Hot Topics in Physical Information (HoTPI-2013)
International Journal of Modern Physics: Conference Series
Vol. 33 (2014) 1460360 (9 pages)

DOI: 10.1142/S2010194514603603

Allison mixtures: Where random digits obey thermodynamic principles

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Published 17 September 2014

Parrondo's paradox is a well-known situation where losing strategies can be combined to win, and is based on a thermodynamic Brownian ratchet principle. We demonstrate a new extension of Parrondo's paradox called the *Allison mixture* where it is possible to paradoxically generate random sequences with nonzero autocovariance, out of subsequences of zero autocovariance. We explain the effect with a ratchet analogy and show the behavior of random digits follows a thermodynamic analogy. As Kish-based cipher techniques rely on the Second Law for their security, this raises the tantalizing open question of whether the Allison mixture can be exploited for new Kish-inspired digital security schemes.

Keywords: Parrondo's paradox; Allison mixture; Kish cipher; Brownian ratchets.

PACS number: 05.40.Jc, 05.70.Ln, 05.70.Ln, 11.30.Qc, 02.50.Cw

1. Introduction

The independence of the samples of a stochastic process appears to be a feature that cannot easily be destroyed. However, it has been suggested[1] that a random mixture of two random sequences of digits can give rise to a resultant sequence with a nonzero autocovariance. Epstein has called this process the *Allison mixture.*[2]

This phenomenon is not dissimilar to *Parrondo's paradox*[3,?], in which games of chance that are individually biased against the player can be mixed in order to achieve a gain overall.

Parrondo's paradox and thermodynamics are closely related; indeed, the former has its genesis in the theory of Brownian ratchets.[4] The flashing ratchet[5,6] transports particles using Brownian motion by repeated alternation of a potential. The potential has the appearance of a sawtooth, allowing net drift in one direction, but when switched on and off the particles drift in the opposite direction.

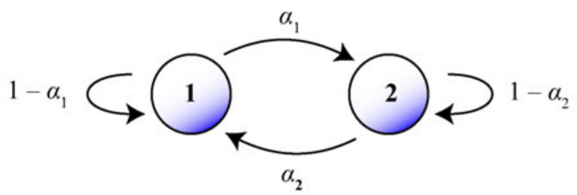


Fig. 1. The Markov chain governing the Allison mixture process. When in state one, its value is equal to that of the first process U, and when in state two to that of the second V.

A related phenomenon in finance is *volatility pumping*[7] whereby one regularly rebalances a portfolio to maintain a 50:50 split between, say, a volatile stock and a mediocre low-performing stock. It is surprising that it results in a theoretical exponential growth in capital.[4] Volatility pumping is the result of an asymmetry that rectifies fluctuations in the market, which means it is acting as a type of Brownian ratchet. The action of maintaining the 50:50 portfolio split guarantees that we are always buying low and selling high, which is a form of ratcheting asymmetry.[4]

The original motivation for the development of the Allison mixture is from the statistical modelling of language.[1] Here, the word repetition intervals are modelled as a Poisson process whose rate parameter occasionally switches between a higher and a lower value according to whether the word is a topic of discussion.

2. The Allison mixture

The Allison mixture is a random process formed by the sampling of a pair of base processes, and is defined as follows:

Definition 2.1. The Allison mixture

Let U_t and V_t be a pair of white, stationary, and independent random processes, and S_t be a binary random process whose values form a two-state Markov chain, without absorbing states. The Allison mixture is a random process X_t such that

$$X_t = U_t S_t + V_t(1 - S_t). \tag{1}$$

That is, $X_t = U_t$ when the Markov chain is in state one, and $X_t = V_t$ when in state two.

The behavior of the Allison mixture is determined by its sampling process, a Markov chain such as in Figure 1. The parameters α_1 and α_2 determine the rate at which the mixture will switch between processes. The requirement that the Markov chain have no absorbing states excludes the cases where $\alpha_1 = 0$ or $\alpha_2 = 0$, in which states one and two respectively are absorbing.

Theorem 2.1. ***The autocovariance function of the Allison mixture***
The Allison mixture X_t has autocovariance function

$$R_{XX}(\tau) = (E[U] - E[V])^2 R_{SS}(\tau). \tag{2}$$

Proof. *The autocovariance function of X is defined by Ref. 8 (p. 289) as*

$$R_{XX}(\tau) = E[X_t X_{t+\tau}] - E[X_t]E[X_{t+\tau}]. \tag{3}$$

Our first step in evaluating this function is to consider the product

$$\begin{aligned} X_t X_{t+\tau} = U_t U_{t+\tau} S_t S_{t+\tau} \\ + V_t V_{t+k}(1 - S_t)(1 - S_{t+\tau}) \\ + U_t V_{t+k} S_t (1 - S_{t+\tau}) \\ + V_t U_{t+k}(1 - S_t) S_{t+\tau}, \end{aligned} \tag{4}$$

and its expectation, which by the independence, whiteness, and stationarity of U and V is equal to

$$\begin{aligned} E[X_t X_{t+\tau}] &= E[U_t]^2 E[S_t S_{t+\tau}] + E[V_t]^2 \left(1 - 2E[S_t] + E[S_t S_{t+\tau}]\right) && (5) \\ &\quad + 2E[U_t]E[V_t] \left(E[S_t] - E[S_t S_{t+\tau}]\right). \\ &= E[U_t]^2 (R_{SS}(\tau) + E[S_t]^2) && (6) \\ &\quad + E[V_t]^2 (R_{SS}(\tau) + (1 - E[S_t])^2) \\ &\quad + 2E[U_t]E[V_t] \left(E[S_t](1 - E[S_t]) - R_{SS}(\tau)\right). \end{aligned}$$

Similarly, we may write

$$\begin{aligned} E[X_t] = E[X_{t+\tau}] &= E[U_t S_t + V_t(1 - S_t)] && (7) \\ &= E[U_t]E[S_t] + E[V_t](1 - E[S_t]), && (8) \end{aligned}$$

and so

$$\begin{aligned} R_{XX}(\tau) &= E[X_t X_{t+\tau}] - E[X_t]E[X_{t+\tau}] && (9) \\ &= E[U_t]^2 (R_{SS}(\tau) + E[S_t]^2) && (10) \\ &\quad + E[V_t]^2 (R_{SS}(\tau) + (1 - E[S_t])^2) \\ &\quad + 2E[U_t]E[V_t] \left(E[S_t](1 - E[S_t]) - R_{SS}(\tau)\right). \\ &\quad - \left(E[U_t]E[S_t] + E[V_t](1 - E[S_t])\right)^2 \\ &= E[U_t]^2 R_{SS}(\tau) + E[V_t]^2 R_{SS}(\tau) + 2E[U_t]E[V_t] R_{SS}(\tau) && (11) \\ &= (E[U] - E[V])^2 R_{SS}(\tau). && (12) \end{aligned}$$

That is to say, the autocovariance of the sampling process is scaled by the squared difference of means. □

We now focus our attention on the Allison mixture as defined above; we determine the single-step autocovariance of the sampling process before applying the scaling factor given by (2).

Theorem 2.2. ***The lag-one autocovariance of the Allison mixture***

The Allison mixture X_t associated with a fully mixed sampling process S_t as in Figure 1 has a lag-one autocovariance of

$$R_{XX}(1) = (E[U] - E[V])^2 \frac{\alpha_1\alpha_2}{(\alpha_1+\alpha_2)^2}(1-\alpha_1-\alpha_2). \tag{13}$$

Proof. *We begin by analysing the Markov process S_t with structure given by Figure 1. As it is fully mixed, the distribution of the states is equal to the stationary distribution*[9] *π_i of the chain. These may be calculated as*

$$\pi_1 = \frac{\alpha_2}{\alpha_1+\alpha_2} \tag{14}$$

$$\pi_2 = \frac{\alpha_1}{\alpha_1+\alpha_2}. \tag{15}$$

We may then write

$$\begin{aligned} E[S_t] &= \pi_2 && (16)\\ E[S_tS_{t+1}] &= \sum_{s_t,s_{t+1}} S_tS_{t+1}P[S_t = s_t \cap S_{t+1} = s_{t+1}] && (17)\\ &= P[S_t = 1 \cap S_{t+1} = 1] && (18)\\ &= P[S_{t+1} = 1 | S_t = 1]\pi_2 && (19)\\ &= (1-\alpha_2)\pi_2 && (20) \end{aligned}$$

and so its lag-one autocovariance

$$\begin{aligned} R_{SS}(1) &= E[S_tS_{t+1}] - E[S_t]E[S_{t+1}] && (21)\\ &= \pi_2(1-\alpha_2-\pi_2) && (22)\\ &= \pi_1\pi_2(1-\alpha_1-\alpha_2) && (23)\\ &= \frac{\alpha_1\alpha_2}{(\alpha_1+\alpha_2)^2}(1-\alpha_1-\alpha_2). && (24) \end{aligned}$$

We substitute this into (2), resulting in autocovariance

$$R_{XX}(1) = (E[U] - E[V])^2 \frac{\alpha_1\alpha_2}{(\alpha_1+\alpha_2)^2}(1-\alpha_1-\alpha_2) \tag{25}$$

as originally stated. □

For simplicity, if we put $\mu_1 = E[U]$ and $\mu_2 = E[V]$, we can write,

$$R_{XX}(1) = (\mu_1-\mu_2)^2 \frac{\alpha_1\alpha_2}{(\alpha_1+\alpha_2)^2}(1-\alpha_1-\alpha_2) \tag{26}$$

From this result we may trivially extract the necessary conditions for correlation.

Corollary 2.1. *Consecutive samples of the Allison mixture are correlated if and only if all of the following are true:*

$$\alpha_1 \neq 0 \tag{27}$$

$$\alpha_2 \neq 0 \tag{28}$$

$$\alpha_1 + \alpha_2 \neq 1 \tag{29}$$

$$\mu_1 \neq \mu_2. \tag{30}$$

This result might at first seem counterintuitive, as the processes from which the elements are drawn exhibit no time dependence. This phenomenon can be explained by imagining two processes U_t and V_t with very different means. Then, from the value of X_t one may determine S_t with reasonable certainty. If the switching probability of the current state is small, then one would expect X_{t+1} to be drawn from the same process, and so be similar in value to X_t. This is the source of the correlation between subsequent values.

3. Numerical results

The relationship between the probabilities α_1 and α_2 and the autocorrelation coefficient $\rho = R_{XX}(1)/\operatorname{Var}(X)$ is shown in Figure 2.

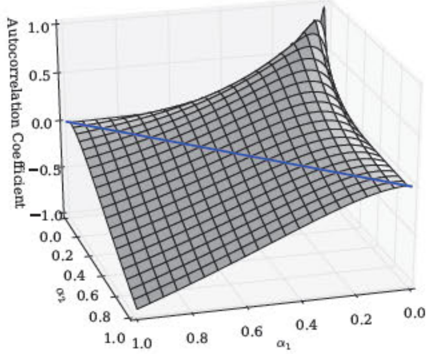
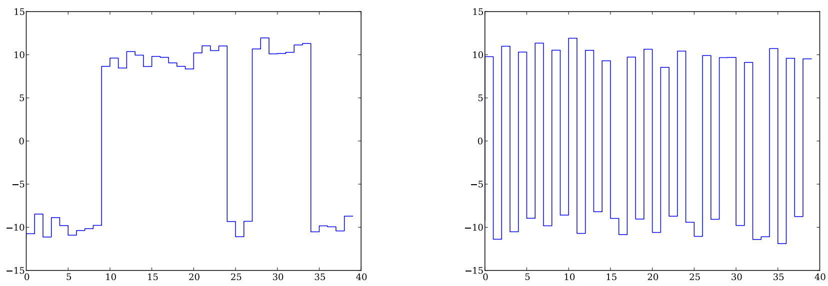


Fig. 2. The autocorrelation coefficient of the Allison mixture of $N(-10, 1)$ and $N(10, 1)$ for various values of α_1 and α_2. The thick line shows the case $\alpha_1 + \alpha_2 = 1$ in which the autocorrelation coefficient is zero.

We show the results of simulation in Figure 3. When the parameters α are small, the process switches states only rarely and has a large positive autocovariance. When the switching probabilities are large, the process will switch almost every time, causing it to flit back and forth between input processes and so have a large negative autocovariance.



(a) $\alpha_1 = \alpha_2 = 0.1$ (b) $\alpha_1 = \alpha_2 = 0.9$

Fig. 3. The Allison mixture of $N(-10, 1)$ and $N(10, 1)$ with varying parameters α_i. In (a) the low probability of switching causes the process to stay with its current input for long periods of time. The autocorrelation coefficient is large and positive. Conversely, in (a) the probability of switching is high, causing the sampling operation to flit between the two processes almost every cycle. The autocovariance is large and negative.

4. The Allison mixture and Brownian ratchets

Let us imagine two sequences of random numbers; we shall call them Sequence 1 and Sequence 2—they are totally random in that they are independent and have zero autocovariance. For simplicity, we can consider these sequences to be random strings of 1s and 0s—however, note that the effect is not limited to binary sequences but is in fact general. If we now mix these two sequences to generate a third new sequence, naively we would expect this resulting sequence to also be completely random.

It turns out that this is not always the case: counterintuitively the final sequence can have a finite autocovariance coefficient ρ even though the ρ's of originating sequences are zero—this is what we call an *Allison mixture.*

Let us now be a little bit more precise about how we actually scramble the two sequences. We start at an arbitrary nth position of Sequence 1. We either move to position $n + 1$ of Sequence 1 with probability $1 - \alpha_1$ or skip to position $n + 1$ of Sequence 2 with probability α_1. Whenever we find ourselves in Sequence 2, we hop to the next location on Sequence 1 with probability α_2 or advance one step within Sequence 2 with probability $1-\alpha_2$. We continue hopping back and forth between the two sequences in this manner and each digit that we sequentially land on is called out to form the new sequence. In this way a third new sequence is generated from the original two sequences by random hopping, using separate transition probabilities α_1 and α_2 to keep everything perfectly general. For the sake of further generality, let the means of the two originating sequences be μ_1 and μ_2.

Our naive expectation is that a random mixture of random sequences should always result in $\rho = 0$—however, Equation 26 reveals that R_{XX}, and hence ρ, is only zero provided $\mu_1 = \mu_2$ or $\alpha_1 + \alpha_2 = 1$. If we break both these conditions, then we can legally produce a sequence with a nonzero ρ. The mathematics dictates to us that this must be the case, but the question is why? And what is the physical picture and basis for what is going on?

First, let us address physically why we must have $\mu_1 \neq \mu_2$, in order to obtain $\rho \neq 0$—the implication is that the means of the sequences are analogous to the temperature of a physical process. Loosely speaking, temperature is some measure that is proportional to the average of all the jiggling within a solid object. In the case of the random sequences, μ_1 and μ_2 are the averages of all the jiggling or varying numbers and play the same role as temperature. Thus when $\mu_1 \neq \mu_2$, we have an irreversible situation—the sequences are irreversibly mixed and we therefore obtain a nonzero autocovariance in the final sequence, because there is information loss.

Recall that the originating sequences are random, and thus are incompressible, and thus contain maximal information in the Chaitin-Kolmogorov sense. Thus by subjecting them to an irreversible process we know from thermodynamics that we must *lose* information, and thus redundancy must have crept into the final sequence leading to $\rho \neq 0$. Now, in the special case, when $\mu_1 = \mu_2$, we have reversible mixing, because this is analogous to thermal equilibrium where $T_1 = T_2$. If the process is reversible then there is no information loss, no redundancy is added, and therefore $\rho = 0$.

However, this is only part of the picture, as Equation 26 also predicts that to obtain a case where $\rho \neq 0$, we must also observe the $\alpha_1 + \alpha_2 \neq 1$ condition. So what is the physical reason why $\alpha_1 + \alpha_2 \neq 1$ is required to obtain nonzero ρ in the final sequence?

Consider the case of Figure 1, for the case when $\alpha_1 + \alpha_2 \neq 1$. Circle 1 represents the state of landing in random Sequence 1 and Circle 2 represents the state we are in when we land on random Sequence 2. We jump between these two sequences to generate a new sequence. The apparent paradox of this Allison mixture is that the resulting sequence has nonzero autocovariance even when the originating sequences have zero autocovariance. Here, α_1 and α_2 are the transition probabilities of jumping between the two sequences. Notice, for example, as $\alpha_1 \to 0$ the probability of a self-transition to stay in Sequence 1 is high—so, if we are already in Sequence 1 we are likely to stay there. This can be thought of as a type of 'memory' of the system, which causes the new sequence to have nonzero autocorrelation. Note: this is *not* a form of memory in the sense that requires storage of a previous state—as there is no clear terminology, in the literature, for our probabilistic type of memory effect, so we hereby call it *memory persistence*

Now imagine what happens to the state diagram, of Figure 1, when we have a balanced case of $\alpha_1 + \alpha_2 = 1$, resulting in no memory persistence. For simplicity, insert $p = \alpha_1 = 1-\alpha_2$ in the diagram, and this clearly reveals that the probability of entering a state exactly balances the probability of self-transition in that state. This detailed balance implies there is no memory persistence effect. Alternatively, we can see that the probability of entering a state is independent of the current state, and so the memory persistence effect vanishes. Hence, the new generated sequence also has zero autocovariance.

In summary, for the case $\alpha_1 + \alpha_2 = 1$, we obtain detailed balance between the probability of entering a state and the probability of staying in a state. (Note that staying within a state is also called a *self-transition*). The detailed balance implies there is no memory persistence and hence $\rho = 0$. An alternative valid explanation is to use the argument that detailed balance implies a reversible process. So essentially we must have $\alpha_1 + \alpha_2 \neq 1$ to ensure irreversibility, which is a necessary condition for obtaining $\rho \neq 0$.

We now begin to see the connection between an Allison mixture and Parrondian effects that both require an asymmetry to interact with random behavior. We have a symmetric case where we have detailed balance, and an asymmetric case where detailed balance is broken. Symmetry breaking is the essence of all Parrondian and Brownian ratchet phenomena. The $\mu_1 \neq \mu_2$ condition is analogous to the $T_1 \neq T_2$ condition that is required for Feynman's ratchet to operate. The $\alpha_1 + \alpha_2 \neq 1$ condition is analogous to a ratchet (eg. an asymmetric sawtooth).

An open question is that of a possible application for Allison mixtures—this remains to be seen, but possible areas of promise might be in encryption and in optimizing file compression. Kish has made the intriguing suggestion that the Second Law of thermodynamics can be exploited to guarantee information security and has proposed analog schemes to achieve this. [10] However, can this now be extended to purely digital manipulations by exploiting the thermodynamic properties of the Allison mixture? This remains a tantalizing open question.

Another open question to ask is if there are any links between Allison mixtures and biological evolution or genetics? Could it be that the redundancy that appears in sequences of non-coding (or 'junk') DNA are the result of something along the lines of Allison mixing (i.e. ratcheted random mixing)?

5. Conclusion

We have derived the lag-one autocovariance of the Allison mixture and thereby found a set of conditions both necessary and sufficient for its consecutive samples to become correlated, from random mixtures of random subsequences. We have explained this phenomenon by analogy to the Brownian ratchet, thereby demonstrating that sequences of numbers intriguingly obey thermodynamic principles.

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