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REVERSE AUCTION: THE LOWEST UNIQUE POSITIVE INTEGER GAME

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A reverse auction can be likened to a tendering process where a contract is awarded to the lowest bidder. This is in contrast to a conventional forward auction where the highest bidder wins. In this paper we analyze a minority game version of the reverse auction where an extra condition is imposed that, namely, the bid must be unique. In other words, the bidder with the lowest positive unique integer (LUPI) wins. We examine and compare two extrema, namely, the case when all players are rational and the case when all players make random selections.

Keywords: Reverse auction; game theory; minority game; auction theory; random choice; rational choice; bidding; lowest unique positive integer; LUPI; optimization; assignment problems.

1. Introduction

Whilst auctions date back to Roman, and even Babylonian times [1], surprisingly it is only recently that they have been analyzed in game-theoretic terms in the seminal work of Vickrey [2,3] and Shubik [4]. The theory of auctions has evolved to a point where it is now regarded as one of the most successful branches of economic theory [5].

Nevertheless, analysis of the *reverse auction* is sparse in the game-theoretic literature. In this paper, we examine a specific type of reverse auction where, out of n players, the lowest unique bid wins. For the first time, we provide a simple analysis assuming the players make random decisions and contrast this with the case of rational players. As well as the more traditional applications of game theory to economics and population dynamics, the field is generally becoming of increasing importance in physics for studying the evolution of information in the presence of noise [6].

In a conventional *forward auction* the highest bid determines the price at the end of the bidding process. On the other hand, a reverse auction—in its most general form—is rather like a contract tender process, where a contract is awarded to the lowest bidder. It should not be confused with a Dutch auction, where the called price is progressively lowered and the *first* bidder wins. Algorithms exploiting reverse auction optimization have been shown to be of practical benefit for asymmetric assignment problems, leading to robustness against instability due to 'price war' escalation [7]. Also switching between reverse and forward auctions can be exploited for symmetric assignment problems [7]. The importance of forward and backward bidding processes in e-commerce has been previously pointed out [8]. Reverse auctions over the internet and mobile phones are becoming increasingly common [9] and resource allocation models based on the reverse auction strategy have found applications in grid computing [10].

In this paper, we examine a more specific, but interesting, case of a reverse auction—namely, the *least unique positive integer game.* The added restriction of requiring a bid to be unique, in order to win, brings in the complexity of a *minority game* [11] scenario. The resulting dynamics of adding the element of a minority game has many practical consequences. For example, in a traffic network, travel time is minimized if the driver seeks to be in the minority to avoid congestion. These types of models lead to fascinating counterintuitive phenomena, such as the Braess paradox [12]. In physics one can encounter situations where it is of interest to find a minimum unique energy optimum [13].

In the following sections, we analyze the least unique positive integer game firstly assuming the players select random integers. This then sets the motivation to analyze and compare it with the case of rational players.

2. Random Selection

There are n players who pick a number in the range 1 to n at random. A player wins if he has the lowest unique number. The multinomial distribution is very relevant here. Given n boxes and probabilities p_i $(i = 1 \cdots n)$ that an object is placed in box i, if we repeat m times, the joint probability of there being k_i entries in box i is:

$$\mathrm{P}(k_1, k_2 \cdots k_n) = \frac{n!}{k_1!k_2! \cdots k_n!} p_1^{k_1} p_2^{k_2} \cdots p_n^{k_n} \tag{1}$$

where $m = k_1 + k_2 + \cdots + k_n$.

Let us now examine the probability of a given player winning and let us label this Player A. To make the problem tractable let us simply consider the case where we have n players, and we calculate the probability of winning given that Player A picks the integer 1. We then consider the next case where Player A selects 2, and so on. This will enable us to see the emerging properties of the game under random selection rules, conditioned on Player A's original choice. This will enable us to suggest the best rational choice for Player A, given $n-1$ players that make random choices.

If Player A selects 1, then to win, the other $n-1$ players must select integers in the range 2 to n. The probability of any player picking boxes in the range 2 to n is $(n-1)/n$. The probability, P_1, of Player A winning, is the probability that other players do not select 1:

$$\begin{aligned} P_1 &= \mathrm{P}(\text{A wins}|\text{A selects 1}) = \left(\frac{n-1}{n}\right)^{n-1} \\ &= 0.4288 && (\text{for } n=4) \\ &= 0.3874 && (\text{for } n=10) \\ &= 0.3679 = \mathrm{e}^{-1} && (\text{for } n \to \infty) \end{aligned} \quad (2)$$

where P_1 is the probability of Player A winning.

Table 1. The case when Player A alone selects 1, and thus always wins against $n-1$ other players.

Selected Number	1	2 to n
Players	A only	$(n-1)$ others

If Player A picks 2, then in order to win, the other $n-1$ players must select integers in the range 3 to n, or two or more must select 1 with the other players in the range 3 to n.

Table 2. The case when Player A alone selects 2. There are n players in total and, say, k players that select 1. Player A can only win if $k \neq 1$ players select 1.

Selected Number	1	2	3 to n
Players	k	A only	$(n-k-1)$ others

Given that Player A has selected 2, the probability of the other $n-1$ players picking something else is P_1. However, Player A only wins if $k > 1$ since if $k = 1$ that player will win. The probability of one other player choosing 1 and the other $n-2$ players choosing in the range 3 to n is the multinomial probability Q_2 with the probability of picking 1 being $1/n$ and the probability of picking 3 to n being $p_2 = (n-2)/n$. The losing pattern for Player A occurs when only one player ($k = 1$) selects the integer 1 with probability Q_2, where,

$$Q_2 = \mathrm{P}(\text{A loses}|k=1) = \frac{(n-1)!}{1!(n-2)!}\left(\frac{1}{n})(\frac{n-2}{n}\right)^{n-2} = \left(\frac{n-1}{n}\right)\left(\frac{n-2}{n}\right)^{n-2}. \quad (3)$$

Now, the probability P_2 of Player A winning is,

$$\begin{aligned} P_2 &= P_1 - Q_2 \\ &= 0.2344 && (\text{for } n=4) \\ &= 0.2364 && (\text{for } n=10) \\ &= 0.2325 = \mathrm{e}^{-1} - \mathrm{e}^{-2} && (\text{for } n \to \infty). \end{aligned} \quad (4)$$

If Player A selects 3, then in order to win, the other $n-1$ players must select integers in the range 4 to n, or two or more must pick 1 and 2 with the rest in the range 4 to n.

Table 3. The case when Player A alone selects 3. There are n players in total and, say, k_1 players that select 1 and k_2 players that select 2. Here, k_1 is the number of players that select 1 and k_2 is the number of players that select 2. Player A can only win if $k_1 \neq 1$ players select 1 and $k_2 \neq 1$ players select 2.

Selected Number	1	2	3	4 to n
Players	k_1	k_2	A only	$(n-k_1-k_2-1)$ others

Given that Player A has selected 3, the probability of the other $n-1$ players not choosing 3 is P_1. However, Player A only wins if $k_1 > 1$ and $k_2 > 1$ since if either of these is true then Player A will lose.

- The probability of one other player choosing 1 and the other $n-2$ players choosing 2 or in the range 4 to n is Q_2.
- Similarly, the probability of one other player choosing 2 and the other $n-2$ players choosing 1 or in the range 4 to n is Q_2.

The sum of these two probabilities covers all the situations where either $k_1 = 1$ or $k_2 = 1$, but double counts the case where both are 1. Hence we must account for the probability of this which is Q_3,

$$\begin{aligned} Q_3 &= \mathrm{P}(k_1 = 1, k_2 = 1) \\ &= \frac{(n-1)!}{1!1!(n-3)!}\left(\frac{1}{n}\right)\left(\frac{1}{n}\right)\left(\frac{n-3}{n}\right)^{n-3} = \left(\frac{(n-1)(n-2)}{n^2}\right)\left(\frac{n-3}{n}\right)^{n-3}. \end{aligned} \tag{5}$$

Now, by applying the union probability rule that, if $a = (k_1 = 1)$ and $b = (k_2 = 1)$, then $\mathrm{P}(a+b) = \mathrm{P}(a) + \mathrm{P}(b) - \mathrm{P}(ab) = 2Q_2 - Q_3$ yields the probability of Player A winning, P_3,

$$\begin{aligned} P_3 &= P_1 - 2Q_2 + Q_3 \\ &= 0.1406 && (\text{for } n = 4) \\ &= 0.1447 && (\text{for } n = 10) \\ &= 0.1470 = \mathrm{e}^{-1} - 2\mathrm{e}^{-2} + \mathrm{e}^{-3} && (\text{for } n \to \infty). \end{aligned} \tag{6}$$

If Player A picks 4, then in order to win the other $n-1$ players must select integers in the range 5 to n, or two or more must pick 1 and 2 and 3 with the rest in the range 5 to n.

Table 4. The case when Player A alone selects 4. There are n players in total and, say, k_1 players that select 1, k_2 players that select 2, and k_3 players that select 3. Player A can only win if $k_1 \neq 1$ players select 1 and $k_2 \neq 1$ players select 2 and $k_3 \neq 1$ players select 3.

Selected Number	1	2	3	4	5 to n
Players	k_1	k_2	k_3	A only	$(n-k_1-k_2-k_3-1)$ others

Given that Player A has selected 4, the probability of the other $n-1$ players not picking 4 is P_1. However, Player A only wins if $k_1 > 1$ and $k_2 > 1$ and $k_3 > 1$ since if any of these is true then Player A will lose.

- The probability of one other player choosing 1 and the other $n-2$ players choosing 2, 3 or in the range 5 to n is Q_2.
- The probability of one other player choosing 2 and the other $n-2$ players choosing 1, 3 or in the range 5 to n is Q_2.
- The probability of one other player choosing 3 and the other $n-2$ players choosing 1, 2 or in the range 5 to n is Q_2.

The sum of these three probabilities, $3Q_2$, covers all the situations where either $k_1 = 1$ or $k_2 = 1$ or $k_3 = 1$, but multiply counts the cases where more than two of the three are 1. For example, $(1, 1, x)$ is counted in both Case 1 and Case 2, $(1, x, 1)$ is counted in both Case 1 and Case 3, and $(x, 1, 1)$ is counted in both Case 2 and Case 3. The probabilities of these are Q_3, so we need to subtract $3Q_3$. However, $(1, 1, 1)$ is covered in all three cases and also in each of $(1, 1, x)$, $(1, x, 1)$ and $(x, 1, 1)$ so we need to add Q_4, which is given by,

$$\begin{aligned} Q_4 &= \mathrm{P}(k_1 = 1, k_2 = 1, k_3 = 1) \\ &= \frac{(n-1)!}{1!1!1!(n-4)!}\left(\frac{1}{n}\right)^3\left(\frac{n-4}{n}\right)^{n-4} = \left(\frac{(n-1)(n-2)(n-3)}{n^3}\right)\left(\frac{n-4}{n}\right)^{n-4}. \end{aligned} \tag{7}$$

Now, by applying the union probability rule that, if $a = (k_1 = 1)$, $b = (k_2 = 1)$, and $c = (k_3 = 1)$ then, $\mathrm{P}(a+b+c) = \mathrm{P}(a) + \mathrm{P}(b) + \mathrm{P}(c) - \mathrm{P}(ab) - \mathrm{P}(bc) - \mathrm{P}(ac) + \mathrm{P}(abc) = 3Q_2 - 3Q_3 + Q_4$ yields the probability of Player A winning, P_4,

$$\begin{aligned} P_4 &= P_1 - 3Q_2 + 3Q_3 - Q_4 \\ &= 0.0469 && \text{(for } n = 4) \\ &= 0.0888 && \text{(for } n = 10) \\ &= 0.0929 = \mathrm{e}^{-1} - 3\mathrm{e}^{-2} + 3\mathrm{e}^{-3} - \mathrm{e}^{-4} && \text{(for } n \to \infty). \end{aligned} \tag{8}$$

For the general case when Player A initially selects the integer r, the probability of winning is,

$$\mathrm{P}(\text{Player Awins}|r) = P_1 - \sum_{i=1}^{r-1}(-1)^i \binom{r-1}{i} Q_{i+1}, \tag{9}$$

where,

$$Q_i = \frac{(n-1)(n-2)\cdots(n-r+1)}{n^{r-1}}\left(\frac{n-r}{n}\right)^{n-r}. \tag{10}$$

In general, the probability of Player A winning given any initial choice of integer can be found, using this approach. As we see from the above four examples, the trend that emerges is that the probability of winning reduces the higher the initial chosen integer. Hence if all other players choose randomly, it pays Player A to choose 1. In other words, if Player A is the sole rational player amongst $n-1$ random players then the best strategy is always to choose 1. This trend is summarized in Table 5, where the probabilities of winning $\{P_1 \cdots P_4\}$ are listed as n varies from 3 to ∞—the asymptotic approach to ∞ is rapid enough that the case $n = 10$ very closely approximates the infinite case.

Table 5. The table summarizes the probabilities of Player A winning, depending on whether 1,2,3 or 4 is chosen. As can be seen, choosing the lowest integer 1 has the best chances of winning assuming random players. The columns show these probabilities for different numbers of total players n, showing that the probability of winning always reduces for a higher total population n. However, interestingly, it appears that a relatively small $n = 10$ closely approximates the case when $n = \infty$. Note that the bottom of the first column has no entry as, by definition, we cannot have four players when $n = 3$. Note also that the probabilities in, say, the first column do not sum to one—this is because of the case when all players select 1 and no one wins. However, we do expect $\sum P_i \to 1$, as the number of players $n \to \infty$, because the probability that an infinite number of players all select the same number $\to 0$.

	3	4	10	100	1000	∞
P_1	0.44444	0.42188	0.38742	0.36973	0.36806	0.36788
P_2	0.22222	0.23438	0.23643	0.23302	0.23259	0.23254
P_3	0.22222	0.14063	0.14473	0.14687	0.14698	0.14700
P_4	-	0.04688	0.08881	0.09257	0.09288	0.09292

3. Rational Selection

3.1. *Specific cases*

As before, let there be n players and let each player pick a positive integer from 1 to n. The player with the lowest unique integer wins. Let the winning amount be $x > 0$ dollars, otherwise the player receives no pay-off.

Example 1. Suppose there are two players. If the selected integers are (1, 1) or (2, 2) then neither player wins. If it is (1, 2), Player A wins x dollars. If it is (2, 1), then Player B wins x dollars. What is the optimal strategy for Player A? Selecting 2 does not work as the best outcome is a draw, in which Player A gains nothing. So Player A will select 1. Similarly, Player B will select 1 as well. The equilibrium outcome is a certain draw at (1, 1) and no one wants to deviate.

Example 2. Suppose there are three players—this gives rise to 27 possible scenarios. A draw situation occurs when the chosen integers are (1, 1, 1), (2, 2, 2), (3, 3, 3), otherwise there will be a winner. What is the optimal strategy for Player A? Let us analyze the winning situations and winning strategies for Player A, who wins (among the nine situations picked by the other two players):

- Players B & C pick (2, 3) or (3, 2) and Player A picks one;
- Players B & C pick (1, 1) and Player A picks either 2 or 3;
- Players B & C pick (2, 2) and Player A picks either 1 or 3;
- Players B & C pick (3, 3) and Player picks either 1 or 2;

otherwise Player A will receive a pay-off of zero. To simplify the analysis, we make the assumption that each player is indifferent between the two strategies conditioned on the other players' choices and a player will always pick the lowest number. Arguably this might be too strong. However, one may defend this on reflection that the players really need a lower number to win. Furthermore, this assumption simplifies the analysis and it appears this is without loss of generality.

So under the situation that the other two players pick the same number, Player A will pick 2 under (1, 1) or 1 under (2, 2) or (3, 3). In other words, selection of 3 is never an optimal strategy under any situation because the players can always pick a lower number to achieve the same results. Because the players are symmetric, each player will only consider picking either 1 or 2. Now the situation is that there are two possible choices for Player A. So the optimal strategy is a mixed one. Namely, with some probability π Player A will pick 1, and with probability $1 - \pi$ Player A will pick 2. Again because of the symmetry, this probability is the same for the other two players. Player A will choose the probability to maximize the expected payoff.

What is the expected payoff? There are only two situations in which Player A will win, namely, (1, 2, 2) and (2, 1, 1)—and as argued above, no player will pick 3. So the expected payoff is:

$$x[\pi(1-\pi)^2 + (1-\pi)\pi^2] = x(\pi - \pi^2). \tag{11}$$

The optimal choice is $\pi = 0.5$, and the expected payoff for each player is $0.25x$.

Example 3. Suppose there are four players. Again, the tie situations are (1, 1, 1, 1), (2, 2, 2, 2), (3, 3, 3, 3), (4, 4, 4, 4), in which nobody wins. Otherwise there will always be a winner. Using similar arguments as above, 4 is not an optimal choice—this is because the only situation where 4 is winning is when the other three pick (1, 1, 1), (2, 2, 2), (3, 3, 3), but then there are lower numbers (either 1 or 2) that will achieve the same results. So if we still make our original assumption, only 1 or 2 will be selected. Again, the situations in which Player A wins are,

- Choose 1, if the other three players pick (2, 2, 2);
- Choose 2, if the other three players pick (1, 1, 1).

Again, if the probability of picking 1 is π and that of 2 is $1-\pi$, then the expected payoff of Player A is:

$$x[\pi(1-\pi)^3 + (1-\pi)\pi^3] = x\pi[1 - 3\pi + 4\pi^2 - 2\pi^3]. \tag{12}$$

As before, $\pi = 0.5$ is the optimal choice, and the expected payoff is 1/8.

3.2. *General results*

For the general situations when there are n players, the optimal strategy for any player will be either one or two. The expected payoff is:

$$x[\pi(1-\pi)^{n-1} + (1-\pi)\pi^{n-1}]. \tag{13}$$

Let us consider the term in the bracket only. The first derivative is:

$$(1-\pi)^{n-1} - \pi(n-1)(1-\pi)^{n-2} - \pi^{n-2} + (1-\pi)(n-1)\pi^{n-2}. \qquad (14)$$

Note that $\pi = 0.5$ is always an optimal choice. Whether or not there are other solutions remains to be seen, though intuitively it would seem the result is unique because of the symmetry. The expected payoff is $x/2^{n-1}$.

4. Conclusion

We have performed simple analyses of the lowest unique positive integer (LUPI) game, under two extrema: (i) all players make random choices, except for Player A and (ii) all players make rational choices under the assumption of the lowest number rule. The results show that, in case (i), the winning strategy is always to select 1; whereas, in case (ii) the strategy is to select 1 or 2 with probability $\pi = 0.5$. Of course, in practice, *mixed* behavior may occur: some choices may be rational and some random within the population. Performing a sensitivity analysis to see the robustness of the outcomes as a function of the ratio of rational to random players is an interesting open question for future study.

The definition of equilibrium must include a player's expectation of the strategy of other players. It is possible that one player wants to deviate from this. So another extension is to check the robustness of this deviation from the expected optimal strategy. There are other interesting future extensions worth examining: (i) comparing when players know the value of n versus ignorance of the bound, (ii) the case of asymmetric information when some players know n and others do not, (iii) the case when the population n varies stochastically between each round, and (iv) the case of collusion between players, where the assumption of independence breaks down.

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COMMENTS ON 'REVERSE AUCTION: THE LOWEST UNIQUE POSITIVE INTEGER GAME'

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In Zeng *et al.* [Fluct. Noise Lett. **7** (2007) L439–L447] the analysis of the lowest unique positive integer game is simplified by some reasonable assumptions that make the problem tractable for arbitrary numbers of players. However, here we show that the solution obtained for rational players is not a Nash equilibrium and that a rational utility maximizer with full computational capability would arrive at a solution with a superior expected payoff. An exact solution is presented for the three- and four-player cases and an approximate solution for an arbitrary number of players.

Keywords: Reverse auction; game theory; minority game; rational choice; LUPI.

1. Introduction

The lowest unique positive integer game can be briefly described as follows: Each of n players secretly selects an integer x in the range $[1, n]$ with the player selecting the smallest unique integer receiving a utility of one, while the other players score nothing. If there is no lowest unique integer then all players score zero.

In Zeng *et al.* [1] the assumption is made that "...a player is indifferent between two strategies conditioned on the other players' choices and a player will always pick the lowest number. Arguably this might be too strong." In the following section we show the latter assumption is indeed too strong.

We make use of the following game-theoretic concepts: A *strategy profile* is a set of strategies, one for each player; a *Nash equilibrium* (NE) is a strategy profile from which no player can improve their payoff by a unilateral change in strategy; a *Pareto optimal* (PO) strategy profile is one from which no player can improve their result without someone else being worse off.

2. Nash Equilibrium for the $n = 3, 4$ Player Cases

It is easy to show that the solution of Zeng *et al* is not a NE for small n. In the three player case, if Bob and Charles adopt the strategy $(\frac{1}{2}, \frac{1}{2}, 0)$, where the ith number in the parentheses is the probability of selecting the integer i, Alice can maximize her payoff by selecting the strategy $(0, 0, 1)$. In this case Alice wins when ever the other two choose the same integer. Hence her expected payoff is $\frac{1}{2}$, double that obtained by selecting the strategy $(\frac{1}{2}, \frac{1}{2}, 0)$. Bob and Charles win in only $\frac{1}{4}$ of the cases. This result is an (asymmetric) NE. Given that $\$_{\text{A}} + \$_{\text{B}} + \$_{\text{C}} = 1$ the result is also PO: the sum of the payoffs is maximal so no other strategy profile can give one player a higher payoff without someone else being worse off.

For $n = 4$ there is an analogous solution. If Bob, Charles and Debra play the strategy $(\frac{1}{2}, \frac{1}{2}, 0, 0)$, Alice's optimal play is to select '3' with probability one. Then she wins if the others have all selected '1' or all selected '2'. The expected payoff to all players is $\frac{1}{4}$ and so the equilibrium is fair to all players. Again this solution is a NE and is PO with the maximum possible sum of payoffs (one). For $n > 4$, the strategy "always choose '3' " is no longer optimal against a group of players choosing a mixed '1' or '2' strategy and there is no simple analogue to the above NE strategy profiles.

Asymmetric strategy profiles such as those given above are difficult to realize in practice since in the absence of communication it is not possible to decide on who plays the odd strategy. We will now search for a symmetric NE strategy profile where all the players choose the same (mixed) strategy. Suppose all players but Alice choose the strategy $(p_1, p_2, \ldots, p_n)$, while Alice plays $(\pi_1, \pi_2, \ldots, \pi_n)$, with the normalization conditions[1] $\Sigma p_i = \Sigma \pi_i = 1$. In the end we will set $\pi_i = p_i \ \forall i$ to give a symmetric strategy profile. For Alice's strategy to yield her maximum payoff (given the others' strategies) it is necessary, though not sufficient, for $d\$_{\text{A}}/d\pi_i = 0, \ \forall i$.

Using the normalization conditions to substitute for p_n and π_n, we can write Alice's expected winnings as

$$\begin{aligned} \$_{\text{A}} = \pi_1(1-p_1)^{n-1} &+ \left(1 - \sum_{k=1}^{n-1} \pi_k\right) \sum_{j=1}^{n-1} p_j^{n-1} \\ &+ \sum_{i=2}^{n-1} \pi_i \left[(1 - \sum_{j=1}^{i} p_j)^{n-1} + \sum_{j=1}^{i-1} p_j^{n-1} \right] . \end{aligned} \tag{1}$$

By differentiating with respect to each of the π_i and setting the result equal to zero $n-1$ non-linear coupled equations in the $n-1$ variables $p_1, \ldots p_{n-1}$ are obtained. Amongst the simultaneous solutions of these equations will be one that is maximal for Alice. By setting $\pi_i = p_i \ \forall \, i$ we obtain a strategy that is maximal for all players and is thus a NE. We note that the derivatives of $\$_{\text{A}}$ do not involve the π_i.

For the case of $n = 3$ we have

$$\$ = \pi_1(1-p_1)^2 \; + \; \pi_2 \left(p_1^2 + (1-p_1-p_2)^2\right) \; + \; (1-\pi_1-\pi_2)(p_1^2+p_2^2) \, , \tag{2}$$

[1]In addition, if Alice picks either n or $n-1$ she can only win if all the other players have chosen the same integer. This will mean that for the NE strategy $\pi_{n-1} = \pi_n$. However, in the following analysis we shall not make use of this relation.

(where the subscript A has been dropped for simplicity) resulting in

$$\frac{d\$}{d\pi_1} = 1 - 2p_1 - p_2^2\,, \tag{3a}$$

$$\frac{d\$}{d\pi_2} = 1 - 2p_1 + p_1^2 - 2p_2 + 2p_1p_2\,. \tag{3b}$$

This has the unique (for the physical range of p_1, p_2) solution

$$p_1 = 2\sqrt{3} - 3\,, \qquad p_2 = 2 - \sqrt{3}\,. \tag{4}$$

We note that $p_2 = 1 - p_1 - p_2$, as observed in the earlier footnote. When Bob and Charles play the strategy (4), that is, when they select '1' with probability $2\sqrt{3} - 3 \approx 0.464$ and '2' or '3' each with probability $2 - \sqrt{3} \approx 0.268$, Alice's payoff is independent of her strategy. The game being symmetric, the same is true for any of the players when the other two choose (4). Thus, no player can improve their strategy by a unilateral change in strategy, demonstrating that (4) is a NE. When all players select this strategy, the expected payoff to each is $4(7 - 4\sqrt{3}) \approx 0.287$, which is higher than the payoff of 0.25 that results when each player selects only between '1' or '2' with equal probability, the "rational" player result of Ref. [1]. It is interesting, and some what anti-intuitive, that the solution involves a non-zero value for $p_3 = 1 - p_1 - p_2$ since '3' can never be the lowest integer, though it can be the only unique integer.

Proceeding in the same manner for $n = 4$, (1) reduces to

$$\begin{aligned} \$ &= \pi_1(1-p_1)^3 + \pi_2[p_1^3 + (1-p_1-p_2)^3] \\ &\quad + \pi_3\,[p_1^3 + p_2^3 + (1-p_1-p_2-p_3)^3] + (1-\pi_1-\pi_2-\pi_3)(p_1^3+p_2^3+p_3^3). \end{aligned} \tag{5}$$

Differentiating with respect to each of π_1, π_2, and π_3 and setting the results equal to zero gives the unique (physical) solution

$$p_1 \approx 0.488\,, \qquad p_2 \approx 0.250\,, \qquad p_3 \approx 0.131\,, \tag{6}$$

again with the relationship $p_3 = 1 - p_1 - p_2 - p_3$. The exact values for the p_i are complicated and unilluminating. The payoff to each player when they all choose the strategy (6), that is, when each player selects '1' with probability ≈ 0.488, '2' with probability ≈ 0.250 and '3' or '4' each with probability ≈ 0.131, is approximately 0.134. This is higher than that obtainable if all the players simply select between '1' and '2' (0.125). Again, when three players choose (6) the payoff to the fourth player is independent of their strategy, demonstrating that the strategy profile is a NE. Note the symmetric mixed strategy NE profiles have lower average payoffs than the asymmetric ones found earlier.

3. Approximate Solution for an Arbitrary Number of Players

In general, since we have $n - 1$ coupled equations of degree $n - 1$, for $n > 5$ no analytic solution will be possible, and for $n = 5$ the solution will be problematic. By inspection of (4) and (6) the mixed strategy with

$$\pi_i = \frac{1}{2^i} \quad \text{for } i < n, \qquad \pi_n = \pi_{n-1} = \frac{1}{2^{n-1}}\,, \tag{7}$$

Table 1. The payoff for the approximate symmetric Nash equilibrium solution of strategy (7) along with the rational player payoffs from Ref. [1] and the exact symmetric Nash equilibrium payoffs of strategies (4) and (6), for the three- and four-player cases, respectively. Exact solutions for the other cases have not be calculated. Payoffs have been rounded to three significant figures.

n	3	4	5	6	7	8
Equation (8)	0.281	0.133	0.0645	0.0317	0.0157	0.00784
Reference [1]	0.25	0.125	0.0625	0.0313	0.0156	0.00781
Exact	0.287	0.134				

is an approximation to the symmetric NE solutions for $n = 3, 4$. Equation (7) is in keeping with our intuition by giving higher weights to the selection of smaller integers. The payoff to each player for $n > 2$ if all select (7) is

$$\$ = \sum_{k=1}^{n-1} \left[\frac{1}{2^k} \sum_{j=1}^{k} \left(\frac{1}{2^j} \right)^{n-1} \right] + \frac{1}{2^{n-1}} \sum_{j=1}^{n-1} \left(\frac{1}{2^j} \right)^{n-1} . \tag{8}$$

For $n = 3$ the payoff is $\frac{9}{32} \approx 0.281$ and for $n = 4$ it is ≈ 0.133, both very close to the values for the exact symmetric NE given in the previous section. The payoff (8) as a function of n is shown in Table 1, along with the payoffs from Ref. [1] of $1/2^{n-1}$ and the exact solutions for the $n = 3$ and 4 cases. The payoff given in Ref. [1] is slightly smaller than the payoff given by (8) but will asymptote to it as n increases.

4. Conclusion

We have found both asymmetric and symmetric NE strategy profiles for a three- and four-player lowest unique positive integer game with payoffs superior to that resulting from the simplifying assumption of Ref. [1]. In particular the assumption that a player will always choose the lowest integer in a situation where they have a choice results in a strategy that is not a NE. The asymmetric NE are also PO, and in the case of $n = 4$, is fair to all players. The symmetric solutions are unique amongst symmetric strategy profiles but yield a lower payoff than the asymmetric solutions. Anti-intuitively, the NE strategy profiles includes a non-zero probability for selecting the largest integer since this may be the only unique integer.

For arbitrary n, we propose a simple symmetric strategy profile with geometrically decreasing probabilities of selecting higher integers. This gives very close to the payoffs of the exact symmetric NE solutions for the two cases for which exact solutions were obtained. The rational player solution of Ref. [1] is simpler than ours but gives payoffs slightly smaller.

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References

[1] Q. Zeng, B. R. Davis and D. Abbott, Reverse auction game: the lowest unique positive integer game, *Fluct. Noise Lett.* **7** (2007) L439–L447.