switching strategies*

Here we use the version of Parrondo's games that consist of two games, Game A and Game B as described in Sec. 1.1. At discrete-time step n, only one game will be played, either Game A or Game B. The algorithm or pattern used to decide which game to play at discrete-time step n is defined as the switching strategy. In the original Parrondo's games, the switching strategies utilize random or periodic sequences [3]. In this paper, chaotic switching of Game A and Game B based on several chaotic sequences is investigated through simulations.

2. Games with Chaotic Switching Strategy

To play Parrondo's games with chaotic switching, a chosen chaotic generator is used to generate a sequence, X. Sequence X is used to decide if either Game A or B is played at discrete-time step n. There are many ways to carry out this task, but the easiest way is to compare each value of X with a constant γ. On each round (round n) of Parrondo's games, a value from the chaotic sequence, x_n is compared with γ, if $x_n \leq \gamma$, Game A will be played, but if $x_n > \gamma$, Game B will be played. This simple procedure is adopted in this paper. For a random switching strategy, γ is equivalent to the proportion of Game A played after n discrete-time steps. However, this is not necessarily true for a chaotic switching strategy.

2.1. *Chaotic generators*

The outcomes of Parrondo's games will be affected by the different chaotic switching strategies applied. Before this aspect is investigated, the parameters that affect the behaviors of the chaotic sequence have to be identified. The properties of a particular chaotic sequence from a chaotic generator depend on two elements: coefficient(s) of the chaotic generator and initial conditions.

2.1.1. *Coefficient(s) of a chaotic generator*

The phase space of a chaotic signal changes with the coefficient(s) defining its chaotic map as in Eq. (2) to Eq. (7). That is the relationship between consecutive points of a chaotic sequence changes with the coefficient(s) and this can lead the chaotic sequence to either a chaotic or stable state [19]. For example, the a coefficient in a Logistic map will decide the state of the sequence generated, whether in a stable or chaotic state. This result can be looked up from the bifurcation diagram of the Logistic map as shown in Fig. 2(a). The bifurcation diagram is also known as the Feigenbaum diagram. The regions with continuous points correspond to chaotic states, while those with distinct points correspond to stable states [19]. The bifurcation diagrams of the one-dimensional chaotic maps can be easily plotted as shown in Fig. 2(a) and Fig. 2(b). However, the complete bifurcation diagram of two-dimensional chaotic maps are more complicated since four parameters are involved.

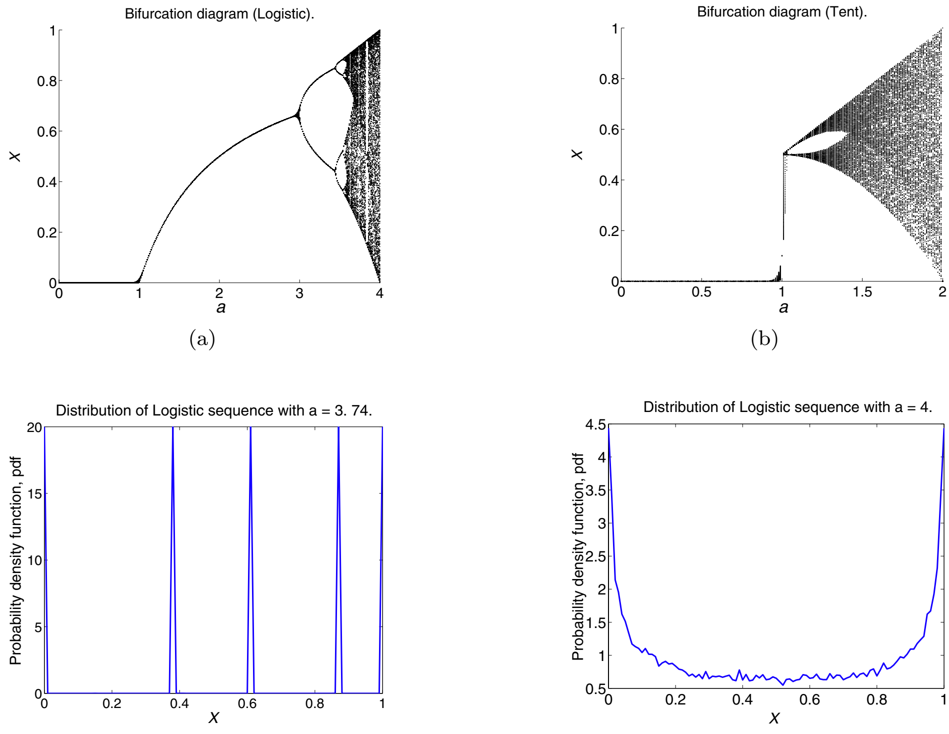


Fig 2. (a) Bifurcation diagram of Logistic map. There are 250 samples for each of the coefficient range from 0 to 4 with a step of 0.01. (b) Bifurcation diagram of Tent map. There are 250 samples for each of the coefficient range from 0 to 2 with a step of 0.01. (c) Distribution of Logistic sequence in a stable state ($a = 3.74$). The pdf is constructed with an initial condition of 0.1 using 50,000 samples. (d) Distribution of Logistic sequence in an unstable state ($a = 4$). The pdf is constructed with an initial condition, $x_0 = 0.1$ using 50,000 samples.

The distribution of x_n of a chaotic sequence between 0 and 1 is determined by the coefficients a and b as defined in Eq. (2) to Eq. (7). With coefficients that correspond to stable states of a chaotic generator, only a finite number of distinct values of x are observed. The pdf of x_n for a Logistic sequence in a stable state is plotted as shown in Fig. 2(c). In a chaotic state, the spread of x_n is more even over the range from 0 to 1. This is shown with a Logistic sequence and $a = 4$, in Fig. 2(d).

2.1.2. *Initial conditions*

Different initial conditions give the same phase space plot of the chaotic maps. However, initial conditions affect the way the phase space is constructed. A small fluctuation in the initial condition starts a "snow ball" effect on the chaotic sequence, which affects the values of the whole sequence after a few iterations. This is one of the famous properties of a chaotic sequence [18].

There are some initial conditions that map the chaotic sequence to a constant value. An example to illustrate this behavior is to use Logistic map with $a = 4$. From simulation results, initial conditions of 0, 1/4, 1/2 and 3/4 map the sequence to a constant value of either 0 or 3/4. This can be explained from the phase space plot of the Logistic map as seen in Fig. 1(d). The intersections of the phase space plot and the line $x_{n+1} = x_n$ occur at $x_{n+1} = x_n = 0$ and $x_{n+1} = x_n = 3/4$. A tiny deviation from these intersection points would lead to a chaotic behavior. They are called unstable fixed points [20]. Hence, the choice of initial conditions is vital for obtaining an oscillating sequence in order to carry out effective chaotic switching of Parrondo's games.

2.2. γ *values*

To decide whether to play Game A or B at each discrete-time step n, we utilize the γ parameter. The γ value sets a threshold on selection of games to be played on each round. On the other hand, the γ value is important to make Parrondo's paradox appear. The effect of the γ value on the rate of winning of Parrondo's games with chaotic switching strategies is investigated through simulations. Before a chaotic sequence x_n is used for a switching strategy, it is normalized to have values between 0 and 1. This is done by taking the minimum value of a sequence from each value of a sequence ($x_n - \min(x_n)$, for all n), then divide by the range of the sequence, where the range of a sequence is calculated as $\max(x_n) - \min(x_n)$.

3. Effect of the Coefficient(s) of Chaotic Generator on the Rate of Winning

The rate of winning, $R(n)$, is given below [4], where $\pi_j(n)$ is the stationary probability of being in state j at discrete-time step n.

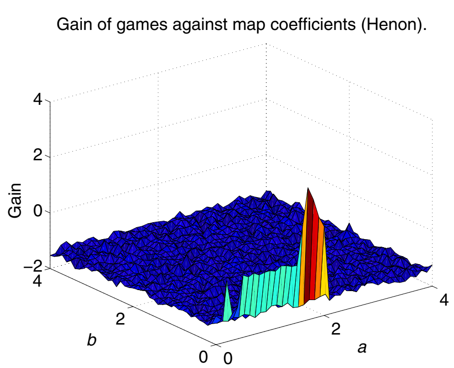
$$R(n) = E[J_{n+1} - J_n] = E[J_{n+1}] - E[J_n] = \sum_{j=-\infty}^{\infty} j[\pi_j(n+1) - \pi_j(n)]. \qquad (8)$$

The coefficient(s) of a chaotic generator determines the stability of the chaotic system. In the stable regions, the system shows periodic behavior. The simulation results show that the maximum rate of winning of Parrondo's games occurs when the chaotic generator used for switching tends toward periodic behavior. On the other hand, when a chaotic generator behaves truly chaotically, the rate of winning is smaller compared to a periodic case. Hence, under periodic or stable state of a chaotic sequence, and properly tuned initial conditions and γ value as discussed in the next section, the rate of winning obtained can be higher than the one achieved by random switching strategy. However, to identify the exact periodic sequence that gives the highest rate of winning is a complicated problem. In Cleuren and Van den Broeck's model, which exhibits Parrondo's paradox, the periodic switching strategy that gives the maximal gain has been proposed [21].

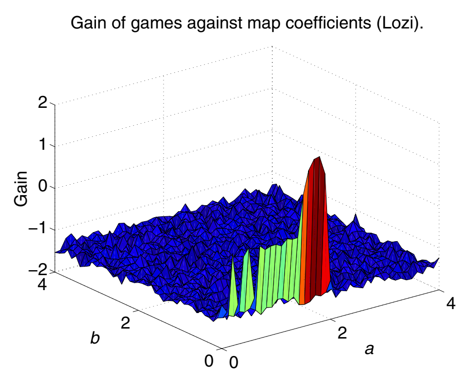
For two-dimensional maps such as the Henon map and Lozi map, there are two coefficients, a and b that control the behavior of the sequence. Hence, they determine the rate of winning of Parrondo's games. From Fig. 3(a) and Fig. (3b), the gains after 100 games are plotted with different combinations of a and b values. It is found that both the maps give maximum gain after 100 games when $a = 1.7$ and $b = 0$. For $b = 0$, Eq. 6 and Eq. 7 are simplified to Eq. 9 and Eq. 10 respectively, which are one-dimensional.

$$x_{n+1} = 1 - a{x_n}^2, \tag{9}$$

$$x_{n+1} = 1 - a|x_n|. \tag{10}$$



(a)



(b)

Fig 3. (a) Gain of Parrondo's games with a Henon switching strategy after 100 games against different a & b coefficients. To construct this plot, both the coefficients are run from 0 to 4 with a step of 0.1, γ is set to 0.5, initial condition is set to $(x, y) = (0, 0)$ and 5,000 trials is averaged. (b) Gain of Parrondo's games with a Lozi switching strategy after 100 games against different a & b coefficients. Similar to the setting for the case of Henon switching, both the coefficients are run from 0 to 4 with a step of 0.1, γ is set to 0.5, initial condition is set to $(x, y) = (0, 0)$ and 5,000 trials is averaged.

4. Effect of Initial Conditions and γ Value on the Rate of Winning

Different combinations of the initial conditions and γ values give different rates of winning for Parrondo's game. For some initial conditions, chaotic switching strategy causes the games to lose. This occurs when the initial conditions drive the chaotic sequence towards its attractors as discussed in Sec. 2.1.2. This situation can be explained as playing Game A or Game B individually since the sequence stays at a constant value. However, the other initial conditions give the same rate of winning for a given particular value of γ and coefficient(s) of the chaotic generator. Since the γ value decides the proportion of games played, the γ value can make the games either win or lose, as long as the initial condition is not in the losing region. To obtain optimized or maximum rate of winning, the capital of the games after 100 games averaged over 5,000 trials against initial conditions and γ value is plotted. These 3-dimensional diagrams are shown in Fig. 4(a) to Fig. 4(d).

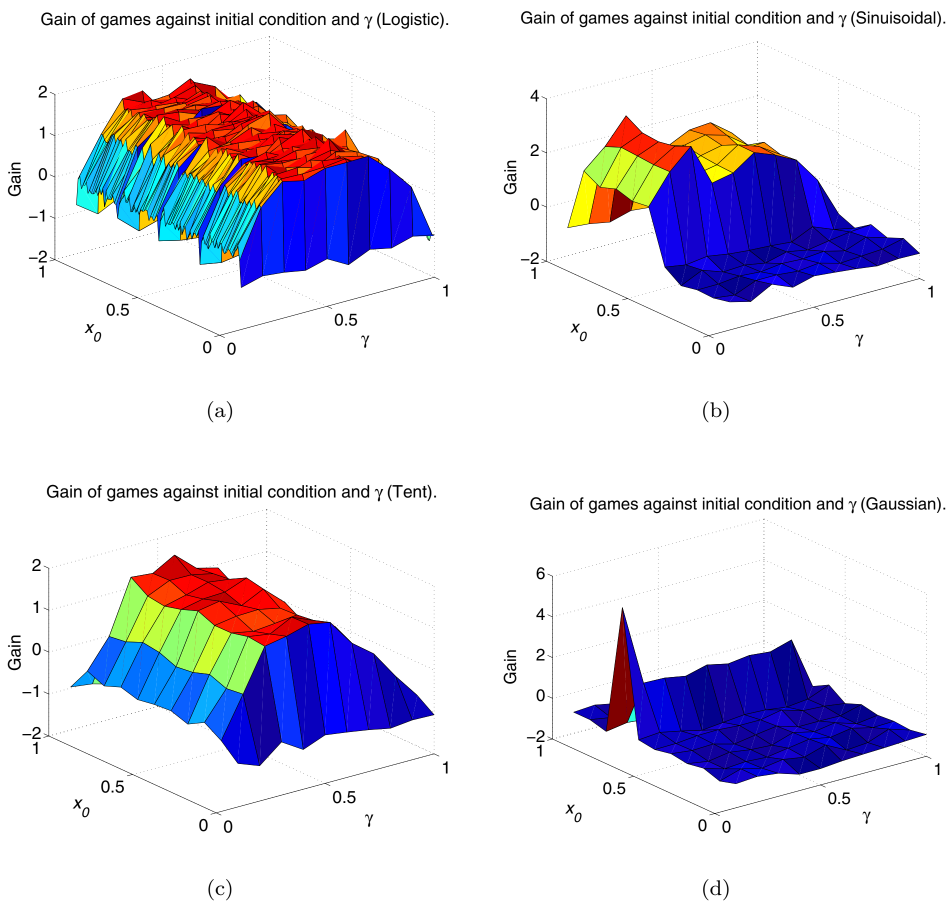
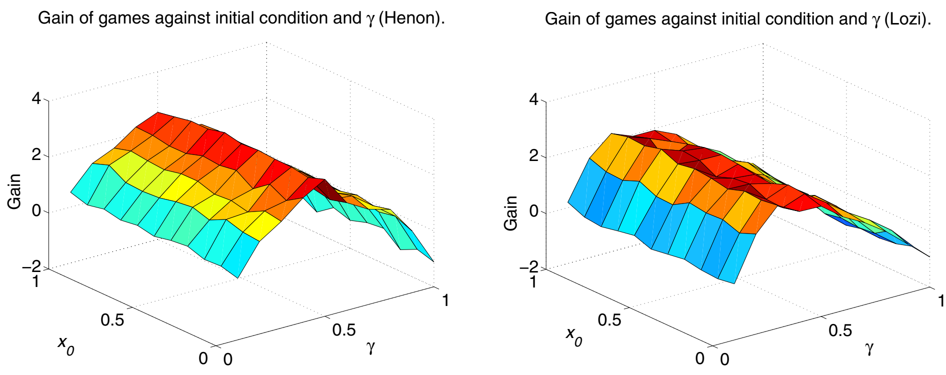


Fig 4. (a) Gain of Parrondo's games with Logistic switching strategy after 100 games against initial condition and γ. (b) Gain of Parrondo's games with Sinusoidal switching strategy after 100 games against initial condition and γ. (c) Gain of Parrondo's games with Tent switching strategy after 100 games against initial condition and γ. (d) Gain of Parrondo's games with Gaussian switching strategy after 100 games against initial condition and γ.



(a) (b)

Fig 5. (a) Gain of Parrondo's games with Henon switching strategy ($a = 1.7$, $b = 0$) after 100 games against initial condition and γ. (b) Gain of Parrondo's games with Lozi switching strategy ($a = 1.7$, $b = 0$) after 100 games against initial condition and γ.

For Henon and Lozi maps, the 3-dimensional diagrams of gain after 100 games against initial conditions and γ are plotted using the simplified maps explained in Sec. 3 in order to obtain maximum rate of winning. They are shown in Fig. 5(a) and Fig. 5(b).

When γ and the initial condition, x_0 are set to 0.5 and 0.1 respectively for a Logistic sequence with $a = 3.74$, the maximum capital averaged over 50,000 trials, is found to have value of 6.2 after 100 games. This is the maximum capital after 100 games of all the simulations carried out.

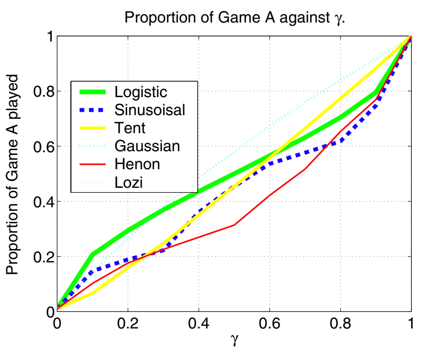
5. Effect of Initial Conditions and γ on the Proportion of Game A Played

The initial conditions of all the chaotic generators have no affect on the proportion of Game A played. However, the proportion of Game A played is significantly dependent on γ value. This is because the γ value is acting as a threshold value on deciding whether the next game played should be Game A or B.

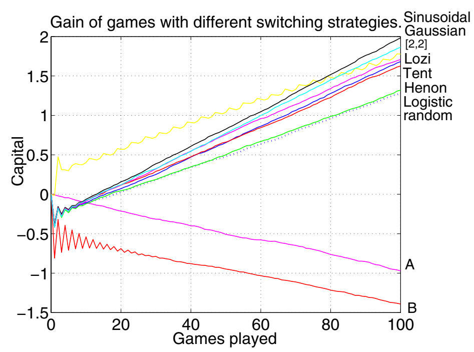
6. Comparing Different Switching Strategies Under Same Proportion of Game A Played

The performance of different switching strategies is based on the rate of winning Parrondo's games. The higher the rate of winning, the better the performance is. To compare the performance of the switching strategies in a fair manner, a normalization procedure has to be properly carried out. One suggested way is to compare switching strategies with the same proportion of Game A and B played. Since the proportion of Game A and B played depends on γ, for all switching strategies γ is used to adjust the proportion of Game A played to a certain fixed value (say 0.5). The graph showing the relationship of proportion of Game A played and γ value for chaotic switching strategies is plotted in Fig. 6(a).

The chosen fixed proportion of Game A played for all the chaotic switching strategies is 0.5, since the proportion of Game A played for periodic sequence of [AABB...] or [2,2], is 0.5. Hence, γ is used to obtain 0.5 proportion of Game A played for all chaotic and random switching strategies. The γ value that corresponds to 0.5 proportion of Game A played for respective chaotic switching strategy can be found in Table 1 and Fig. 6(a). The a, b and initial condition, x_0 in Table 1 and Fig. 6(a) are chosen to maximize the rate of winning of games for each chaotic switching strategy. Under these conditions, the rates of winning of the games with different switching strategies can be properly compared. The simulation results of all chaotic switching strategies discussed together with random and periodic [2,2] switching strategies are plotted in Fig. 6(b).



(a)



(b)

Fig 6. (a) Plot of effect of γ on the proportion of Game A played. This plot shows that the proportion of Game A played is not linear in γ for chaotic switching strategies. However, the expected proportion of Game A played against γ line for a uniform random switching is a straight line joining points (0,0) and (1,1). (b) Capital under different switching regimes for 100 games (averaged over 50,000 trials). It is noted from the plot that all curves with chaotic switching strategies are higher than the random switching curve. However, the comparison of capital under periodic and chaotic switching strategies is inconclusive. Parrondo's games with sinusoidal switching gives the highest capital after 100 games.

This shows that Parrondo's games with chaotic switching strategies can give higher rate of winning compared to one with random switching strategy, but may or may not be higher than one with periodic switching strategy.[a] It is found that a particular chaotic switching strategy gives an increased rate of winning when its sequence is having periodic behavior with short period.

[a]It is hard to compare chaotic and periodic switching strategies in a fair manner. This is because a chaotic switching strategy contains periodic behavior and there are an infinite number of ways of constructing a periodic switching strategy.

Table 1. Parameters set up for simulation of the proportion of Game A played and capital distribution after 100 games under different switching regimes.

Maps	***a***	***b***	γ	**Initial condition**	**No. of trials**
Logistic	4	-	0.50	0.1	50,000
Sinusoidal	2.27	-	0.55	0.5	50,000
Tent	1.9	-	0.55	0.8	50,000
Gaussian	-	-	0.41	0.701	50,000
Henon	1.7	0	0.68	[x,y]=[0,0]	50,000
Lozi	1.7	0	0.55	[x,y]=[0,0]	50,000

7. Conclusion

The proportion of Game A and B played must be equal for all switching strategies in order to compare Parrondo's games in a fair manner. Parrondo's games with chaotic switching strategy can give higher rate of winning compared to a random switching strategy. The rate of winning obtained from a chaotic switching strategy is controlled by the coefficient(s) defining the chaotic generator, initial conditions and proportion of Game A played. When a chaotic switching strategy approaches periodic behavior with a short period, it gives an increased rate of winning for Parrondo's games. From simulation results, combination of Game A and B in the pattern [ABABB...] is found to give the highest rate of winning.[b]

Acknowledgements

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