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Optimal weighted suprathreshold stochastic resonance with multigroup saturating sensors

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HIGHLIGHTS

- A weighted summing array of independently noisy binary comparators is investigated.
- We present an optimal linearly weighted decoding scheme for combining the comparator responses.
- We solve for the optimal weights by applying least squares regression to simulated data.
- We find that the MSE distortion of weighting before summation is superior to unweighted summation of comparator responses.
- For some parameter regions, the decrease in MSE distortion due to weighting is negligible.

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ABSTRACT

Suprathreshold stochastic resonance (SSR) describes a noise-enhanced effect that occurs, not in a single element, but rather in an array of nonlinear elements when the signal is no longer subthreshold. Within the context of SSR, we investigate the optimization problem of signal recovery through an array of saturating sensors where the response of each element can be optimally weighted prior to summation, with a performance measure of mean square error (MSE). We consider groups of sensors. Individual sensors within each group have identical parameters, but each group has distinct parameters. We find that optimally weighting the sensor responses provides a lower MSE in comparison with the unweighted case for weak and moderate noise intensities. Moreover, as the slope parameter of the nonlinear sensors increases, the MSE superiority of the optimally weighted array shows a peak, and then tends to a fixed value. These results indicate that SSR with optimal weights, as a general mechanism of enhancement by noise, is of potential interest to signal recovery.

1. Introduction

Stochastic resonance (SR) essentially represents a class of phenomenon where, for certain types of nonlinear coupling between signal and noise, the presence or the addition of noise provides an improved performance over the absence of noise [1–5]. This counter-intuitive effect was first introduced in the field of atmospheric science [1], and it has gradually been observed in a wide variety of fields, for instance, physics, biology and electronic engineering [2–26]. Initial studies

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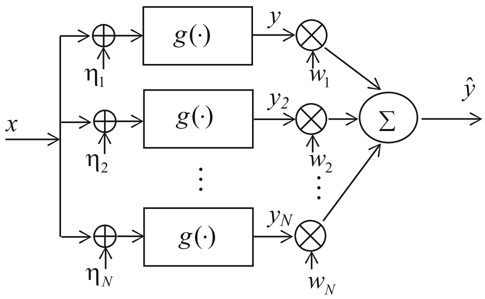

Fig. 1. Weighted summing array of N identical noisy nonlinear elements, $g(\cdot)$. Each of these N elements operates on a common signal x subject to additive noise η_i. The output of each individual element y_i is multiplied by the weighting coefficient w_i, and the overall output $\hat{y} = \sum_{i=1}^{N} w_i y_i$.

of SR mainly focused on dynamical systems with a nonlinearity due to a simple threshold operation or a potential barrier [1–26], where the enhanced response of a weak signal results from noise-assisted threshold or barrier crossings. With a growing interest in SR, the threshold-free or barrier-free nonlinear system is also found to demonstrate noise-enhanced signal transmission effects [27–31]. Especially, some nonlinearities with saturation have been observed to exhibit SR with improved signal-to-noise ratio, cross-correlation and mutual information [30,31]. Furthermore, another distinct SR mechanism was explored in coupled or uncoupled parallel arrays of nonlinear systems which significantly extended the concept of SR to broader conditions [32–48]. These related results show that the summed response of parallel nonlinear elements with uncorrelated noise can be more efficient, under certain performance measures, than using a single nonlinearity with no noise. Of course, each element in these parallel arrays can be threshold-free [44–48] or can perform a thresholding operation [32–43], and in the latter case, suprathreshold stochastic resonance (SSR) occurs for a summing network of threshold comparators [33–35].

For the case of arrays of identical threshold comparators, SSR can be also described in terms of stochastic signal quantization [39–42]. In line with this concept, McDonnell et al. have examined SSR with emphasis on finding the optimality of the quantization, in terms of lossy source coding and quantization theory, by using the mean square error (MSE) as the measure of distortion [39–41]. Recently, we investigated the decoding of a quantized signal and proposed an optimal weighted decoding scheme [42]. Our previous study showed that, for particular noise levels and threshold value distributions, the performance of optimally weighting the quantizer responses is superior to the original unweighted array [42].

However, the previous work [42] restricts the optimal weighted decoding scheme only to threshold nonlinearities. In this paper, we will generalize the optimal weighted decoding scheme to arrays composed of arbitrary nonlinearities, and derive the expression of the decoding output for multigroup parameter settings. Specifically, we apply the decoding approach to arrays of saturating sensors. By dividing the nonlinear elements into different sized groups, we compare optimally weighting the element responses with the original unweighted arrays. The results show that, with regular intervals of shifted parameters in multigroups, the MSE performance of optimal weighted decoding is superior to that of the original unweighted arrays for weak and moderate noise intensities. Moreover, this superiority of the MSE distortion becomes more evident with increasing group size. In addition, as the slope parameter varies, the MSE difference of two different decoding schemes reaches a maximum, and then decays to a fixed value. Finally, at a given noise level and for the group size of two, we also compare two different decoding schemes in the case of optimized shifted parameters within each group.

2. Model and method

We consider a weighted summing array of N noisy nonlinear elements receiving an input random signal $x(t)$ with standard deviation σ_x, as shown in Fig. 1. Each element of the array is endowed with the same input–output characteristic, modeled by the static (memoryless) function g. The ith nonlinear element is subject to independent and identically distributed (i.i.d.) additive noise component η_i with standard deviation σ_η, which is independent of the signal $x(t)$. Accordingly, each element produces the output signal $y_i(t) = \mathrm{g}[x(t) + \eta_i(t)]$. The output signal $y_i(t)$ is multiplied by the weighting coefficient w_i ($w_i \in \Re$), so as to yield the weighted output $w_i y_i$. Then, all weighted outputs are summed to give the overall output of array $\hat{y} = \sum_{i=1}^{N} w_i y_i$.

When all weighting coefficients w_i are equal to unity, the decoding method is performed by weighting after summation for $i = 1, 2, \ldots, N$. When the weighting coefficient w_i is arbitrarily chosen, the decoding function is performed by weighting before summation. It is known that if the reconstruction points are linearly spaced, then the optimal reconstruction points are given by Wiener decoding [49] which is carried out by weighting after summation. For the case of $\mathrm{E}[x] = 0$, the reconstructed value $\hat{y}_\mathrm{w}$ with Wiener linear decoding is expressed by [41,49]

$$\hat{y}_\mathrm{w} = \frac{\mathrm{E}[xy]}{\mathrm{var}[y]}(y - \mathrm{E}[y]), \tag{1}$$

where $y = \sum_{i=1}^{N} y_i$ represents the unweighted sum of the array response and $\mathrm{var}[y]$ is the variance of y.

We now consider the model shown in Fig. 1 for arbitrary weights w_i applied to the element outputs y_i. The MSE distortion between the decoded signal and the input is given by

$$\mathrm{MSE} = \mathrm{E}[(x-\hat{y})^2] = \mathrm{E}[\hat{y}^2] - 2\mathrm{E}[x\hat{y}] + \mathrm{E}[x^2]. \tag{2}$$

The optimal weights, $\mathbf{w}^o = [w_1^o, w_2^o, \ldots, w_N^o]^\top$ for minimizing the MSE distortion are given in Appendix A by Eq. (A.5). Then, the decoding $\hat{y}$ for input signal x can be given by

$$\hat{y} = \sum_{i=1}^{N} w_i^o (y_i - \bar{y}_i), \tag{3}$$

where $\bar{y}_i$ is the mean value of each element output. This decoding scheme is termed optimal weighted decoding [42].

We divide the set of nonlinear elements into M ($M \le N$) groups. All nonlinear elements are identical when $M = 1$ and they are all unique when $M = N$ [42]. Generally, in each group with size N_m ($m = 1, 2, \ldots, M$, and $\sum_{m=1}^{M} N_m = N$), we assume all elements have the same system parameters, while the different groups have different system parameters. According to the optimal weighted decoding equation of Eq. (3), the decoding output for this case of grouped nonlinearities can be written as

$$\begin{aligned}\hat{y} &= \sum_{i=1}^{N_1} w_{i,1}^o (y_{i,1} - \bar{y}_i) + \sum_{j=1}^{N_2} w_{j,2}^o (y_{j,2} - \bar{y}_j) + \cdots + \sum_{k=1}^{N_M} w_{k,M}^o (y_{k,M} - \bar{y}_k) \\ &= \sum_{m=1}^{M}\sum_{q=1}^{N_m} w_{q,m}^o (y_{q,m} - \bar{y}_q).\end{aligned} \tag{4}$$

Within each summation on the right side of Eq. (4), the parameters of elements are identical, and therefore the means are equal. Since there is no way to distinguish among each $y_{q,m}$, the weights can be assigned to any $y_{q,m}$. This is equivalent to giving all $y_{q,m}$ the same weight, provided it is equal to the average of the weights returned by the optimization. Thus, the reconstructed signal $\hat{y}$ for M groups is given by

$$\hat{y} = \sum_{m=1}^{M} a_m (y_m - b_m), \tag{5}$$

where the outputs of the elements in each group are $y_m = \sum_{q=1}^{N_m} y_{q,m}$, and the constants a_m and b_m can be expressed as $a_m = (\sum_{q=1}^{N_m} w_q^o)/N_m$ and $b_m = \sum_{q=1}^{N_m} \bar{y}_q$, respectively.

For the case of the group size $M = 1$, the decoding equation of Eq. (5) can be simplified as

$$\hat{y} = a(y - b), \tag{6}$$

where $y = \sum_{i=1}^{N} y_i$, $a = (\sum_{i=1}^{N} w_i^o)/N$, and $b = \sum_{i=1}^{N} \bar{y}_i = \mathrm{E}[y]$. It is proved in Appendix B that, for this case of $M = 1$, the optimal weighted decoding will achieve the same performance as Wiener linear decoding.

3. Numerical results

It is interesting to note that the above-mentioned decoding scheme can be applied to an array composed of arbitrary nonlinear elements. Here, we consider a typical threshold-free element, this is the saturating sensor

$$y_i(t) = \tanh[\beta(x(t) + \eta_i(t))], \quad i = 1, 2, \ldots, N \tag{7}$$

with slope $\beta > 0$, which is linear in βu for small $|u| \ll 1/\beta$ and saturates at ± 1 for large $|u| \gg 1/\beta$. In neuronal information transmission, neurons exhibit saturation in their response except near their threshold [30]. For example, in the perception of visual information in the presence of a very high level of ambient light or in the hearing of acoustic signals in the presence of suddenly high aerostatic pressure, neurons are close to saturation [30]. Some illustrative curves of $y = \tanh(\beta u)$ for various values of β are shown in Fig. 2. It is noted that, as $\beta \to \infty$, the nonlinearity in Eq. (7) tends to the threshold nonlinearity $y = \mathrm{sign}(x)$. Thus, Eq. (7) can actually manifest threshold-free or threshold properties, dependent on β, and is a candidate for evaluating the performance of the optimal weighted decoding scheme.

In the following, we assume that both the signal $x(t)$ and the noise $\eta(t)$ are zero-mean Gaussian distributed with standard deviations σ_x and σ_η, respectively. Then, we explore the elements with this form of nonlinearity of Eq. (7) to examine the MSE distortion of the optimal weighted decoding scheme for various parameter settings, as the ratio $\sigma = \sigma_\eta/\sigma_x$ varies.

3.1. Case of $M = 1$

We first consider the group size $M = 1$, i.e. all elements with identical parameters. For the saturating sensor in Eq. (7), the MSE distortion of the optimal weighted decoding is shown in Fig. 3 as a function of σ for various array sizes N and the

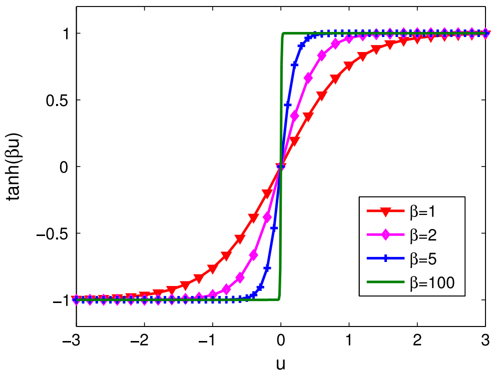

Fig. 2. Plots of $\tanh(\beta u)$ for various values of β.

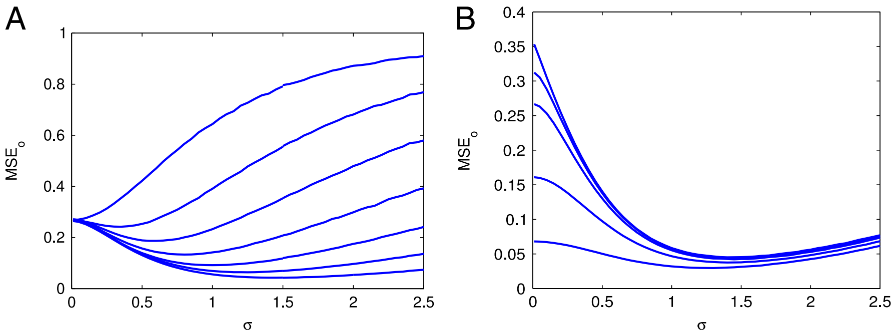

Fig. 3. (A) MSE distortion of optimal weighted decoding against increasing σ in the case of elements with identical nonlinearities, i.e. the group size $M = 1$. Here, from top to bottom, the curves of MSE distortion are plotted for $N = 1, 3, 7, 15, 31, 63, 127$ and the slope $\beta = 5$; (B) MSE distortion of optimal weighted decoding versus σ for $\beta = 1, 2, 5, 10, 100$ (from bottom to top). Here, the array size $N = 127$ and the group size $M = 1$.

slope $\beta = 5$. The main feature visible in Fig. 3(A) is that, as the noise level σ increases, the MSE experiences a nonmonotonic evolution, i.e. SSR effect, except a single sensor with $N = 1$.

For various β and $N = 127$, Fig. 3(B) demonstrates the nonmonotonic MSE distortion versus σ. As β increases, the noise-aided improvement is more pronounced. While, in the absence of noise ($\sigma = 0$), the MSE distortion becomes larger. The reason is that, for larger β, the linear region of Eq. (7) becomes narrower, and the saturating region enlarges, which gives rise to larger distortion of the reconstructed signal. However, the SSR effect of noise-assisted transmission is preserved even if β varies.

3.2. *Case of $M > 1$*

For the case of one group $M = 1$, the performance of optimal weighted decoding shows no superiority over Wiener linear decoding, as indicated in Eq. (B.1). Next, we investigate the case of multigroup, $M > 1$. In order to compare the MSE distortions of the two schemes, we define the percentage difference $P = (\mathrm{MSE_w} - \mathrm{MSE_o})/\mathrm{MSE_o}$, where $\mathrm{MSE_w}$ and $\mathrm{MSE_o}$ are the distortion values for Wiener linear decoding and optimal weighted decoding, respectively. When $P > 0$, it means that the performance of optimal weighted decoding is improved over Wiener linear decoding. Otherwise, Wiener linear decoding is superior.

We now consider the characteristic of multigroups with different shifted parameters

$$g_{q,m}(t) = \tanh[\beta(x(t) + \eta_{q,m}(t) + c_m)], \tag{8}$$

for $m = 1, 2, \ldots, M$. For simplicity, we consider the number of elements in each group is equal, i.e. $N_m = N/M$, and the shifted parameters are set with equal intervals

$$c_m = m/(M + 1). \tag{9}$$

This equal interval method is easily implemented, and has been widely used in A/D converters.

Fig. 4(A) shows the percentage difference P between two decoding schemes against σ for various group sizes M and the slope $\beta = 5$. For comparison, the percentage difference for the case of $M = 1$ is also plotted in Fig. 4(A). It clearly illustrates that optimal weighted decoding is always better than the case of Wiener linear decoding except $M = 1$ and $M = 2$. Moreover, as number of groups M increases, the percentage difference P becomes larger and larger, and reaches the maximum as $M = N$. Notably, for the case of $M = 2$, the value of P is very small and close to zero, and the curve of $M = 1$

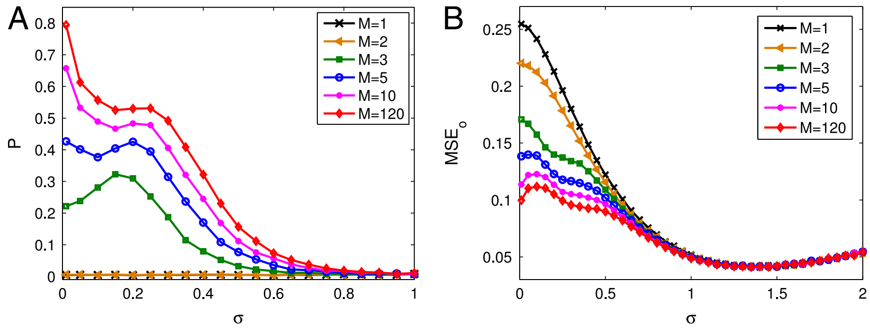

Fig. 4. (A) MSE distortion percentage difference P between optimal weighted decoding and Wiener linear decoding against σ for various group sizes $M = 1, 2, 3, 5, 10, 120$ (from bottom to top). $P = (\mathrm{MSE_w} - \mathrm{MSE_o})/\mathrm{MSE_o}$. (B) MSE distortion of optimal weighted decoding against increasing σ for $M = 1, 2, 3, 5, 10, 120$ (from bottom to top). The shifted parameters $c_m = m/(M+1)$ for $m = 1, 2, \ldots, M$ $(M \le N)$, array sizes $N = 120$, and the slope $\beta = 5$.

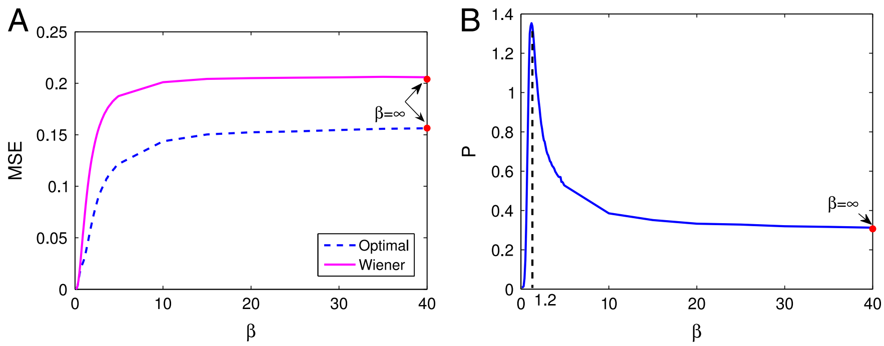

Fig. 5. (A) MSE distortion of optimal weighted decoding compared to Wiener linear decoding against β. (B) MSE distortion percentage difference P between optimal weighted decoding and Wiener linear decoding against β. The group size $M = N = 120$, and $\sigma = 0.1$.

almost overlaps with that of $M = 2$. We also note that the curves in Fig. 4(A) directly reflect the MSE relative change rates between two decoding schemes, and the nonmonotonic behavior of P is closely related to the saturation slope β.

The MSE distortions for optimal weighted decoding are also clearly exhibited in Fig. 4(B) for various group sizes. It is illustrated in Fig. 4(B) that, for small noise levels, the decoding performance greatly improves as the group size M increases—the performance for unique settings ($M = N$) is optimal and identical settings ($M = 1$) are worst. While for very large noise levels, the MSE distortions for all grouped settings tend to that of the identical nonlinearity settings. It is noted that, for weak and moderate noise intensities, the optimal weighted decoding with multigroup is meaningful, otherwise, for large noise levels, this decoding scheme is not effective [40,41].

3.3. Comparison of performance for various slope β

In the above studies, we choose the slope $\beta = 5$ in Eq. (8). It is interesting to compare the performance between optimal weighted decoding and Wiener linear decoding for different values of β. Fig. 5(A) shows the MSE distortion between two decoding schemes against β for group sizes $M = 120$, the array size $N = M$ and $\sigma = 0.1$. It apparently shows that, for both two decoding schemes, the MSE distortion rapidly grows as β increases. As $\beta = \infty$, the saturating sensor becomes the threshold characteristic $y = \mathrm{sign}(x)$, and the corresponding MSE values of two decoding schemes are also marked by solid points at $\beta = 40$. It is seen that, as $\beta > 30$, the performance for saturating nonlinearities gets closer to that of the threshold characteristic. The percentage differences P for the two decoding schemes are also clearly illustrated in Fig. 5(B) for various β. It is seen that, as the slope β increases, the percentage difference P presents a maximum peak at $\beta = 1.2$. For the case of $\beta > 30$, the value of P is almost invariant and $P = 0.317$.

3.4. Tunable shifted parameters c_m

In the above cases, for given group sizes M, the shifted parameters c_m in each group are fixed, as indicated in Eq. (9). However, if the group size M, the number N_m of elements in each group, and shifted parameters c_m are tunable, which scheme has a lower MSE distortion? This is an unconstrained multidimensional optimization problem that deserves to be investigated in the further studies.

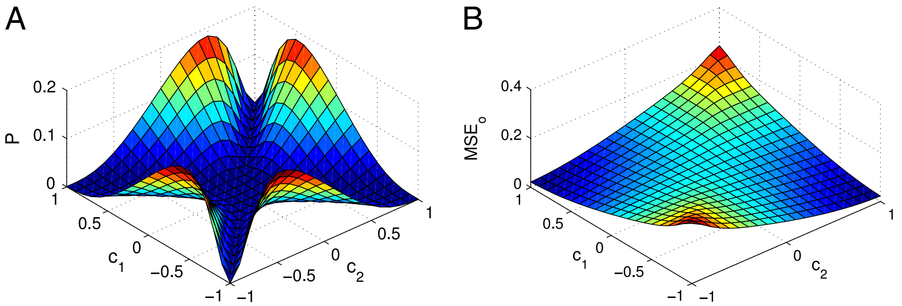

Fig. 6. (A) MSE distortion percentage difference P between optimal weighted decoding and Wiener linear decoding against shifted parameters c_1 and c_2. (B) MSE distortion of optimal weighted decoding against shifted parameters c_1 and c_2. Here, $\sigma = 0.65$, $N_1 = N_2 = 64$, and $\beta = 5$.

Fortunately, for the case of $M = 2$, we can numerically investigate the MSE distortion against parameter pairs of c_1 and c_2 for given noise levels and group sizes. For instance, Fig. 6(A) displays the percentage difference P versus pairs of c_1 and c_2 for given $N_1 = N_2 = 64$ and $\sigma = 0.65$. It is seen that the percentage difference values P are always positive, which means that the performance of optimal weighted decoding is always better than that of Wiener linear decoding. However, for certain shifted pairs of c_1 and c_2, the value of P is very large, but the corresponding MSE distortion of optimal weighted decoding is also very high. For example, it is shown in Fig. 6(B) that, at $c_1 = 0.4$ and $c_2 = 1$, the percentage difference values $P = 0.18$, while the value of MSE is as high as 0.13, with respect to $\text{MSE}_o = 0.022$ ($c_1 = 1$ and $c_2 = -1$). Moreover, for certain pairs of c_1 and c_2, the MSE difference between two decoding schemes is extremely small, i.e. $P \to 0$. Therefore, in the considered case $M = 2$, the performance of Wiener linear decoding is close to that of optimal weighted decoding for the optimized parameters c_1 and c_2. Thus, due to the computational complexity required by optimal weights, the Wiener linear decoding scheme is preferable. When the group size M and the number N_m, as well as c_m, are all tunable, it will be very interesting to examine how much lower the minimum MSE distortion of optimal weighted decoding can be achieved compared to that of Wiener linear decoding.

4. Conclusion and discussions

In this paper, we generalize the optimal weighted decoding scheme to arbitrary nonlinear arrays, and derive the expression of the decoding output for multigroups. We especially apply this optimal weighted decoding scheme to a parallel array of saturating sensors. For various parameter settings, the MSE distortions between the optimal weighted decoding and Wiener linear decoding are compared in detail. The results show that (1) arrays of saturating sensors can elicit the SSR effect as a function of the noise level and the array size; (2) for the case of the group size $M > 2$ with equal interval shifted parameters, the optimal weighted decoding scheme is superior to Wiener linear decoding. Moreover, the MSE distortions significantly decrease as the group size increases, and achieve the minimum when group sizes M are equal to array sizes N; (3) for the case of various slope β, the MSE distortion of optimal weighted decoding is also improved over Wiener linear decoding except very small β; (4) In the case of $M = 2$ with tunable shifted parameters, Wiener linear decoding provides an almost identical MSE distortion as the optimal weighted decoding, but offers a much simpler signal recovery approach.

Notably, in the case of $M > 2$ with equal interval shifted parameters, the optimal weighted decoding scheme provides a much lower MSE in comparison with Wiener linear decoding. This is a significant result for many potential applications inspired by the SSR mechanism, because, in comparison with tunable shifted parameters, the equal interval method is easily implemented, and the MSE distortion performance is greatly improved for weak and moderate noise intensities.

We here assume the internal noise of sensor arrays is additive white noise, i.e. uncorrelated noise. In many practical situations, the idealization of white noise is never exactly realized [24]. An open question is to study the least mean square (LMS) algorithm to optimize the weights aiming to further minimise the MSE distortions. The correlated noise is a case for our current work. Another interesting question is how large the array size N needs to be to acquire satisfactory MSE. The larger N is, the smaller the MSE distortions are, but the more processing power is required. It is valuable to study, for a required constraint on MSE, how large the difference is between the required processing powers for the considered decoding schemes.

The present study contributes to establishing the optimal weighted SSR as a general mechanism of enhancement by noise, with significance for various signal processing tasks, for instance signal reconstruction as considered here. The presented method may be applicable to A/D converters, DIMUS sonar arrays and cochlear implants [20,23,34,41], etc.

Acknowledgments

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Appendix A. The optimal weights

In the appendix, we review the optimal weights derived in the previous work [42]. Let a vector $\mathbf{x}$ of size $(K \times 1)$ denote a sequence of K independent samples drawn from the probability distribution of input signal, a matrix $\mathbf{Y}$ of size $(K \times N)$ denote N responses for each of the K input samples, $\mathbf{w} = [w_1, w_2, \ldots, w_N]^\top$ denote an arbitrary vector of weights. Ideally we seek to find $\mathbf{w}^o$ that satisfies

$$\mathbf{Y}\mathbf{w}^o = \mathbf{x}. \tag{A.1}$$

Although an exact solution potentially exists if $K = N$, it is usually the case that $K > N$ (in practice we desire $K \gg N$) so that the system is overcomplete. In this situation, we follow the standard approach of solving $\mathbf{w}^o$ that minimizes the MSE, i.e., the solution to the optimization problem

$$\mathbf{w}^o = \underset{\mathbf{w}}{\operatorname{argmin}} \sum_{k=1}^{K} (\mathbf{y}_k \mathbf{w} - x_k)^2, \tag{A.2}$$

where $\mathbf{y}_k$ is the kth row of $\mathbf{Y}$.

The minimum of the sum of squares in Eq. (A.2) is found by setting the gradient to zero, and we have

$$\mathbf{Y}^\top \mathbf{x} = \mathbf{Y}^\top \mathbf{Y}\mathbf{w}^o. \tag{A.3}$$

If we assume that the inverse matrix $(\mathbf{Y}^\top\mathbf{Y})^{-1}$ exists, the optimal weights can be written as

$$\mathbf{w}^o = (\mathbf{Y}^\top\mathbf{Y})^{-1}\mathbf{Y}^\top\mathbf{x}. \tag{A.4}$$

Here, $\mathbf{Y}^\top\mathbf{Y}$ is a symmetric matrix, and we assume $\det(\mathbf{Y}^\top\mathbf{Y}) \neq 0$. Otherwise, $(\mathbf{Y}^\top\mathbf{Y})^{-1}$ does not exist. For example, in an array of N identical thresholds of Refs. [34,35], all responses of threshold devices may be all zeros or unities in the absence of noise. The problem that can arise due to the non-existence of $(\mathbf{Y}^\top\mathbf{Y})^{-1}$ can be avoided by regularizing Eq. (A.4) to an approximation of the form [50]

$$\mathbf{w}^o = (\mathbf{Y}^\top\mathbf{Y} + \lambda\mathbf{I})^{-1}\mathbf{Y}^\top\mathbf{x}, \tag{A.5}$$

where λ is a parameter that can be optimized using cross-validation, and $\mathbf{I}$ is the $N \times N$ identity matrix.

Appendix B. The group size $M = 1$

Substituting Eq. (6) into Eq. (2) and noting $\mathrm{E}[x] = 0$, we have the MSE distortion

$$\mathrm{MSE} = a^2\mathrm{E}[y^2] - 2a\mathrm{E}[xy] + \mathrm{E}[x^2] - a^2\mathrm{E}^2[y]. \tag{B.1}$$

When differentiating Eq. (B.1) with respect to a, and setting the result to zero, an optimal expression for a is

$$a^o = \frac{\mathrm{E}[xy]}{\mathrm{var}[y]}. \tag{B.2}$$

From Eq. (B.2), it is seen that Eq. (6) is actually the same as Eq. (1) when this expression for a^o is used. This result indicates that, for the case of arrays with identical elements, the optimal weighted decoding will achieve the same performance as Wiener linear decoding.

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