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Quantum walks with history dependence

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Abstract
We introduce a multi-coin discrete quantum walk where the amplitude for a coin flip depends upon previous tosses. Although the corresponding classical random walk is unbiased, a bias can be introduced into the quantum walk by varying the history dependence. By mixing the biased walk with an unbiased one, the direction of the bias can be reversed leading to a new quantum version of Parrondo's paradox.

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1. Introduction

Random walks have long been a powerful tool in mathematics, have a number of applications in theoretical computer science [1, 2] and form the basis for much computational physics, such as Monte Carlo simulations. The recent flourish of interest in quantum computation and quantum information [3, 4] has lead to a number of studies of quantum walks both in continuous [5, 6] and in discrete time [7–11]. Meyer has shown that a discrete time, discrete space, quantum walk requires an additional degree of freedom [8], or quantum 'coin', and can be modelled by a quantum lattice gas automaton [12]. Quantum walks reveal a number of startling differences to their classical counterparts. In particular, diffusion on a line is quadratically faster [13, 14]. Quantum walks show promise as a means of implementing quantum algorithms. Childs *et al* [15] prove that a continuous-time quantum walk can find its way across some types of graphs exponentially faster than any classical random walk, while a discrete-time, coined quantum walk is able to find a specific item in an unsorted database with a quadratic speedup over the best classical algorithm [16], a performance equal to Grover's algorithm. Several methods for implementing quantum walks have been proposed, including in an ion trap computer [14], on an optical lattice [17], and using cavity quantum electrodynamics [18]. A simple continuous-time quantum walk has been experimentally demonstrated in a nuclear magnetic resonance machine [19]. A recent overview of quantum walks is given by Kempe [20].

Parrondo's paradox arises where a combination of two losing games result in a winning game [21–23]. Such an effect can occur when one game has a form of feedback, for example, through a dependence on the game state [24], through the outcomes of previous games [25], or through the states of neighbours [26], that leads to a negative bias. When this feedback is disrupted by mixing the play with a second losing game that acts as a source of noise, a net positive bias may result. The recent attention attracted by classical versions of Parrondo's games is motivated by their relation to physical systems such as flashing ratchets or Brownian motors [27–30], or systems of interacting spins [31]. Applications in fields as diverse as population genetics [23], biogenesis [32], economics and biochemistry [33] have been suggested. Quantum equivalents to Parrondo's games with a pay-off dependence [12] or a history dependence [34, 35] have been demonstrated. A link between quantum Parrondo's games and quantum algorithms has been discussed [36, 37]. Recent reviews of classical and quantum Parrondo's games can be found in [38] and [39], respectively. In this paper we develop a model of a quantum walk with history dependence and detail its main features. We show that this can lead to a new quantum version of Parrondo's paradox.

The paper is organized as follows. Section 2 gives a brief summary of the classical Parrondo's games and their quantum analogues, section 3 sets out the mathematical formalism of our scheme, section 4 gives some results for the random walk of a single particle on a line with this scheme, while section 5 demonstrates a new quantum Parrondo effect.

2. Parrondo's games

The original Parrondo's games were cast in the form of a pair of gambling games: game A the toss of a simple biased coin with winning probability $p = \frac{1}{2} - \epsilon$, and game B consisting of two biased coins, the selection of which depends upon the state of the game. Coin B_1, with winning probability p_1, is selected when the capital is a multiple of 3, while coin B_2, with winning probability p_2, is chosen otherwise. Each coin toss results in the gain or loss of one unit of capital. With, for example,

$$p_1 = 1/10 - \epsilon \qquad p_2 = 3/4 - \epsilon \qquad \epsilon > 0 \tag{1}$$

game B is losing since the 'bad' coin B_1 is played more often than the one-third of the time that one would naively expect. By interspersing plays of games A and B, the probability of selecting B_1 approaches $\frac{1}{3}$, and that game produces a net positive result that can more than offset the small loss from game A, when ϵ is small. The combination of the two losing games to form a winning one is the essence of the apparent paradox first described by Parrondo [21].

Meyer and Blumer [12] were the first to present a quantum version of this effect. In their model, the quantum analogue of the capital is the discretization of the position of a particle undergoing Brownian motion in one dimension. Each play of the game changes the particle position by ± 1 unit in the x direction. The biases of game A and B are achieved by the application of potentials

$$\begin{aligned} V_A(x) &= \alpha x \qquad \alpha > 0 \\ V_B(x) &= V_A(x) + \beta\big(1 - \tfrac{1}{2}(x \bmod 3)\big) \qquad \beta > 0 \end{aligned} \tag{2}$$

respectively. By adjusting the parameters of the potentials, the quantum games A and B can be made to yield similar negative biases to their classical counterparts. When switching between the potentials is introduced, the bias can be reversed for certain mixtures of A and B. For the

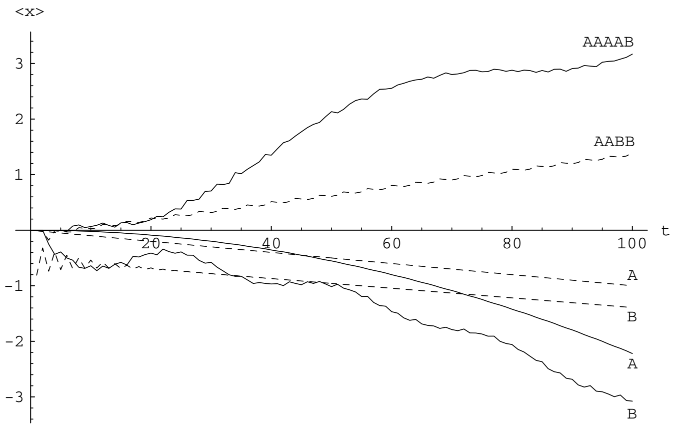

Figure 1. Mean position $\langle x \rangle$ as a function of time (in number of coin tosses) for (dashed lines) the classical games A, B and the repeated sequence AABB with $\epsilon = 0.005$ in equation (1), and (solid lines) the quantum games A, B and the repeated sequence AAAAB with $\alpha = \pi/2500$ and $\beta = \pi/3$ in equation (2). In the classical case, x is the player's capital with \$1 awarded for each winning coin toss and $-\$1$ for each losing toss. Here, x is the particle position and we assume full coherence is maintained in the quantum case. The difference in payoffs between the classical and quantum examples is due to the particular parameters chosen. However, interference in the quantum case produces a greater turn around in x than is obtainable in the classical situation.

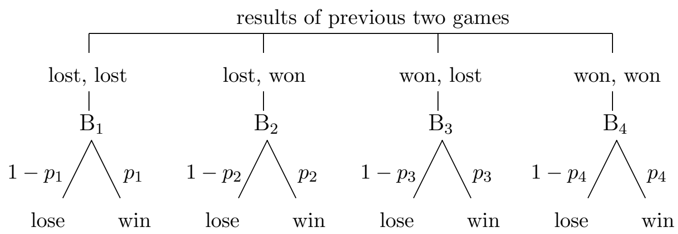

Figure 2. In the classical history-dependent Parrondo's game B, the selection of coins B_1 to B_4 depends upon the results of the last two plays, as shown. The probabilities of winning (increasing the player's capital by 1) are p_1 to p_4 and of losing (decreasing the player's capital by 1) are $1 - p_1$ to $1 - p_4$. The overall payoff for a series of games is the player's final capital.

classical and quantum versions, comparisons of the expectations for the individual games and an example of a winning combination are given in figure 1. For details of the classical case see Harmer and Abbott [24] and for the quantum case Meyer and Blumer [12].

A history-dependent game can be substituted for the above game B to produce a variant of Parrondo's games. Game B consists of four coins whose choice is determined by the results of the previous two games, as indicated in figure 2. An analysis of this game for

$$p_1 = 7/10 - \epsilon \qquad p_2 = p_3 = 1/4 - \epsilon \qquad p_4 = 9/10 - \epsilon \tag{3}$$

indicates that the game is losing for $\epsilon > 0$ [25]. Mixing this with game A or a different history-dependent game B [40] can yield an overall winning result. A direct quantization of this scheme is given by Flitney *et al* [35]. The quantum effects in this model depend upon the selection of a suitable superposition as an initial state. Interference can then arise since there may be more than one way of obtaining a particular state. Without interference, this scheme gives the same results as the classical history-dependent Parrondo's game. The

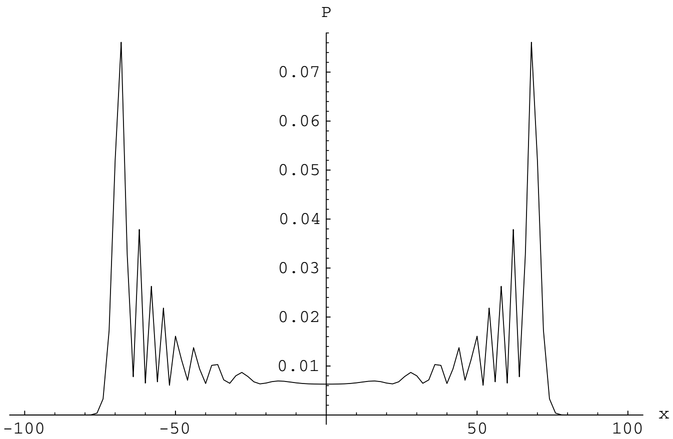

Figure 3. The distribution of probability density $P(x) = |\psi(x)|^2$ at toss $t = 100$ for an unbiased, single coin quantum walk with $|\psi_0\rangle = \frac{1}{\sqrt{2}}(|0, L\rangle - |0, R\rangle)$. Only even positions are plotted since $\psi(x)$ is zero for odd x at $t = 100$. The total area under the graph is equal to 1.

method presented in this paper uses an alternative approach, a discrete quantum walk or quantum lattice gas automaton.

3. Scheme formalism

A direct translation of a classical discrete random walk into the quantum domain is not possible. If a quantum particle moving along a line is updated at each step, in superposition, to the left and right, the global process is necessarily non-unitary. However, the addition of a second degree of freedom, the chirality, taking values L and R, allows interesting quantum walks to be constructed. Consider a particle whose position is discretized in one dimension. Let $\mathcal{H}_{\mathrm{P}}$ be the Hilbert space of particle positions, spanned by the basis $\{|x\rangle : x \in \mathbf{Z}\}$. In each time-step the particle will move either to the left or right depending on its chirality. Let $\mathcal{H}_{\mathrm{C}}$ be the Hilbert space of chirality, or 'coin' states, spanned by the orthonormal basis $\{|L\rangle, |R\rangle\}$. A simple quantum walk in the Hilbert space $\mathcal{H}_{\mathrm{P}} \otimes \mathcal{H}_{\mathrm{C}}$ consists of a quantum mechanical 'coin toss', a unitary operation $\hat{U}$ on the coin state, followed by the updating of the position to the left or right:

$$\hat{E} = (\hat{S} \otimes \hat{P}_{\mathrm{R}} + \hat{S}^{-1} \otimes \hat{P}_{\mathrm{L}})(\hat{I}_{\mathrm{P}} \otimes \hat{U}) \tag{4}$$

where $\hat{S}$ is the shift operator in position space, $\hat{S}|x\rangle = |x + 1\rangle$, $\hat{I}_P$ is the identity operator in position space, and $\hat{P}_R$ and $\hat{P}_L$ are projection operators on the coin space with $\hat{P}_R + \hat{P}_L = \hat{I}_C$, the coin identity operator. For example, a walk controlled by an unbiased quantum coin is carried out by the transformations

$$\begin{aligned} |x, L\rangle &\to \tfrac{1}{\sqrt{2}}\,(|x - 1, L\rangle + \mathrm{i}|x + 1, R\rangle) \\ |x, R\rangle &\to \tfrac{1}{\sqrt{2}}\,(\mathrm{i}|x - 1, L\rangle + |x + 1, R\rangle)\,. \end{aligned} \tag{5}$$

Figure 3 shows the distribution of probability density after 100 steps of equation (5) with the initial state $|\psi_0\rangle = (|0, L\rangle - |0, R\rangle)/\sqrt{2}$.[3] This initial state is chosen so that a symmetrical distribution results. In fact the states $|0, R\rangle$ and $|0, L\rangle$ evolve independently. We can see this

[3] This scheme is equivalent to the Hadamard quantum walk with initial state $\frac{1}{\sqrt{2}}(|0, L\rangle + \mathrm{i}|0, R\rangle)$.

since any flip $|R\rangle \leftrightarrow |L\rangle$ involves multiplication by a factor of i. Thus, any $|x, L\rangle$ state that started from $|0, R\rangle$ will be multiplied by an odd power of i and is orthogonal to any $|x, L\rangle$ state that originated from $|0, L\rangle$ (and similarly for the $|x, R\rangle$ states).

To construct a quantum walk with history dependence requires an extension of the Hilbert space by additional coin states. Where we have a dependence on the last $M-1$ results, the total system Hilbert space is a direct product between the particle position in one dimension and M coin states:

$$\mathcal{H} = \mathcal{H}_{\mathrm{P}} \otimes (\mathcal{H}_{\mathrm{C}}{}^{\otimes M}). \tag{6}$$

The M coins represent the results of tosses at times $t-1, t-2, \ldots, t-M$. A single step in the walk consists of tossing the Mth coin, updating the position depending on the result of the toss, and then re-ordering the coins so that the newly tossed coin is in the first (most recent) position. In general, the unitary coin operator $\hat{U}$ can be specified, up to an overall phase that is not observable, by three parameters, two of which are phases. In the single coin case the effect of the phases can be completely mimicked by changes to $|\psi_0\rangle$ [11]. This does not carry over to our multi-coin history-dependent scheme. However, for the sake of simplicity we shall omit the phases and simply write

$$\hat{U}(\rho) = \begin{pmatrix} \sqrt{\rho} & \mathrm{i}\sqrt{1-\rho} \\ \mathrm{i}\sqrt{1-\rho} & \sqrt{\rho} \end{pmatrix} \tag{7}$$

where $1-\rho$ is the classical probability that the coin changes state, with $\rho = \frac{1}{2}$ being an unbiased coin. To allow for history dependence, ρ will depend upon the results of the last $M-1$ coin tosses, so that a single toss is effected by the operator

$$\begin{aligned} \hat{E} = {} & \big(\hat{S} \otimes \hat{I}_{\mathrm{C}}{}^{\otimes(M-1)} \otimes \hat{P}_{\mathrm{R}} + \hat{S}^{-1} \otimes \hat{I}_{\mathrm{C}}{}^{\otimes(M-1)} \otimes \hat{P}_{\mathrm{L}}\big) \\ & \times \left(\hat{I}_P \otimes \sum_{j_1,\ldots,j_{M-1} \in \{L,R\}} \hat{P}^*_{j_1\ldots j_{M-1}} \otimes \hat{U}\big(\rho_{j_1\ldots j_{M-1}}\big) \right) \end{aligned} \tag{8}$$

where $\hat{P}_j$, $j \in \{L, R\}$ is the projection operator of the Mth coin onto the state $|j\rangle$ and $\hat{P}^*_{j_1\ldots j_{M-1}}$, $j_k \in \{L, R\}$ is the projection operator of the first $M-1$ coins onto the state $|j_1 \ldots j_{M-1}\rangle$. The second parenthesized term in (8) flips the Mth coin with a parameter ρ that depends upon the state of the first $M-1$ coins, while the first term updates the particle position depending on the result of the flip. Re-ordering of the coins is then achieved by

$$\hat{O} = \hat{I}_P \otimes \sum_{j_1,\ldots,j_M \in \{L,R\}} |j_M j_1 \ldots j_{M-1}\rangle\langle j_1 \ldots j_{M-1} j_M|. \tag{9}$$

This scheme is distinguished from Brun *et al*'s work on quantum walks with multiple coins [41] where the walk is carried out by cycling through a given sequence of M coins, $\hat{U}(\rho_1), \ldots, \hat{U}(\rho_M)$. In Brun's scheme, a coin toss is performed by

$$\hat{E} = \big(\hat{S} \otimes \hat{I}_{\mathrm{C}}{}^{\otimes(M-1)} \otimes \hat{P}_{\mathrm{R}} + \hat{S}^{-1} \otimes \hat{I}_{\mathrm{C}}{}^{\otimes(M-1)} \otimes \hat{P}_{\mathrm{L}}\big)\big(\hat{I}_{\mathrm{P}} \otimes \hat{I}_{\mathrm{C}}{}^{\otimes(M-1)} \otimes \hat{U}(\rho_k)\big) \tag{10}$$

where $k = (t \bmod M)$, and the step is completed by the $\hat{O}$ operator as before. The scheme has memory but not the dependence on history of the current method. The two schemes are only equivalent when all the ρ_k and $\rho_{j_1\ldots j_{M-1}}$ are equal, for example, when all the coins are

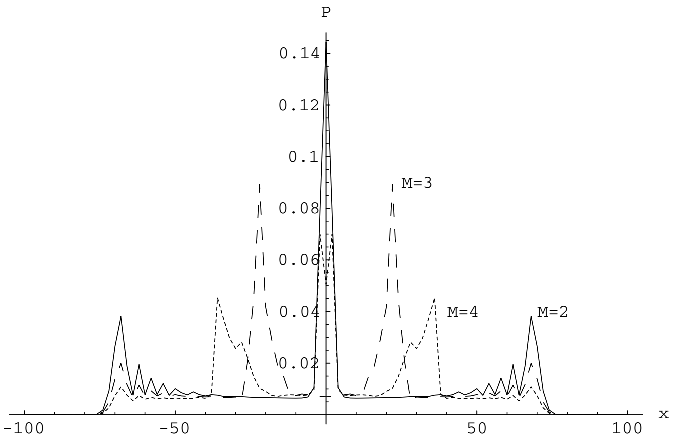

Figure 4. The probability density distributions $P(x) = |\psi(x)|^2$ at toss $t = 100$, for the 2- (——), 3- (– – –) and 4- (- - - -) coin unbiased, symmetrical, quantum walks. Only even positions are plotted since $\psi(x)$ is zero for odd x at $t = 100$. The area under each curve is equal to 1.

unbiased. This amounts to asserting that the scheme of Brun *et al* does not display Parrondian behaviour.

4. Results

The probability density distributions for unbiased 2, 3, and 4 coin history-dependent quantum walks, with initial states that are an equal superposition of the possible coin states antisymmetric as $L \leftrightarrow R$[4] are shown in figure 4. These distributions are essentially symmetric versions of the graphs of Brun *et al* [41] that result from an initial state $|\psi_0\rangle = |R\rangle^{\otimes M}$.

For arbitrary M we have, as for the $M = 1$ case, two parts of the initial state that evolve without interacting. Thus, for $M = 2$ for example, states arising from $|0, LL\rangle$ and $|0, RR\rangle$ will interfere, as will states arising from $|0, LR\rangle$ and $|0, RL\rangle$, but the two groups evolve into states that are orthogonal, for any given x. For the M coin quantum walk there are $M + 1$ peaks with even values of M having a central peak, the others necessarily being symmetrically placed around $x = 0$ by our choice of initial state. The outermost pair of peaks are in the same position as the peaks for $M = 1$ (figure 3) at $x(t) \approx 0.68t$. All the peaks are interference phenomenon, the central one being the easiest to understand. It arises since there are states centred on $x = 0$ that cycle back to themselves (i.e. that are stationary states over a certain time period). With $M = 2$, the simplest cycle over $t = 2$ is proportional to

$$\begin{aligned} |0, LR\rangle - |0, RL\rangle &\rightarrow \tfrac{1}{\sqrt{2}}(|{+1}, RL\rangle + \mathrm{i}|{-1}, LL\rangle - |{-1}, LR\rangle - \mathrm{i}|{+1}, RR\rangle) \\ &\rightarrow |0, LR\rangle - |0, RL\rangle. \end{aligned} \tag{11}$$

At the second step, complete destructive interference occurs for the states with $x = \pm 2$, so that there is no probability flux leaving the central three x values. In practice, the central region

[4] For example, with $M = 2$, the initial state is $|\psi_0\rangle = (|0, LL\rangle - |0, LR\rangle - |0, RL\rangle + |0, LL\rangle)/2$. For the purposes of this paper we could equally well have chosen an initial state that was symmetrical as $L \leftrightarrow R$. However, the antisymmetric starting state is the one that gives the correct behaviour in the presence of a potential. The state $|\psi_0\rangle$ is the quantum equivalent of the average over past histories that is taken in the classical history-dependent Parrondo game.

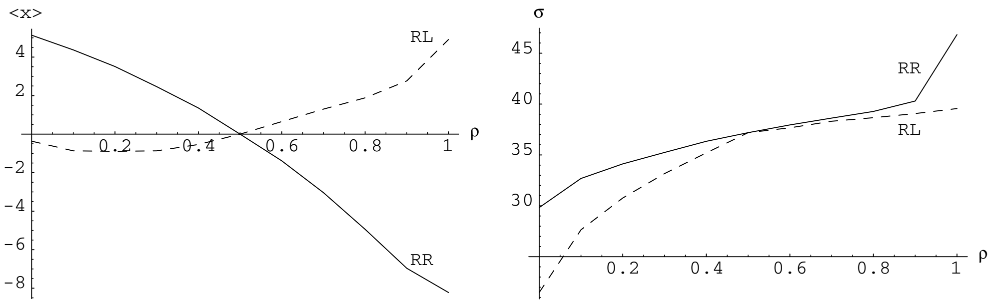

Figure 5. For the $M = 3$ quantum history-dependent walk, $\langle x\rangle$ and σ_x at time-step $t = 100$ as a function of ρ_{RR} (solid line) or ρ_{RL} (dashed line) while the other ρ_{ij} are kept constant at $\frac{1}{2}$. Varying ρ_{LL} has the opposite effect on $\langle x\rangle$ and the same on σ_x as varying ρ_{RR}. Similarly for ρ_{LR} compared to ρ_{RL}.

asymptotically approaches a more complex stationary cycle than (11), such as the $t = 2$ cycle

$$\begin{aligned}|\psi_{\mathrm{centre}}\rangle \propto{} & (a\mathrm{i} - b)(|-2, LL\rangle + |+2, RR\rangle) + (1 - a - \mathrm{i} + b\mathrm{i})(|-2, LR\rangle + |+2, RL\rangle) \\ & + (\mathrm{i} - 1)(|-2, RL\rangle + |+2, LR\rangle) + (b - a\mathrm{i})(|0, LL\rangle + |0, RR\rangle) \\ & + (a + b\mathrm{i})(|0, LR\rangle + |0, RL\rangle)\end{aligned} \tag{12}$$

where a and b are real.

Adjusting the values of the various ρ can introduce a bias into the walk. To create a quantum walk analogous to the history-dependent game B of section 2, requires $M = 3$, giving four parameters, $\rho_{\mathrm{RR}}, \rho_{\mathrm{RL}}, \rho_{\mathrm{LR}}$ and ρ_{LL}. Figure 5 shows the effect of individual variations in these parameters on the expectation value and standard deviation of the position after 100 time-steps.

5. Quantum Parrondo effect

It is useful to consider the classical limit to our quantum scheme. That is, the random walk that would result if the scattering amplitudes were replaced by classical probabilities. As an example consider the $M = 2$ case, with winning probabilities $1 - \rho_{\mathrm{L}}$ and $1 - \rho_{\mathrm{R}}$. The analysis below follows that of Harmer and Abbott [38]. Markov chain methods cannot be used directly because of the history dependence of the scheme. If, however, we form the vector

$$y(t) = [x(t-1) - x(t-2), x(t) - x(t-1)] \tag{13}$$

where $x(t)$ is the position at time t, then $y(t)$ forms a discrete time Markov chain between the states $[-1, -1]$, $[-1, +1]$, $[+1, -1]$ and $[+1, +1]$ with a transition matrix

$$T = \begin{pmatrix} \rho_{\mathrm{L}} & 1-\rho_{\mathrm{L}} & 0 & 0 \\ 0 & 0 & \rho_{\mathrm{R}} & 1-\rho_{\mathrm{R}} \\ 1-\rho_{\mathrm{L}} & \rho_{\mathrm{L}} & 0 & 0 \\ 0 & 0 & 1-\rho_{\mathrm{R}} & \rho_{\mathrm{R}} \end{pmatrix}. \tag{14}$$

Define $\pi_{ij}(t)$ to be the probability of $y(t) = [i, j], i, j \in \{-1, +1\}$. A state is now transformed by $T\pi$ at each time-step. Having represented the history-dependent game as a discrete time Markov chain, the standard Markov techniques can be applied. The equilibrium distribution is found by solving $T\boldsymbol{\pi}_{\mathrm{s}} = \boldsymbol{\pi}_{\mathrm{s}}$. This yields $\boldsymbol{\pi}_{\mathrm{s}} = [1, 1, 1, 1]/4$, giving a process with no net bias to the left or right irrespective of the values of ρ_{L} and ρ_{R}. The same analysis holds for $M > 2$. However, interference in the quantum case presents an entirely different picture.

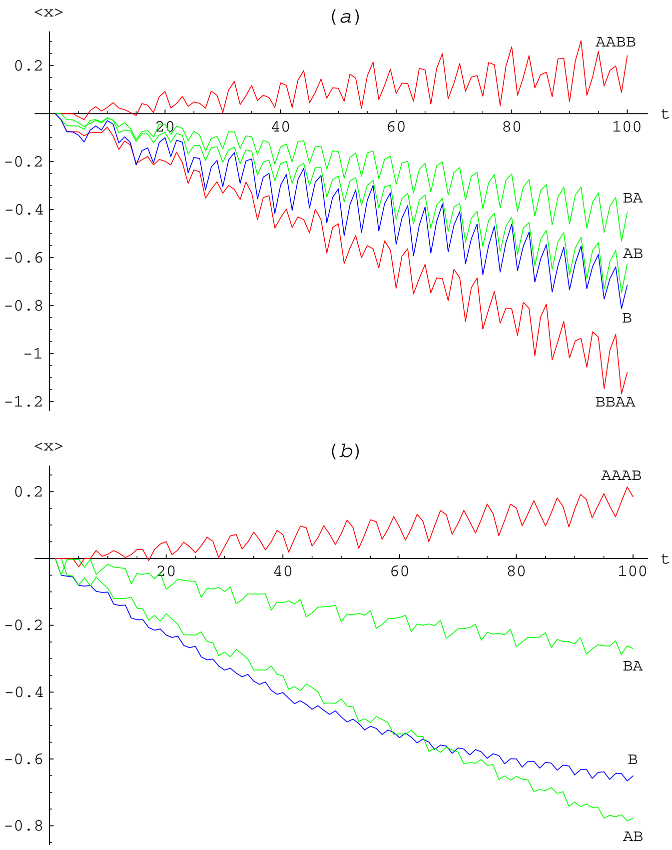

Figure 6. (Colour online) An example of a Parrondo effect for the $M = 3$ history-dependent quantum walk where game B has (*a*) $\rho_{\mathrm{RR}} = 0.55$ or (*b*) $\rho_{\mathrm{LR}} = 0.6$, with the other $\rho_{ij} = 0.5$, $i, j \in \{L, R\}$. Game A has all $\rho_{ij} = 0.5$ (unbiased). The letters next to each curve represent the sequence of games played repeatedly. For example, AB means apply $\hat{A}$ and then $\hat{B}$ to the state, repeating this sequence 50 times to get to $t = 100$.

(This figure is in colour only in the electronic version)

The comparison with the classical history-dependent Parrondo game requires $M = 3$. For game A, select the unbiased game, $\rho_{\mathrm{LL}} = \rho_{\mathrm{LR}} = \rho_{\mathrm{RL}} = \rho_{\mathrm{RR}} = 1/2$. For game B, choose, for example, $\rho_{\mathrm{RR}} = 0.55$ or $\rho_{\mathrm{LR}} = 0.6$ to produce a suitable bias (see figure 5). The operators associated with A and B are applied repeatedly, in some fixed sequence, to the state $|\psi\rangle$. For example, the results of the game sequence AABB after t time-steps is

$$|\psi(t)\rangle = (\hat{B}\hat{B}\hat{A}\hat{A})^{t/4}|\psi(0)\rangle. \tag{15}$$

Figure 6 displays $\langle x\rangle$ for various sequences. Of sequences up to length 4, with game B biased by $\rho_{\mathrm{RR}} > 0.5$ only AABB and AAB give a positive expectation, while when game B is biased by $\rho_{\mathrm{LR}} > 0.5$ only AAAB is positive. These results hold for ρ up to approximately 0.6, above which there are no positive sequences of length less than or equal to 4.

The sequences AABB and BBAA can be considered the same but with different initial states. That is, if instead of $|\psi_0\rangle$, we start with $|\psi_0'\rangle = \hat{A}\hat{A}|\psi_0\rangle$, BBAA gives the same results (displaced by two time-steps) as AABB does with the original starting state. In the classical case, altering the order of the sequence results in the same trend but with a small offset, as one might expect. However, as figure 6 indicates, the change of order in the quantum case can produce radically different results. This feature also appears in Meyer and Blumer's quantum Parrondo model—recall their model is based on a payoff-dependent scheme rather than a history-dependent one as in the present case.

6. Conclusion

A scheme for a discrete quantum walk with history dependence has been presented. Our system involves the use of multiple quantum coins. By suitable selection of the amplitudes for coin flips dependent on certain histories, the walk can be biased to give positive or negative $\langle x \rangle$. In common with many other properties of quantum walks, the bias results from interference, since the classical equivalent of our walks are unbiased. With a starting state averaged over possible histories, the average spread of probability density in our multi-coin scheme is slower than in the single coin case, with the appearance of multiple peaks in the distribution. For even numbers of coins there is a substantial probability of $x \approx 0$. However, the positions of the outermost peaks are the same as those of a single coin quantum walk. As the memory effect increases, the dispersion of the quantum walk decreases. One may speculate that this feature may be relevant to an understanding of decoherence, here considered as loss of coherence within the central portion of the graph around $x \approx 0$. In particular, the dispersion in the wavefunction decreases as we move from a first-order Markov system to a non-first-order Markov system, that is, one with memory. This is consistent with the idea that the Markovian approximations tend to over-estimate the decoherence of the system. A recent study has indicated that the form of a classical distribution is quickly approached as the quantum coins decohere [42].

Our scheme is the quantum analogue of the history-dependent game in a form of Parrondo's paradox. The quantum history-dependent walk also exhibits a Parrondo effect, where the disruption of the history dependence in a biased walk by mixing with a second, unbiased walk can reverse the bias. In distinction to the classical case, the effect seen here is very sensitive to the exact sequence of operations, a quality it shares with other forms of quantum Parrondo's games. This sensitivity is consistent with the idea that the effect relies on full coherence over space and in time.

We have only considered a quantum walk on a line. The effect of memory driven quantum walks on networks with different topologies and whether the memory structure can be chosen to optimize the path in such networks are open questions.

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