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An analysis of noise enhanced information transmission in an array of comparators

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Abstract

An array of N comparators subject to the same input signal and independent additive noise, with the outputs from each comparator summed, is a useful noise model for a range of systems including flash analog-to-digital converters, Digital Multibeam Steering sonar arrays and parallel neurons. It has previously been shown that for certain threshold configurations the transmitted information through such an array is maximised for non-zero noise. This behaviour has been termed Suprathreshold Stochastic Resonance (SSR) [1] and in this paper we show that SSR occurs for a number of different signal and noise distributions. Also presented is an analysis of the variance of the quantisation error incurred when all thresholds are set equal to the signal mean, for Gaussian and uniform distributions. It is shown that the minimum error variance is given by a non-zero value of noise.

Keywords: Comparator; Threshold device; Optimal quantisation; Digital multibeam steering; Flash; ADC; Transmitted information; Mutual information; Stochastic resonance; Suprathreshold stochastic resonance; Neuron model; Dithering

1. Introduction

Consider an array of N comparators, subject to the same continuously value input signal, x, as shown in Fig. 1. The ith device is subject to independent continuously valued additive noise, η_i $(i = 1, \ldots, N)$. The output from each device is 1 if the input signal plus the noise is greater than the threshold θ_i of that device and 0 otherwise. The outputs from the devices are summed to give the output signal y. Hence y is a discrete signal taking on integer values from 0 to N and can be considered as the number of devices that are currently 'on'.

Such an array has many similarities to parallel neuron configurations, such as a summing network of N FitzHugh–Nagumo neurons [2,3]. It is known that sensory neurons can be very noisy, with some papers reporting signal to noise ratios of 0 dB [4], yet still effectively transfer information. It is of interest to investigate this phenomenon to see if it is possible to utilise it to improve non-linear electronic devices such as motion detection systems [5,6]. Such arrays are also good models of flash (parallel) analog-to-digital (A/D) converters [7] (when the thresholds are uniformly distributed across the signal space) and DIMUS (Digital Multibeam Steering) sonar arrays, in the 'on target' position [8,9].

Our aim is to investigate the conditions under which the performance of such an array can be optimised by a certain non-zero noise setting. The phenomenon of a non-linear system performing optimally for non-zero noise levels is known as stochastic resonance [10–14]. Stochastic resonance was first reported as an explanation for the periodicity of ice ages [15]. Since then, it has been shown to occur in many non-linear systems, such as electronic devices [16], ring lasers [17], SQUIDS (super conducting quantum interference devices) [18] and in biological sensory neurons [19] and ion channels [20].

A number of methods have been used to quantify 'optimal performance' in the literature. Originally, stochastic resonance was defined as an increase in output signal to noise ratio for a weak periodic signal in a non-linear system. However, it was later shown that stochastic resonant phenomena could occur for broadband signals. This is known as Aperiodic Stochastic Resonance [21,22]. For such broadband signals, cross-correlation measures [23], transmitted information (Shannon information) [1,21], Fisher

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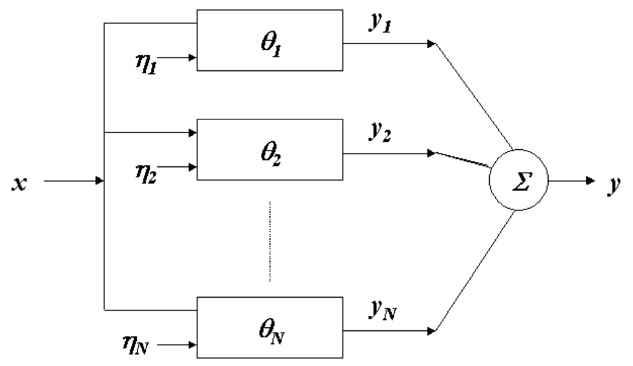

Fig. 1. Array of N summing comparators.

information [24], Kullback entropy [25], ϕ-divergences [26,27] and channel capacity [28,29] have all been shown to possess maxima for non-zero noise values in various systems.

In this paper, we consider only the (Shannon) transmitted information [30]. For general conditions, such a quantity is more robust for broadband excitation signals than quantities such as signal to noise ratio and cross-correlation coefficient for non-linear systems where the signal is large compared to the noise [31]. A brief summary of this paper follows. Section 2 gives mathematical descriptions of the transmitted information for the array of threshold devices, and of various probability densities on which the transmitted information depends. Four different probabilistic distributions of the signal and noise amplitude are considered. In particular, a general formula is presented for calculating the probability that n comparators are 'on' for a given signal x.

Section 3 presents results obtained showing how the transmitted information varies with the noise intensity for two different threshold configurations and the four probability distributions. An alternative approach to analysing the array is given in Section 4. Here we consider the output to be an estimator of the input, and derive a formula for the variance of the error between the output and the input. It is shown that this variance has a minimum for non-zero noise intensity when $N > 1$ and hence displays stochastic resonant-like behaviour. Finally, Section 5 summarises the paper and presents some conclusions and future directions for this work.

2. Calculating information transmitted through the array

The array of N comparators is shown in Fig. 1. The output of device i is given by

$$y_i = \begin{cases} 1 & \text{if } x + \eta_i > \theta_i, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, the output of the array is $y = \sum_{i=1}^{N} y_i$. We consider the array to be an information channel. The transmitted information I through a channel is given by the entropy $H(y)$ of the output less the conditional entropy $H(y|x)$ of the output given the input as

$$I = H(y) - H(y|x). \tag{1}$$

As noted by Stocks [1], $H(y|x)$ can be interpreted as the amount of *encoded* information about the input signal lost through the channel. Since the input to the array is continuously valued and the output is discretely valued, we can consider the channel to be semi-continuous [32]. The transmitted information through such a channel is given by

$$I = -\sum_{n=0}^{N} Q(n)\log_2 Q(n) - \left(-\int_{-\infty}^{\infty} P(x) \sum_{n=0}^{N} P(n|x)\log_2 P(n|x)\mathrm{d}x \right), \tag{2}$$

where $P(x)$ is the probability density of the input signal x, $Q(n)$ is the probability of the output signal y being equal to n $(n = 0, 1, \ldots, N)$ and $P(n|x)$ the conditional probability that the output is n given the input is x [1,31,33].

We have also the equation

$$Q(n) = \int_{-\infty}^{\infty} P(n|x)P(x)\mathrm{d}x, \tag{3}$$

relating $Q(n)$ and $P(n|x)$. Hence, the transmitted information can be expressed in terms of only $P(x)$ and $P(n|x)$. In turn, $P(n|x)$ is determined by $P(x)$ and the channel characteristics, that is, the number N of threshold devices, the values θ_i of the thresholds and the noise probability density $R(\eta)$. Section 2.1 describes briefly the method we use to calculate $P(n|x)$ given the channel characteristics. We shall consider four different signal and noise probability distributions: uniform, Gaussian, Rayleigh and exponential. Expressions for these probability distributions are given in subsequent sections.

2.1. Calculating $P(n|x)$

Following the notation of Stocks [1], let $P_{1|x,i}$ be the probability of device i being 'on' (that is, signal plus noise exceeding the threshold θ_i), given the input signal x. Then

$$P_{1|x,i} = \int_{\theta_i - x}^{\infty} R(\eta)\mathrm{d}\eta = 1 - F_R(\theta_i - x) \tag{4}$$

$(i = 1, \ldots, N)$,

where F_R is the cumulative distribution function of the noise.

Given a noise density and threshold value, $P_{1|x,i}$ can be calculated numerically for any value of x from Eq. (4). Assuming $P_{1|x,i}$ has been calculated for desired values of x, a convenient way of numerically calculating the probabilities $P(n|x)$ for a given number N of devices is as follows. Let $T^k_{n|x}$ denote the probability that n of the devices ($n = 1,\dots,k$) are 'on', given x. Then $T^1_{0|x} = 1 - P_{1|x,1}$ and $T^1_{1|x} = P_{1|x,1}$ and we have the recursive formulae

$$T^{k+1}_{0|x} = (1 - P_{1|x,k+1})T^k_{0|x},$$

$$T^{k+1}_{n|x} = P_{1|x,k+1}T^k_{n-1|x} + (1 - P_{1|x,k+1})T^k_{n|x} \qquad (n = 1,\dots,k),$$

$$T^{k+1}_{k+1|x} = P_{1|x,k+1}T^k_{k|x}. \tag{5}$$

We have $P(n|x)$ given by $T^N_{n|x}$. An alternative evaluation is the coefficient of z^n in the power series expansion of

$$\prod_{i=1}^{N}[1 - P_{1|x,i} + zP_{1|x,i}].$$

In particular, when all the thresholds have the same value, then each $P_{1|x,i}$ has the same value $P_{1|x}$ and we have the binomial distribution

$$P(n|x) = \binom{N}{n}(P_{1|x})^n(1 - P_{1|x})^{N-n} \qquad (0 \le n \le N).$$

Thus, for any arbitrary threshold settings and signal and noise probability distributions, $P(n|x)$ can be easily calculated from Eqs. (4) and (5) and therefore the transmitted information can be calculated by numerical integration of Eq. (2). In previous papers, numerical integration of Eq. (2) has been verified by digital simulations [1,31]. However, digital simulation becomes very difficult for large N, so that N was limited to be less than 100 [31]. The approach given here has the benefit that $P(n|x)$ can be found even for very large values of N, for any given threshold settings.

The following sections give expressions for four different probability densities (uniform, Gaussian, Rayleigh, and exponential), and for each of these gives expressions for $P_{1|x,i}$ obtained directly from Eq. (4). The results in this paper are independent of the value of the mean of the signal or noise, so we have chosen the mean for each distribution below for convenience.

2.2. Uniformly distributed signal and noise

If the input signal, x is uniformly distributed between $-\sigma_p/2$ and $\sigma_p/2$ with zero mean, then

$$P(x) = \begin{cases} 1/\sigma_p & \text{for } -\sigma_p/2 \le x \le \sigma_p/2, \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

If the independent noise η in each device is uniformly distributed between $-\sigma_r/2$ and $\sigma_r/2$ with zero mean, then

$$R(\eta) = \begin{cases} 1/\sigma_r & \text{for } -\sigma_r/2 \le \eta \le \sigma_r/2, \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

Substituting Eq. (7) into Eq. (4) gives

$$P_{1|x,i} = \begin{cases} 0 & \text{for } x < \theta_i - \sigma_r/2, \\ x/\sigma_r + 1/2 - \theta_i/\sigma_r & \text{for } \theta_i - \sigma_r/2 \le x \le \theta_i + \sigma_r/2, \\ 1 & \text{for } x > \theta_i + \sigma_r/2. \end{cases} \tag{8}$$

2.3. Gaussian signal and noise

If the input signal has a Gaussian distribution with zero mean and variance σ_p^2, then

$$P(x) = \frac{1}{\sqrt{2\pi\sigma_p^2}}\exp\left(-\frac{x^2}{2\sigma_p^2}\right). \tag{9}$$

If the independent noise in each device is Gaussian with zero mean and variance σ_r^2, then

$$R(\eta) = \frac{1}{\sqrt{2\pi\sigma_r^2}}\exp\left(-\frac{\eta^2}{2\sigma_r^2}\right). \tag{10}$$

Substituting Eq. (10) into Eq. (4) gives

$$P_{1|x,i} = 0.5\,\mathrm{erfc}\left(\frac{\theta_i - x}{\sqrt{2\sigma_r^2}}\right),$$

where erfc is the complementary error function [34].

2.4. Rayleigh signal and noise

If the input signal x (≥ 0) has a Rayleigh distribution with mean $\sigma_p\sqrt{\pi/2}$, then

$$P(x) = \frac{x}{\sigma_p^2}\exp\left(-\frac{x^2}{2\sigma_p^2}\right). \tag{11}$$

If the independent noise η (≥ 0) in each device has a Rayleigh distribution with mean $\sigma_r\sqrt{\pi/2}$, then

$$R(\eta) = \frac{\eta}{\sigma_r^2}\exp\left(-\frac{\eta^2}{2\sigma_r^2}\right). \tag{12}$$

Substituting Eq. (12) into Eq. (4) gives

$$P_{1|x,i} = \begin{cases} \exp\left(-\dfrac{(\theta_i - x)^2}{2\sigma_r^2}\right) & \text{for } x < \theta_i, \\ 1 & \text{for } x \le \theta_i. \end{cases}$$

2.5. Exponential signal and noise

If the input signal x (≥ 0) has an exponential distribution with mean σ_p then

$$P(x) = \frac{1}{\sigma_p}\exp\left(-\frac{x}{\sigma_p}\right). \tag{13}$$

If the independent noise η (≥ 0) in each device has an exponential distribution with mean σ_r, then

$$R(\eta) = \frac{1}{\sigma_r}\exp\left(-\frac{\eta}{\sigma_r}\right). \quad (14)$$

Substituting Eq. (14) into Eq. (4) gives

$$P_{1|x,i} = \begin{cases} \exp\left(-\frac{(\theta_i - x)}{\sigma_r}\right) & \text{for } x < \theta_i, \\ 1 & \text{for } x \geq \theta_i. \end{cases}$$

3. Comparison of transmitted information for different threshold settings and noise distributions

3.1. Thresholds distributed optimally for zero noise

For the case where all comparators are noiseless $H(y|x)$ is zero, since the output of the array is completely deterministic given the input. Therefore, from Eq. (1), the transmitted information is simply the entropy $H(y)$ of the output signal. Maximizing the output entropy is achieved by ensuring all output states are equally probable, that is, $Q(n) = 1/(N+1)$ for all n [30]. In this case, from Eq. (2), the transmitted information is given by $\log_2(N+1)$ bits per sample. Since there is no noise, $P_{1|x,i}$ is zero when $x < \theta_i$ and unity otherwise. Therefore, $P(n|x)$ is equal to unity when x is between θ_n and θ_{n+1} and zero otherwise. If the sequence $(\theta_n)_{n=1}^N$ is increasing, then from Eq. (3)

$$\int_{\theta_i}^{\theta_{i+1}} P(x)\mathrm{d}x = \frac{1}{N+1} \Leftrightarrow F_P(\theta_i) = \frac{i}{N+1} \Leftrightarrow \theta_i = F_P^{-1}\left(\frac{i}{N+1}\right),$$

where $i = 0, \ldots, N$ and F_P^{-1} is the inverse cumulative distribution function of the input signal.

In the case of a uniformly distributed signal, as in Eq. (6), the thresholds are

$$\theta_i = \sigma_p\left(\frac{i}{N+1} - \frac{1}{2}\right) \qquad (i = 1, \ldots, N).$$

In the case of a Gaussian signal distribution, as in Eq. (9), the thresholds are

$$\theta_i = \sqrt{2}\sigma_p \,\mathrm{erf}^{-1}\left(\frac{2i}{N+1} - 1\right) \qquad (i = 1, \ldots, N),$$

where erf^{-1} is the inverse of the error function [34].

In the case of a Rayleigh signal distribution, as in Eq. (11), the thresholds are

$$\theta_i = \sigma_p\sqrt{-2\ln\left(1 - \frac{i}{N+1}\right)} \qquad (i = 1, \ldots, N).$$

In the case of an exponential signal distribution, as in Eq. (13), the thresholds are

$$\theta_i = -\sigma_p \ln\left(1 - \frac{i}{N+1}\right) \qquad (i = 1, \ldots, N).$$

3.2. Thresholds equal to the signal mean

The second configuration of thresholds we shall consider is the case where all thresholds are set equal to the signal mean. This setting will give a transmitted information of exactly 1 bit when there is no noise, since all of the threshold devices will be simultaneously either 'on' or 'off' and y can be only 0 or N, each value occurring with probability of 0.5. Setting all of the thresholds to values other than the signal mean will give transmitted information of less than 1 for zero noise.

3.3. Results

Figs. 2–5 show plots of transmitted information against $\sigma = \sigma_r/\sigma_p$ for the case where the signal and noise are identically distributed. Hence, σ is the ratio of the signal standard deviation to the noise standard deviation for all four probability densities. In all figures σ_p is set equal to one. The value of σ_r is varied between 0 and 1.6 and the transmitted information calculated numerically from Eq. (2). It is important to note, however, that since we have plotted the transmitted information against the ratio of σ_r to σ_p that the results plotted below are valid for any size (characterised by the variance) of the signal.

It can be seen from the figures that when the thresholds are set so that the transmitted information is maximised for zero noise, the transmitted information in the absence of noise ($\sigma = 0$) is indeed given by $\log_2(N+1)$ bits per sample. As σ increases, the transmitted information decreases

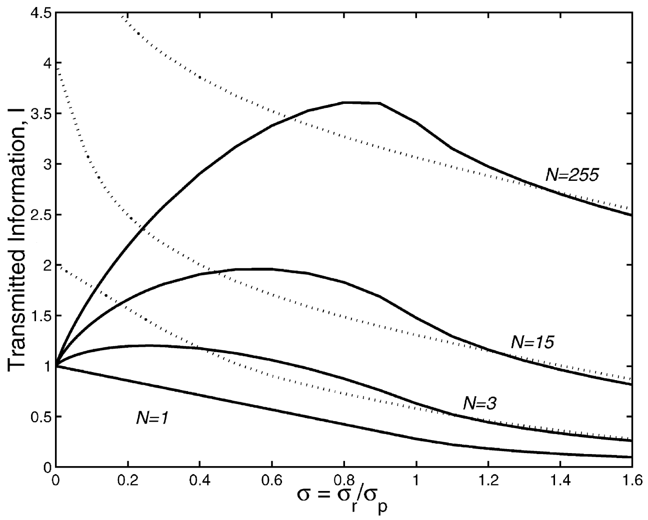

Fig. 2. Plot of transmitted information against σ for various values of N and uniformly distributed signal and noise. The solid lines are the case where all thresholds are set to the signal mean. The dotted lines are the case where the thresholds are set to optimise the noiseless transmitted information.

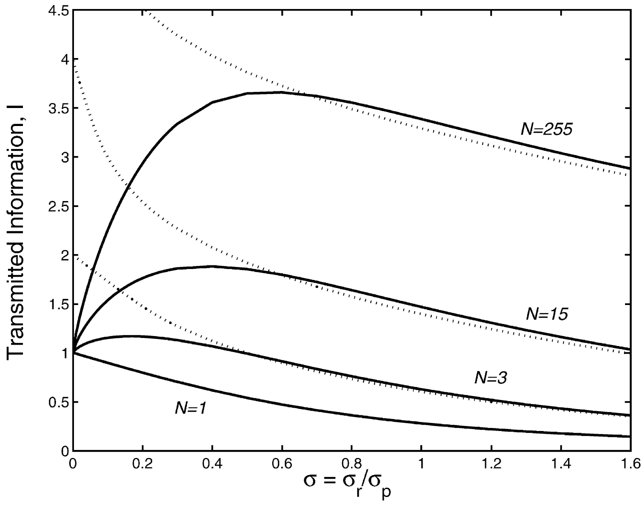

Fig. 3. Plot of transmitted information against σ for various values of N and Gaussian signal and noise. The solid lines are the case where all thresholds are set to the signal mean. The dotted lines are the case where the thresholds are set to optimise the noiseless transmitted information.

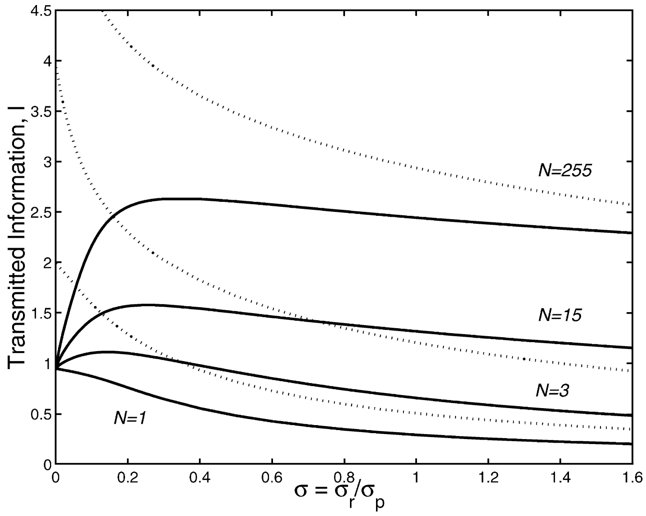

Fig. 5. Plot of transmitted information against σ for various values of N and exponentially distributed signal and noise. The solid lines are the case where all thresholds are set to the signal mean. The dotted lines are the case where the thresholds are set to optimise the noiseless transmitted information.

monotonically from this value. However, for the case of the thresholds all set to the signal mean, the transmitted information is always 1 bit per sample for zero noise. For $N > 1$, as the noise intensity increases from zero, the transmitted information also increases until it reaches a maximum before decreasing again. As N becomes large, the value of σ that gives the maximum transmitted information increases towards unity.

These results confirm those reported by Stocks [33] and furthermore we have extended them to show that a noise-induced maximum occurs in the transmitted information for the Rayleigh and exponential distributions. As pointed out by Stocks, since these results are valid for any size of input signal, they contrast with classical stochastic resonance results where the signal has to be subthreshold for stochastic resonance phenomena to occur. Stocks has coined the term Suprathreshold Stochastic Resonance (SSR) to describe this new result.

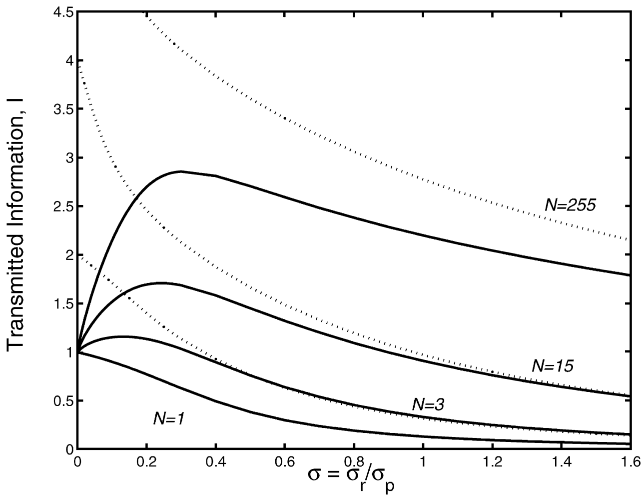

Fig. 4. Plot of transmitted information against σ for various values of N and Rayleigh distributed signal and noise. The solid lines are the case where all thresholds are set to the signal mean. The dotted lines are the case where the thresholds are set to optimise the noiseless transmitted information.

3.4. Analysis of results

For the case of all thresholds set to the signal mean, the reason that only 1 bit is transmitted in the absence of noise is that only the two output states 0 and N are possible. Hence, only 1 bit of information can be transmitted, as this is equivalent to a binary signal. As noise becomes non-zero, the other output states become accessible and hence more than 1 bit of information can be transmitted per sample. This is true for $N > 1$. For $N = 1$, at most 1 bit per sample can be transmitted. This illustrates how using more than one device in parallel can give improvements in signal transfer.

The increase in the transmitted information for non-zero noise can be explained as follows. A probabilistic input signal has a high information content. In the absence of noise, the transmitted information is limited to 1 bit (for all thresholds equal to the signal mean) and much information is lost due to the nature of the channel. As non-zero noise is added independently in each device, all output states become accessible and hence $H(y)$ increases towards a maximum before decreasing again. However, $H(y|x)$ increases monotonically with increasing noise. Therefore, the transmitted information also contains a maximum [31]. As the number N of threshold devices is increased, the transmitted information also increases, since the number of output states available increases.

In the case where the thresholds are distributed optimally for zero noise, this setting is not optimal for many non-zero values of noise. It is clear from the figures that for moderate values of σ the transmitted information, in the case of all thresholds set to the signal mean, can increase above that for the thresholds set optimally for zero noise. This implies that A/D converters designed for optimum performance for random input signals, in the absence of noise, may not perform optimally when non-zero independent noise is present in each comparator. If one had knowledge of the probability distribution of the input signal and the noise at each comparator, then optimising the A/D converters could entail setting the thresholds to zero, and possibly *increasing* the noise variance.

The above discussion leads to a further problem; that of finding the threshold values that maximise the transmitted information for a given number of comparators and non-zero values of σ. It is anticipated that the recursive formula given by Eq. (5) will be of benefit in future work on algorithms that calculate or approximate such maximal threshold values.

4. Variance analysis

In this section we consider only zero mean uniform or Gaussian signal and noise, and all thresholds set equal to the signal mean, i.e. zero. The analysis is simplified for these distributions, since they are even functions. The fact that the transmitted information increases to a maximum with increasing noise when all thresholds are set to the signal mean can be seen qualitatively by considering an output time-series of the array. If the output signal $y \in \{0, 1, \ldots, N\}$ is normalised so that it takes on values between $-c$ and c, it becomes a digital approximation to the input signal. We will call this normalised signal

$$\hat{y} = c\left(\frac{2y}{N} - 1\right).$$

Let $\varepsilon = \hat{y} - x$. We show in Appendix A that $E[\hat{y}]$ is zero for both the uniform and Gaussian cases (indeed, for any even zero mean density function), and hence $E[\varepsilon]$ is zero. The variance of ε then gives an indicator of the quantisation error between the input and output signals when $\hat{y}$ is taken as an estimate for x. We can derive the variance of ε theoretically when all thresholds are equal to the signal mean for the case of uniform and Gaussian signal and noise.

From the definition of variance, and given $E[\hat{y}] = 0$, $\text{var}[\varepsilon] = E[\hat{y}^2] + \text{var}[x] - 2E[x\hat{y}]$, i.e. the sum of the variance of the input and output, less twice the correlation of x and $\hat{y}$. Hence to find the variance of the quantisation error, ε, we need only to find $E[\hat{y}^2]$ and $E[x\hat{y}]$. The details of these derivations is given in Appendix A.

4.1. Uniform signal and noise

For uniform signal and noise we get

$$E[\hat{y}^2] = \begin{cases} \dfrac{\sigma_\text{p}^2}{4}\left[\dfrac{1}{N} + \dfrac{N-1}{N}\left(1 - \dfrac{2\sigma}{3}\right)\right] & (\sigma \le 1), \\ \dfrac{\sigma_\text{p}^2}{4}\left[\dfrac{1}{N} + \dfrac{N-1}{N}\left(\dfrac{1}{3\sigma^2}\right)\right] & (\sigma \ge 1), \end{cases}$$

and

$$E[x\hat{y}] = \begin{cases} \sigma_\text{p}^2\left(\dfrac{1}{8} - \dfrac{\sigma^2}{24}\right) & (\sigma \le 1), \\ \sigma_\text{p}^2\left(\dfrac{1}{12\sigma}\right) & (\sigma \ge 1). \end{cases}$$

Therefore

$$\text{var}[\varepsilon] = \begin{cases} \sigma_\text{p}^2\left(\dfrac{\sigma(1-N)}{6N} + \dfrac{\sigma^2+1}{12}\right) & (\sigma \le 1), \\ \sigma_\text{p}^2\left(\dfrac{\sigma^2(3+N) - 2N\sigma + N - 1}{12N\sigma^2}\right) & (\sigma \ge 1). \end{cases} \quad (15)$$

Note from Eq. (15) that for a given non-zero value of σ_p, the variance of the error for $\sigma \le 1$, is a quadratic function of σ and has a minimum of $\sigma_\text{p}^2(2N-1)/(12N^2)$ at $\sigma = (N-1)/N$. For $\sigma \ge 1$, it is straightforward to show that the variance is strictly increasing. Hence, the variance of the error is minimised for a non-zero value of σ and is independent of the size of the signal variance.

As N becomes large, the variance of ε approaches the variance

$$\lim_{N\to\infty} \text{var}[\varepsilon] = \begin{cases} \dfrac{(\sigma_\text{r} - \sigma_\text{p})^2}{12} & (\sigma \le 1), \\ \dfrac{(\sigma_\text{r} - \sigma_\text{p})^2}{12\sigma^2} & (\sigma \ge 1), \end{cases} \quad (16)$$

of a uniform distribution. We saw earlier that as $N \to \infty$, the value of σ that gave the maxima in the transmitted information approaches one. This is also the case here, where from Eq. (16) the variance approaches 0 as $\sigma \to 1$.

Plots of the variance of ε for various N are shown in Fig. 6. The theoretical calculation was verified by digital simulation.

Interestingly, the value of σ that minimises $\hat{y} - x$ is not the same as that which maximises the transmitted information. However, the minimum of the variance of ε has the same qualitative behaviour as the maximum in the transmitted information. Stocks calculated an approximation for the value of σ which maximises I for large N and $\sigma < 1$ [31]. This is given by $\sqrt{N+1}/(\sqrt{N+1} + 3.297)$ which clearly is not equal to $(N-1)/N$, although both approach unity as N becomes large.

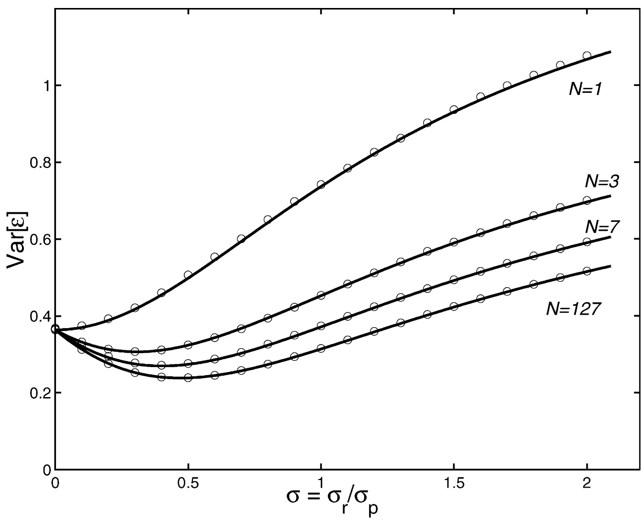

Fig. 6. Plot of the variance of $\hat{y} - x$ against σ for various values of N and uniformly distributed signal and noise, with $\sigma_{\mathrm{p}} = 1$. The solid line plots are the exact value of the variance from Eq. (15) and the circles are from digital simulation. The minimum value of the variance occurs for non-zero σ for $N > 1$ and approaches 1 as N becomes large.

4.2. Gaussian signal and noise

Let y be normalised so that $\hat{y}$ is between $\pm k$ signal standard deviations, i. e. $c = k\sigma_{\mathrm{p}}$. Then for Gaussian signal and noise we get

$$E[\hat{y}^2] = k^2\sigma_{\mathrm{p}}^2\left(\frac{1}{N} + \frac{2(N-1)}{N\pi}\arcsin\left(\frac{1}{\sigma^2+1}\right)\right)$$

and

$$E[x\hat{y}] = k\sigma_{\mathrm{p}}^2\sqrt{\frac{2}{\pi(1+\sigma^2)}}.$$

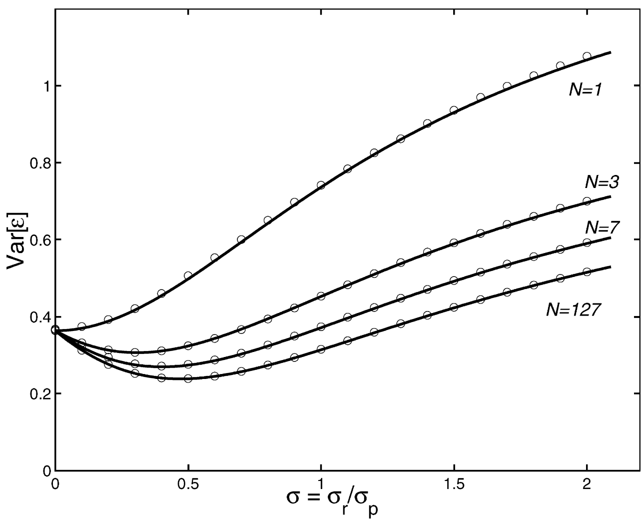

Fig. 7. Plot of the variance of $\hat{y} - x$ against σ for various values of N and Gaussian distributed signal and noise, with $\sigma_{\mathrm{p}} = 1$ and $k = \sqrt{2/\pi}$. The solid line plots are the exact value of the variance from Eq. (17) and the circles are from digital simulation.

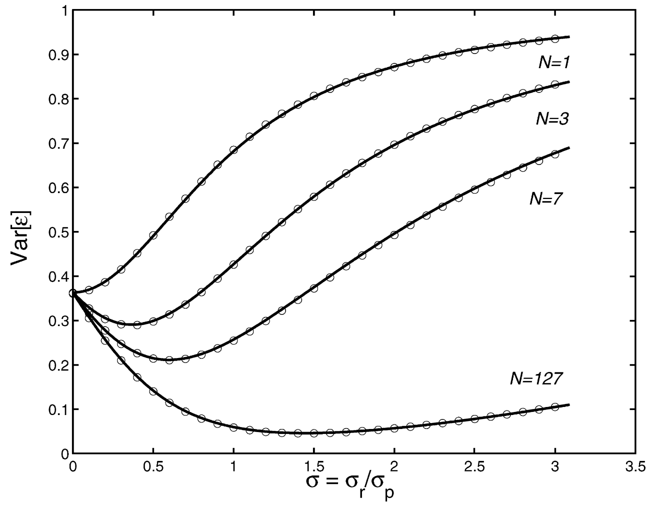

Fig. 8. Plot of the variance of $\hat{y} - x$ against σ for various values of N and Gaussian distributed signal and noise, with $\sigma_{\mathrm{p}} = 1$ and k set to minimise the variance for each value of σ. The solid line plots are the exact value of the variance from Eq. (17) and the circles are from digital simulation.

Therefore

$$\mathrm{var}[\varepsilon] = \sigma_{\mathrm{p}}^2\left(k^2\left(\frac{1}{N} + \frac{2(N-1)}{N\pi}\arcsin\left(\frac{1}{\sigma^2+1}\right)\right) - 2k\sqrt{\frac{2}{\pi(1+\sigma^2)}} + 1\right). \quad (17)$$

If σ is a fixed value, then the minimum value of $\mathrm{var}[\varepsilon]$ becomes a function of k. We can obtain this function by differentiating Eq. (17) with respect to k and setting to zero. For instance, when $\sigma = 0$ (the noiseless case):

$$\mathrm{var}[\varepsilon] = \sigma_{\mathrm{p}}^2(k^2 - 2\sqrt{2/\pi}k + 1)$$

and $\mathrm{var}[\varepsilon]$ has a minimum value of $\sigma_{\mathrm{p}}^2(1 - 2/\pi)$ at $k = \sqrt{2/\pi}$.

Plots of the variance of ε for various N are shown in Fig. 7 for $k = \sqrt{2/\pi}$, and in Fig. 8 for k set to minimise the variance for each value of σ. The theoretical calculation was verified by digital simulation. It is clear that for $N > 1$, there is again a minimum in the error variance for a non-zero value of σ. This occurs whether k is a constant, or set to minimise the variance for each σ. Note from Fig. 8 that as N increases, the variance significantly decreases when compared with Fig. 7.

5. Conclusions and further work

In this paper we have examined the problem of optimising the transmitted information through a summing array of N comparators, where each comparator is subject to additive noise. We have verified the results of Stocks which show that for uniform or Gaussian signal and noise there is a maximum in the transmitted information at a non-zero value of noise when all thresholds are set to the signal mean. This phenomena is known as SSR. We have also shown that SSR occurs when the signal and noise have a Rayleigh or exponential distribution and derived a convenient method of calculating the transmitted information for any value of N, and any given set of threshold values and noise and signal probability densities. Furthermore, we showed that the variance of the difference between the input and output signals is minimised for non-zero values of noise when the signal and noise have uniform or Gaussian distributions.

It has previously been pointed out that for threshold devices, stochastic resonance is related to, or indeed equivalent to dithering in A/D converters [35,36]. The direction of future work is aimed towards finding the optimal threshold settings for a given noise variance, and to investigate the relationship of the results presented here to dithering. Also of interest is whether SSR occurs when the input signal is a non-random signal, and where the distributions of the signal and noise are non-identical.

In recent work, it was shown that under certain conditions both SR and SSR occurred in a network of motion detectors [5]. We hope to apply knowledge gained by this study of SSR in a simple array of comparators to the more complex problem of determining whether SSR could be usefully applied in motion detection systems.

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Appendix A. A derivation of error variances

We consider zero mean uniform and Gaussian signal and noise distributions, with all thresholds set equal to the signal mean, i.e. zero. Hence the probability densities of each distribution are even functions.

Let $\varepsilon = \hat{y} - x$. Then

$$\begin{aligned}\mathrm{var}[\varepsilon] &= \mathrm{var}[\hat{y}-x] = E[(\hat{y}-x)^2] - E[\hat{y}-x]^2 \\ &= E[\hat{y}^2] - 2E[x\hat{y}] + E[x^2] - E[\hat{y}]^2 + 2E[\hat{y}]E[x] - E[x]^2 \\ &= \mathrm{var}[\hat{y}] - 2E[x\hat{y}] + 2E[\hat{y}]E[x] + \mathrm{var}[x] \\ &= \mathrm{var}[\hat{y}] + \mathrm{var}[x] + 2\,\mathrm{cov}[\hat{y}, -x].\end{aligned}$$

Note that $E[x\hat{y}]$ is the *correlation* of x and $\hat{y}$. When $E[\hat{y}] = 0$, then $\mathrm{var}[\hat{y}] = E[\hat{y}^2]$, and

$$\mathrm{var}[\varepsilon] = E[\hat{y}^2] + \mathrm{var}[x] - 2E[x\hat{y}].$$

Hence, $\mathrm{var}[\varepsilon]$ is the sum of the mean squared value of $\hat{y}$ and the variance of x, less twice the correlation of x and $\hat{y}$.

We denote the probability that a device is on, given x, as $P_{1|x}$. That is:

$$P_{1|x} = \mathrm{prob}(x + \eta \geq 0|x) = \int_{-x}^{\infty} R(\eta)\mathrm{d}\eta.$$

Since $R(\eta)$ is even with mean zero we can write

$$P_{1|x} = \int_{-\infty}^{x} R(\eta)\mathrm{d}\eta = \frac{1}{2} + \int_0^x R(\eta)\mathrm{d}\eta.$$

The expected value of $P_{1|x}$ over the signal distribution is then

$$\begin{aligned}E[P_{1|x}] &= \frac{1}{2} + E\left[\int_0^x R(\eta)\mathrm{d}\eta\right] \\ &= \frac{1}{2} + \int_{-\infty}^{\infty}\left(\int_0^x R(\eta)\mathrm{d}\eta\right)P(x)\mathrm{d}x.\end{aligned}$$

Since $R(\eta)$ is even, $\int_0^x R(\eta)\mathrm{d}\eta$ is odd and therefore the integral above is zero, since $P(x)$ is even. Thus, $E[P_{1|x}] = 0.5$.

Suppose the noise in all N devices is independent and identically distributed. If the output y is normalised to $\hat{y}$ so that when $y = N$, $\hat{y} = c$, then we can write

$$\hat{y}(x, \eta_1, \eta_2, \ldots, \eta_N) = \frac{c}{N}\sum_{i=1}^{N} \mathrm{sign}(x + \eta_i),$$

where η_i is the noise in the ith device.

The expected value of $\hat{y}$ given x is then

$$E[\hat{y}|x] = \frac{c}{N}E\left[\sum_{i=1}^{N}\mathrm{sign}(x+\eta_i)|x\right] = cE[\mathrm{sign}(x+\eta)|x],$$

(since all η are independent and identically distributed)

$$E[\hat{y}|x] = c(-1(1 - P_{1|x}) + 1P_{1|x}) = c(2P_{1|x} - 1).$$

Therefore, the expected value of $\hat{y}$ is:

$$E[\hat{y}] = E[E[\hat{y}|x]] = 2cE[P_{1|x}] - c = 0. \tag{A1}$$

The mean square value of $\hat{y}$ given x is

$$\begin{aligned}E[\hat{y}^2|x] &= \frac{c^2}{N^2}E\left[\left(\sum_{i=1}^{N}\mathrm{sign}(x+\eta_i)\right)^2\Bigg| x\right] \\ &= \frac{c^2}{N^2}(NE[(\mathrm{sign}(x+\eta))^2|x] + N(N-1) \\ &\quad \times E[\mathrm{sign}(x+\eta_i)\mathrm{sign}(x+\eta_j)|x]) \\ &= \frac{c^2}{N^2}(N + N(N-1)(-2P_{1|x}(1-P_{1|x}) \\ &\quad + (1-P_{1|x})^2 + P_{1|x}^2)) \\ &= \frac{c^2}{N^2}(N + N(N-1)(2P_{1|x}-1)^2).\end{aligned}$$

Therefore, the mean square value of $\hat{y}$ is

$$\begin{aligned} E[\hat{y}^2] &= E[E[\hat{y}^2|x]] \\ &= \frac{c^2}{N} + \frac{c^2(N-1)}{N}E[(2P_{1|x}-1)^2] \\ &= \frac{c^2}{N}(1+(N-1)(4E[P_{1|x}^2]-4E[P_{1|x}]+1)) \\ &= \frac{c^2}{N}(1+(N-1)(4E[P_{1|x}^2]-1)). \end{aligned} \tag{A2}$$

The correlation of x and $\hat{y}$ is

$$\begin{aligned} E[x\hat{y}] &= E[E[x\hat{y}|x]] = E[xE[\hat{y}|x]] \\ &= \int_{-\infty}^{\infty} xE[\hat{y}|x]P(x)\mathrm{d}x \\ &= \int_{-\infty}^{\infty} xP(x)c(2P_{1|x}-1)\mathrm{d}x \\ &= 2c\int_{-\infty}^{\infty} xP(x)P_{1|x}\,\mathrm{d}x - c\int_{-\infty}^{\infty} xP(x)\mathrm{d}x \\ &= 2c\int_{-\infty}^{\infty} xP(x)P_{1|x}\,\mathrm{d}x - cE[x] \\ &= 2c\int_{-\infty}^{\infty} xP(x)P_{1|x}\,\mathrm{d}x. \end{aligned} \tag{A3}$$

A.1. Uniform signal and noise with all thresholds zero

For a uniformly distributed signal, as given by Eq. (6), we wish the output to take on states between $-\sigma_p/2$ and $\sigma_p/2$. Hence, $c = \sigma_p/2$. From Eq. (8), $P_{1|x}$ is 0 for $x \le -\sigma_r/2$, a function of x for $-\sigma_r/2 \le x \le \sigma_r/2$ and 1 for $x \ge \sigma_r/2$, and $E[\varepsilon]$ depends on whether σ_p is less than or greater than σ_r. Therefore, we require separate derivations of $E[\varepsilon]$ for $\sigma = \sigma_r/\sigma_p$ less than 1 and greater than 1. When $\sigma = 1$, both cases give $E[\varepsilon] = \sigma_p^2/6N$.

A.1.1. Uniform signal and noise with $\sigma \le 1$

We have

$$\begin{aligned} E[P_{1|x}^2] &= \int_{-\infty}^{\infty} P_{1|x}^2 P(x)\mathrm{d}x \\ &= \frac{1}{\sigma_p}\int_{-\sigma_r/2}^{\sigma_r/2}\left(\frac{1}{2}+\frac{x}{\sigma_r}\right)^2\mathrm{d}x + \frac{1}{\sigma_p}\int_{\sigma_r/2}^{\sigma_p/2}\mathrm{d}x \\ &= \frac{1}{2} - \frac{\sigma}{6}. \end{aligned}$$

Therefore from Eq. (A2) we get

$$E[\hat{y}^2] = \frac{\sigma_p^2}{4}\left[\frac{1}{N} + \frac{N-1}{N}\left(1-\frac{2\sigma}{3}\right)\right], \tag{A4}$$

and from Eq. (A3)

$$\begin{aligned} E[x\hat{y}] &= 2c\int_{-\infty}^{\infty} xP(x)P_{1|x}\,\mathrm{d}x \\ &= \sigma_p\int_{-\sigma_r/2}^{\sigma_r/2} x\left(\frac{1}{2}+\frac{x}{\sigma_r}\right)\frac{1}{\sigma_p}\mathrm{d}x + \sigma_p\int_{\sigma_r/2}^{\sigma_p/2}\frac{x}{\sigma_p}\mathrm{d}x \\ &= \frac{\sigma_r^2}{12} + \left(\frac{\sigma_p^2}{8} - \frac{\sigma_r^2}{8}\right) = \mathrm{var}[\eta] + \frac{\sigma_p^2-\sigma_r^2}{8} \\ &= -\frac{\sigma_r^2}{24} + \frac{\sigma_p^2}{8} = \sigma_p^2\left(\frac{1}{8}-\frac{\sigma^2}{24}\right). \end{aligned} \tag{A5}$$

Accordingly, from Eqs. (A4) and (A5)

$$\begin{aligned} \mathrm{var}[\varepsilon] &= \frac{\sigma_p^2}{4}\left[\frac{1}{N}+\frac{N-1}{N}\left(1-\frac{2\sigma}{3}\right)\right] \\ &\quad - 2\sigma_p^2\left(\frac{1}{8}-\frac{\sigma^2}{24}\right) + \frac{\sigma_p^2}{12} \\ &= \sigma_p^2\left(\frac{\sigma(1-N)}{6N} + \frac{\sigma^2+1}{12}\right). \end{aligned}$$

A.1.2. Uniform signal and noise with $\sigma \ge 1$

We have

$$\begin{aligned} E[P_{1|x}^2] &= \int_{-\infty}^{\infty} P_{1|x}^2 P(x)\mathrm{d}x = \frac{1}{\sigma_p}\int_{-\sigma_p/2}^{\sigma_p/2}\left(\frac{1}{2}+\frac{x}{\sigma_r}\right)^2\mathrm{d}x \\ &= \frac{1}{4} + \frac{1}{12\sigma^2}. \end{aligned}$$

Therefore, from Eq. (A2) we get

$$E[\hat{y}^2] = \frac{\sigma_p^2}{4}\left[\frac{1}{N}+\frac{N-1}{N}\left(\frac{1}{3\sigma^2}\right)\right], \tag{A6}$$

and from Eq. (A3)

$$\begin{aligned} E[x\hat{y}] &= 2c\int_{-\infty}^{\infty} xP(x)P_{1|x}\,\mathrm{d}x \\ &= \sigma_p\int_{-\sigma_p/2}^{\sigma_p/2} x\left(\frac{1}{2}+\frac{x}{\sigma_r}\right)\frac{1}{\sigma_p}\mathrm{d}x = \sigma_p^2\left(\frac{1}{12\sigma}\right). \end{aligned} \tag{A7}$$

Accordingly, from Eqs. (A6) and (A7)

$$\begin{aligned} \mathrm{var}[\varepsilon] &= \frac{\sigma_p^2}{4}\left[\frac{1}{N}+\frac{N-1}{N}\left(\frac{1}{3\sigma^2}\right)\right] - 2\sigma_p^2\left(\frac{1}{12\sigma}\right) + \frac{\sigma_p^2}{12} \\ &= \sigma_p^2\left(\frac{\sigma^2(3+N) - 2N\sigma + N - 1}{12N\sigma^2}\right). \end{aligned}$$

A.2. Gaussian

If the signal is Gaussian, and we wish y to be normalised so that $\hat{y}$ spans $\pm k$ signal standard deviations, then $c = k\sigma_p$.

We have

$$
\begin{aligned}
E[P_{1|x}^2] &= \int_{-\infty}^{\infty} P_{1|x}^2 P(x)\mathrm{d}x \\
&= \int_{-\infty}^{\infty} \left(\frac{1}{2}+\frac{1}{2}\mathrm{erf}\left(\frac{x}{\sqrt{2}\sigma_\mathrm{r}}\right)\right)^2 P(x)\mathrm{d}x \\
&= \int_{-\infty}^{\infty} \left(\frac{1}{4}+\mathrm{erf}\left(\frac{x}{\sqrt{2}\sigma_\mathrm{r}}\right)+\frac{1}{4}\mathrm{erf}\left(\frac{x}{\sqrt{2}\sigma_\mathrm{r}}\right)^2\right) P(x)\mathrm{d}x \\
&= \frac{1}{4}\int_{-\infty}^{\infty} P(x)\mathrm{d}x + \int_{-\infty}^{\infty} \mathrm{erf}\left(\frac{x}{\sqrt{2}\sigma_\mathrm{r}}\right)P(x)\mathrm{d}x \\
&\quad + \frac{1}{4}\int_{-\infty}^{\infty} \mathrm{erf}^2\left(\frac{x}{\sqrt{2}\sigma_\mathrm{r}}\right)P(x)\mathrm{d}x \\
&= \frac{1}{4}+0+\frac{1}{4}\int_{-\infty}^{\infty} \mathrm{erf}^2\left(\frac{x}{\sqrt{2}\sigma_\mathrm{r}}\right)P(x)\mathrm{d}x,
\end{aligned}
$$

since $P(x)$ is even and $\mathrm{erf}(x)$ is odd and therefore the second term above is zero. From a table of integrals [37], we have

$$
\int_{-\infty}^{\infty} \exp(-a^2x^2)\mathrm{erf}^2(x)\mathrm{d}x = \frac{2}{a\sqrt{\pi}}\arctan\frac{1}{a\sqrt{a^2+2}}. \tag{A8}
$$

Therefore

$$
\begin{aligned}
E[P_{1|x}^2] &= \frac{1}{4}+\frac{1}{4\sqrt{2\pi\sigma_\mathrm{p}^2}}\int_{-\infty}^{\infty} \mathrm{erf}^2\left(\frac{x}{\sqrt{2}\sigma_\mathrm{r}}\right)\exp\left(-\frac{x^2}{2\sigma_\mathrm{p}^2}\right)\mathrm{d}x \\
&= \frac{1}{4}+\frac{\sigma}{4\sqrt{\pi}}\int_{-\infty}^{\infty} \mathrm{erf}^2(\tau)\exp(-\sigma^2\tau^2)\mathrm{d}\tau \\
&= \frac{1}{4}+\frac{1}{2\pi}\arctan\left(\frac{1}{\sigma\sqrt{\sigma^2+2}}\right) \\
&= \frac{1}{4}+\frac{1}{2\pi}\arcsin\left(\frac{1}{\sigma^2+1}\right),
\end{aligned}
$$

since $\sigma \geq 0$. Therefore, from Eq. (A2)

$$
E[\hat{y}^2] = \frac{c^2}{N}\left(1+\frac{2(N-1)}{\pi}\arcsin\left(\frac{1}{\sigma^2+1}\right)\right), \tag{A9}
$$

and from Eq. (A3)

$$
\begin{aligned}
E[x\hat{y}] &= 2c\int_{-\infty}^{\infty} xP(x)P_{1|x}\mathrm{d}x = 2c\int_{-\infty}^{\infty} xP(x)\left(\int_{-\infty}^{x} R(\eta)\mathrm{d}\eta\right)\mathrm{d}x \\
&= 2c\int_{-\infty}^{\infty} R(\eta)\left(\int_{\eta}^{\infty} xP(x)\mathrm{d}x\right)\mathrm{d}\eta \\
&= 2c\int_{-\infty}^{\infty} R(\eta)\left(\int_{\eta}^{\infty} \frac{x}{\sqrt{2\pi}\sigma_\mathrm{p}}\exp\left(-\frac{x^2}{2\sigma_\mathrm{p}^2}\right)\mathrm{d}x\right)\mathrm{d}\eta \\
&= \frac{2c\sigma_\mathrm{p}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} R(\eta)\exp\left(-\frac{\eta^2}{2\sigma_\mathrm{p}^2}\right)\mathrm{d}\eta \\
&= \frac{2c\sigma_\mathrm{p}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_\mathrm{r}}\exp\left(-\frac{\eta^2}{2\sigma_\mathrm{r}^2}\right)\exp\left(-\frac{\eta^2}{2\sigma_\mathrm{p}^2}\right)\mathrm{d}\eta \\
&= \frac{c}{\pi\sigma}\int_{-\infty}^{\infty} \exp\left(-\frac{\eta^2}{2}\left(\frac{1+\sigma^2}{\sigma_\mathrm{r}^2}\right)\right)\mathrm{d}\eta.
\end{aligned}
$$

The final integrand is a Gaussian density function, with variance $\sigma_\mathrm{r}^2/(1+\sigma^2)$. Hence, the integral from negative to positive infinity is 1 times the normalizing factor, i.e.

$$
\begin{aligned}
E[x\hat{y}] &= \frac{c}{\pi\sigma}\sqrt{2\pi\left(\frac{\sigma_\mathrm{r}^2}{1+\sigma^2}\right)} = c\sigma_\mathrm{p}\sqrt{\frac{2}{\pi(1+\sigma^2)}} \\
&= k\sigma_\mathrm{p}^2\sqrt{\frac{2}{\pi(1+\sigma^2)}}.
\end{aligned} \tag{A10}
$$

Accordingly, from Eqs. (A9) and (A10):

$$
\begin{aligned}
&\mathrm{var}[\varepsilon] \\
&= \frac{c^2}{N}\left(1+\frac{2(N-1)}{\pi}\arcsin\left(\frac{1}{\sigma^2+1}\right)\right)-2c\sigma_\mathrm{p}\sqrt{\frac{2}{\pi(1+\sigma^2)}}+\sigma_\mathrm{p}^2 \\
&= \frac{k^2\sigma_\mathrm{p}^2}{N}\left(1+\frac{2(N-1)}{\pi}\arcsin\left(\frac{1}{\sigma^2+1}\right)\right)-2k\sigma_\mathrm{p}^2\sqrt{\frac{2}{\pi(1+\sigma^2)}}+\sigma_\mathrm{p}^2 \\
&= \sigma_\mathrm{p}^2\left(k^2\left(\frac{1}{N}+\frac{2(N-1)}{N\pi}\arcsin\left(\frac{1}{\sigma^2+1}\right)\right)-2k\sqrt{\frac{2}{\pi(1+\sigma^2)}}+1\right).
\end{aligned}
$$

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