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Non-factorizable joint probabilities and evolutionarily stable strategies in the quantum prisoner's dilemma game

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ABSTRACT

The well-known refinement of the Nash Equilibrium (NE) called an Evolutionarily Stable Strategy (ESS) is investigated in the quantum Prisoner's Dilemma (PD) game that is played using an Einstein–Podolsky–Rosen type setting. Earlier results report that in this scheme the classical NE remains intact as the unique solution of the quantum PD game. In contrast, we show here that interestingly in this scheme a non-classical solution for the ESS emerges for the quantum PD.

1. Introduction

In the area of quantum games [1–38], a result from a recent paper [28] shows that, in the quantization scheme based on performing generalized Einstein–Podolsky–Rosen–Bohm (EPR–Bohm) experiments [39–44], the two-player quantum game of Prisoner's Dilemma (PD) does not offer a new Nash Equilibrium (NE[1]) [45,46], which is different from the classical NE of the game in which both players play the strategy of defection. This quantization scheme constructs the quantum PD in two steps:

(1) The players' payoff relations are re-expressed in terms of joint probabilities corresponding to generalized EPR–Bohm experiments involving a bipartite system shared between two players. In a run each player receives one part of the system while having two observables both of which are dichotomic. A player's strategy is defined to be entirely classical that consists of a linear combination (with real and normalized coefficients) of choosing between his/her two observables. The scheme embeds the classical game within the quantum game by placing constraints on joint probabilities. These constraints guarantee that for factorizable joint probabilities the classical game emerges along with its particular outcome.

(2) As a set of joint probabilities that violates Bell's inequality must always be non-factorizable, the corresponding quantum game is constructed by retaining the constraints on joint probabilities, obtained in the last step, while they can now be non-factorizable.

By constructing quantum games from non-factorizable joint probabilities, which a quantum-mechanical apparatus can provide, this quantization scheme avoids state vectors and brings out the essence of quantum games without referring to quantum mechanics — an important consideration in developing the present approach to quantum games. Game theory finds applications in a range of disciplines [47] and we believe that more accessible approaches to quantum games remain in need of development.

It turns out that in this quantization scheme the constraints on joint probabilities obtained for the game of PD, which embed the classical game within the quantum game, come out to be so strong that the subsequent permitting joint probabilities to become non-factorizable cannot change the outcome of the game. The quantum PD game that is played in this framework, therefore, generates an outcome identical to the one obtained in the classical game in which both players defect. This finding motivates us in the present Letter to investigate if non-factorizable joint probabilities can bring out some non-classical outcome for a refinement of the NE in the PD game, while not affecting the NE itself.

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[1] In the rest of this Letter we use NE to mean Nash Equilibrium or Nash Equilibria. The correct meaning is judged from the context.

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In this Letter we show that surprisingly this indeed is the case. That is, with an EPR–Bohm type setting for playing a quantum game a set of non-factorizable joint probabilities is able to produce a non-classical outcome in the quantum PD game to a well-known refinement on the set of symmetric Nash equilibria — called an Evolutionarily Stable Strategy (ESS) [48–50]. This contrasts interestingly with the reported result [28] that for the same game, non-factorizable joint probabilities are unable to produce a non-classical outcome for a NE and the classical NE remains intact as the unique solution of the quantum PD game.

Using the quantization schemes of Eisert et al. [5] and Marinatto and Weber [11], the game-theoretic concept of an ESS was originally investigated in the area of quantum games by Iqbal and Toor in a series of papers [12,14,17,20,24] and was reviewed by Iqbal and Cheon in a book chapter [33]. The present Letter addresses the issues raised in these publications using the new approach towards constructing quantum games recently proposed by Iqbal and Cheon [28], which exploits non-factorizable property of quantum-mechanical joint probabilities in the construction of quantum games.

In a recent paper [36] we have investigated a quantum version of the Matching Pennies game played in this quantization scheme to find that non-classical NE emerge in this game for sets of (quantum-mechanical) joint probabilities that maximally violate CHSH form of Bell's inequality [43]. The present Letter considers the PD game in this quantization scheme and explores the fate of a well-known refinement of the NE concept in relation to joint probabilities becoming non-factorizable.

2. Evolutionarily stable strategy

An ESS is the central solution concept of evolutionary game theory [49,50] (EGT). In EGT genes are considered players in survival games and players' strategies are the behavioral characteristics imparted by genes to their host organism, while the payoff to a gene is the number of offspring carrying that gene [50]. The players' strategies (which the players genes play until the biological agents carrying those genes die) and their payoffs become related as host organisms having favourable behavioral characteristics are better able to reproduce than others.

Referring to a pool of genes, the notion of an ESS considers a large population of players (genes) in which players are matched in random pair-wise contests. We call the two players in an interaction to be player 1 and player 2. Each player can play the strategy S or the strategy S' in a pair-wise interaction and the payoff matrix for the game is given as

$$\begin{array}{cc} & \text{Player 2} \\ & \begin{array}{cc} S & S' \end{array} \\ \text{Player 1} \quad \begin{array}{c} S \\ S' \end{array} & \begin{pmatrix} (a_1, b_1) & (a_2, b_2) \\ (a_3, b_3) & (a_4, b_4) \end{pmatrix}, \end{array} \tag{1}$$

where the two entries in the bracket are player 1's and player 2's strategies, respectively. For example, player 1's payoff is $\Pi_1(S, S) = a_1$ when both players play the strategy S. It is found useful to define

$$\mathcal{A} = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}, \qquad \mathcal{B} = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \tag{2}$$

to be player 1's and player 2's payoff matrices, respectively. We write players' payoffs as $\Pi_{1,2}(x, y)$ where subscripts 1 or 2 refer to the players and x and y in bracket are player 1's and player 2's strategies, respectively.

An ESS deals with symmetric games in which

$$\Pi_1(x, y) = \Pi_2(y, x) \quad \text{and} \quad \Pi_1(y, x) = \Pi_2(x, y) \tag{3}$$

saying that, for example, player 1's payoff when she/he plays x and player 2 plays y, is same as the player 2's payoff when she/he plays y and player 1 plays x, where x and y can be either S or S'. In words, in a symmetric game a player's payoff is determined by the strategy, and not by the identity, of a player.

For a symmetric game using subscripts in payoff relations becomes redundant as $\Pi(x, y)$ denotes payoff to an x-player against a y-player. This allows not to refer to players at all and to describe $\Pi(x, y)$ as the payoff to x-strategy against the y-strategy. The game given by the matrix (1) is symmetric when $\mathcal{A} = \mathcal{B}^T$. The game of PD is a symmetric game, which is defined by the constraint $a_3 > a_1 > a_4 > a_2$.

Assume that, in random pair-wise contests, the strategy x is played by a number of players whose relative proportion in the population is ϵ whereas the rest of the population plays the strategy $x^\star$. EGT defines the fitnesses [49,50] of the strategies x and $x^\star$ as

$$\begin{aligned} F(x) &= \epsilon \Pi(x, x) + (1 - \epsilon)\Pi(x, y), \\ F(x^\star) &= \epsilon \Pi(x^\star, x) + (1 - \epsilon)\Pi(x^\star, x^\star), \end{aligned} \tag{4}$$

in terms of which the strategy $x^\star$ is called an ESS when $F(x^\star) > F(x)$, i.e.

$$\epsilon \Pi(x^\star, x) + (1 - \epsilon)\Pi(x^\star, x^\star) > \epsilon \Pi(x, x) + (1 - \epsilon)\Pi(x, x^\star). \tag{5}$$

Since $\epsilon \ll 1$, the terms containing ϵ can be ignored effectively. So $F(x^\star) > F(x)$ implies $\Pi(x^\star, x^\star) > \Pi(x, x^\star)$. If, however, $\Pi(x^\star, x^\star) = \Pi(x, x^\star)$, we need to consider the terms containing ϵ. In this case, $F(x^\star) > F(x)$ requires that $\Pi(x^\star, x) > \Pi(x, x)$. We then define an strategy $x^\star$ to be evolutionarily stable iff for all strategies $x \neq x^\star$ either

$$\begin{aligned} &(1)\ \text{either} \quad \Pi(x^\star, x^\star) - \Pi(x, x^\star) > 0 \quad \text{or if} \\ &(2)\ \Pi(x^\star, x^\star) = \Pi(x, x^\star), \quad \text{then} \quad \Pi(x^\star, x) - \Pi(x, x) > 0. \end{aligned} \tag{6}$$

This definition shows that an ESS is a symmetric NE [49,50] satisfying an additional stability property. The stability property ensures that [50] if an ESS establishes itself in a population, it is able to withstand pressures of mutation and selection. Using a game-theoretic wording, an ESS is a refinement on the set of symmetric Nash equilibria and, though being a static solution concept, it describes dynamic evolutionary situations.

3. ESS in prisoner's dilemma when joint probabilities are factorizable

We consider game-theoretic solution-concept of an ESS in quantum-mechanical regime by observing that quantum mechanics can make only probabilistic predictions and any setup for a quantum game must have a probabilistic description. That is, when a quantum game is constructed using joint probabilities, even the so-called one-shot game must first be translated into some appropriate probabilistic version before one considers its quantum version. This translation permits us, in the following step, to introduce (quantum-mechanical) joint probabilities (that may not be factorizable) and to find if and how such probabilities can change the outcome of the game.

To achieve this in view of the ESS concept, we consider an EPR–Bohm type setting [28] consisting of a bipartite dichotomic physical system that the two players share to play the game (1). This system can be described by the following 16 joint probabilities p_i with $1 \leqslant i \leqslant 16$:

$$\begin{aligned} &p_i = \Pr(\pi_1, \pi_2; a, b) \\ &\quad \text{with } i = 1 + \frac{(1 - \pi_2)}{2} + 2\frac{(1 - \pi_1)}{2} + 4(b - 1) + 8(a - 1), \end{aligned} \tag{7}$$

where π_1 is player 1's outcome, that can have a dichotomic value of $+1$ or -1, obtained when she/he plays the strategy S or S'. We associate $S \sim 1$ and $S' \sim 2$ that then assigns a value for a. Similarly, π_2 is player 2's outcome, that can have a dichotomic value of $+1$ or -1, obtained when she/he plays the strategy S or S'. The same association $S \sim 1$ and $S' \sim 2$ then assigns a value for b. For example, the joint probability corresponding to the situation when player 1's outcome π_1 is $+1$ when she/he plays S' (i.e. $a = 2$), while player 2's outcome π_2 is -1 when she/he plays S (i.e. $b = 1$), is obtained from (7) as p_{10}.

We now define players' payoff relations when they play the game (1) using this (probabilistic) physical system to which the 16 joint probabilities (7) correspond,

$$\Pi_{A,B}(x,y) = \begin{pmatrix} x \\ 1-x \end{pmatrix}^T \begin{pmatrix} \Pi_{A,B}(S,S) & \Pi_{A,B}(S,S') \\ \Pi_{A,B}(S',S) & \Pi_{A,B}(S',S') \end{pmatrix} \begin{pmatrix} y \\ 1-y \end{pmatrix}, \tag{8}$$

where

$$\begin{aligned} \Pi_{A,B}(S,S) &= \sum_{i=1}^{4} (a,b)_i p_i, \\ \Pi_{A,B}(S,S') &= \sum_{i=5}^{8} (a,b)_{i-4} p_i, \\ \Pi_{A,B}(S',S) &= \sum_{i=9}^{12} (a,b)_{i-8} p_i, \\ d\Pi_{A,B}(S',S') &= \sum_{i=13}^{16} (a,b)_{i-12} p_i. \end{aligned} \tag{9}$$

Here T indicates transpose and x and y are the probabilities, definable over a large number of runs, with which Alice and Bob choose the strategies S and S', respectively. Joint probabilities are normalized, i.e.

$$\sum_{i=1}^{4} p_i = 1 = \sum_{i=5}^{8} p_i, \qquad \sum_{i=9}^{12} p_i = 1 = \sum_{i=13}^{16} p_i. \tag{10}$$

A Nash equilibrium strategy pair $(x^\star, y^\star)$ is then obtained from the inequalities:

$$\Pi_A(x^\star, y^\star) - \Pi_A(x, y^\star) \geqslant 0, \qquad \Pi_B(x^\star, y^\star) - \Pi_B(x^\star, y) \geqslant 0, \tag{11}$$

and a symmetric game, defined by the conditions (3), is obtained when

$$\begin{aligned} &\Pi_A(S,S) = \Pi_B(S,S), \qquad \Pi_A(S,S') = \Pi_B(S',S), \\ &\Pi_A(S',S) = \Pi_B(S,S'), \qquad \Pi_A(S',S') = \Pi_B(S',S'). \end{aligned} \tag{12}$$

As it is reported in Ref. [28], in case joint probabilities are factorizable one can find $r, s, r', s' \in [0,1]$ such that [28]

$$\begin{aligned} &p_1 = rr', \qquad p_2 = r(1-r'), \quad \ldots, \quad p_8 = (1-r)(1-s'), \\ &p_9 = sr', \qquad p_{10} = s(1-r'), \quad \ldots, \quad p_{16} = (1-s)(1-s'), \end{aligned} \tag{13}$$

and the Nash inequalities (11) are reduced to [28]

$$\begin{aligned} &(\mathfrak{r} - \mathfrak{s})^T \mathcal{A} \{ y^\star (\mathfrak{r}' - \mathfrak{s}') + \mathfrak{s}' \} (x^\star - x) \geqslant 0, \\ &\{ x^\star (\mathfrak{r} - \mathfrak{s})^T + \mathfrak{s}^T \} \mathcal{B} (\mathfrak{r}' - \mathfrak{s}')(y^\star - y) \geqslant 0, \end{aligned} \tag{14}$$

where

$$\mathfrak{r} = \begin{pmatrix} r \\ 1-r \end{pmatrix}, \qquad \mathfrak{s} = \begin{pmatrix} s \\ 1-s \end{pmatrix},$$
$$\mathfrak{r}' = \begin{pmatrix} r' \\ 1-r' \end{pmatrix}, \qquad \mathfrak{s}' = \begin{pmatrix} s' \\ 1-s' \end{pmatrix}.$$

When joint probabilities are factorizable, the conditions (12) to obtain a symmetric game can be shown to reduce to $\mathcal{A} = \mathcal{B}^T$ and the payoff relations (8) are then simplified to

$$\Pi(x,y) = \begin{pmatrix} x \\ 1-x \end{pmatrix}^T \begin{pmatrix} \Pi(S,S) & \Pi(S,S') \\ \Pi(S',S) & \Pi(S',S') \end{pmatrix} \begin{pmatrix} cy \\ 1-y \end{pmatrix}, \tag{15}$$

where

$$\begin{aligned} &\Pi(S,S) = \mathfrak{r}^T \mathcal{M} \mathfrak{r}', \qquad \Pi(S,S') = \mathfrak{r}^T \mathcal{M} \mathfrak{s}', \\ &\Pi(S',S) = \mathfrak{s}^T \mathcal{M} \mathfrak{r}', \qquad \Pi(S',S') = \mathfrak{s}^T \mathcal{M} \mathfrak{s}', \end{aligned} \tag{16}$$

and $\mathcal{M} = \mathcal{A} = \mathcal{B}^T$. The second inequality in (11) is $\Pi_B(x^\star, y^\star) - \Pi_B(x^\star, y) \geqslant 0$ that becomes $\Pi_A(y^\star, x^\star) - \Pi_A(y, x^\star) \geqslant 0$ for a symmetric game. Comparing it to the first inequality in (11) gives $x^\star = y^\star$ and $x = y$ and the definition of a symmetric NE is reduced simply to $\Pi(x^\star, x^\star) - \Pi(x, x^\star) \geqslant 0$.

Evaluating the two parts of the ESS definition (6) from a symmetric game payoff relations (15) we find

$$\begin{aligned} &\Pi(x^\star, x^\star) - \Pi(x, x^\star) = (x^\star - x)(x^\star \Delta_1 + \Delta_2), \\ &\Pi(x^\star, x) - \Pi(x, x) = (x^\star - x)(x \Delta_1 + \Delta_2) \end{aligned} \tag{17}$$

where $\Delta_1 = \Pi(S,S) - \Pi(S',S) - \Pi(S,S') + \Pi(S',S')$ and $\Delta_2 = \Pi(S,S') - \Pi(S',S')$. Now Δ_1 and Δ_2 are evaluated using (16) as

$$\Delta_1 = (r-s)(r'-s')\Omega_1 \quad \text{and} \quad \Delta_2 = (r-s)(s'\Omega_1 - \Omega_2), \tag{18}$$

where $\Omega_1 = a_1 - a_2 - a_3 + a_4$ and $\Omega_2 = a_4 - a_2$. Recall that PD is defined by the constraints $a_3 > a_1 > a_4 > a_2$ and we have $\Omega_2 > 0$, which asks for a natural association of the strategy of defection in PD to the strategy $x^\star = 0$ played in the present setting. When both players play this strategy we obtain from Eq. (15) $\Pi(0,0) = \Pi(S',S')$, which is the payoff to each player in the classical game when they both defect. With this association Eqs. (17) give $\Pi(0,0) - \Pi(x,0) = -x\Delta_2$ and $\Pi(0,x) - \Pi(x,x) = -x(x\Delta_1 + \Delta_2)$, which correspond to the first and second parts of the ESS definition (6), respectively. For this strategy if we take

$$s' = \Omega_2 / \Omega_1, \qquad r - s = r' - s', \tag{19}$$

then the two parts of the ESS definition are reduced to

$$\begin{aligned} &\Pi(0,0) - \Pi(x,0) = 0, \\ &\Pi(0,x) - \Pi(x,x) = -x^2 (r-s)^2 \Omega_1. \end{aligned} \tag{20}$$

As $\Omega_1 = \Omega_2 - \Omega_3$, where $\Omega_3 = (a_3 - a_1)$, for PD both $\Omega_2, \Omega_3 > 0$. As $\Omega_2 > 0$ and $s' = \Omega_2/\Omega_1$ is a probability we require the constraint $1 > \Omega_2/\Omega_1 > 0$, so $\Omega_1 > \Omega_2 > 0$, from which one obtains $\Omega_2 > \Omega_3$, i.e.

$$a_4 - a_2 > a_3 - a_1 \tag{21}$$

along with this, of course, we also have $a_3 > a_1 > a_4 > a_2$. The extra requirement (21) defines a subset of the games that are put under the name of a generalized PD. For this game the result (20) states that the strategy $x^\star = 0$ is not an ESS, though it is a symmetric NE, when joint probabilities are factorizable, in the sense described by (13), and have the constraints (19) imposed on them.

4. Obtaining the quantum game

There can be several different possible routes in obtaining a quantum game. The general idea is to establish correspondence, as a first step, between classical feature of a physical system and a classical game in the sense that classical game results because of those features. In the following step, the classical feature are replaced by quantum feature, while the obtained correspondence in the first step is retained. One then looks at the impact which the quantum feature has on the solution/outcome of the game under consideration. As the mentioned correspondence can be established in several possible ways, there can be many different routes in obtaining a quantum game.

To consider ESS in quantum PD we translate playing of this game in terms of factorizable joint probabilities, which is achieved in the previous section. We then find constraints on these probabilities ensuring that the classical game remains embedded within the quantum game, which is achieved by Eq. (19). For factorizable joint probabilities Eqs. (13) hold that permit us to translate the constraints (19) in terms of joint probabilities. In the following step, the joint probabilities are allowed to be non-factorizable, while they continue to be restricted by the obtained constraints.

Joint probabilities p_i become non-factorizable when one cannot find $r, s, r', s' \in [0, 1]$ such that p_i can be expressed in terms of them, i.e. as given in (13). The same payoff relations (8), therefore, correspond to the quantum game, whose parts are given by (9), and players' strategies remain exactly the same.

We require that the constraints (19), when they are re-expressed using (13) in terms of joint probabilities p_i, remain valid while p_i are allowed to be non-factorizable. We notice that Eqs. (13) allow re-expressing the constraints (19) in terms of p_i as

$$\begin{aligned} &r = p_1 + p_2, \qquad r' = p_1 + p_3, \\ &s = p_9 + p_{10}, \qquad s' = p_5 + p_7, \end{aligned} \tag{22}$$

and the constraints (19) take the form

$$\begin{aligned} &p_5 + p_7 = \Omega_2/\Omega_1, \\ &p_1 + p_2 - p_9 - p_{10} = p_1 + p_3 - p_5 - p_7. \end{aligned} \tag{23}$$

At this stage we refer to the analysis of joint probabilities in generalized EPR–Bohm experiments by Cereceda [44] reporting that eight out of sixteen joint probabilities can be eliminated using the normalization constraints (10) and the causal communication constraints given as follows:

$$\begin{aligned} &p_1 + p_2 = p_5 + p_6, \qquad p_1 + p_3 = p_9 + p_{11}, \\ &p_9 + p_{10} = p_{13} + p_{14}, \qquad p_5 + p_7 = p_{13} + p_{15}, \\ &p_3 + p_4 = p_7 + p_8, \qquad p_{11} + p_{12} = p_{15} + p_{16}, \\ &p_2 + p_4 = p_{10} + p_{12}, \qquad p_6 + p_8 = p_{14} + p_{16}. \end{aligned} \tag{24}$$

The constraints (10), (24), of course, do hold for factorizable joint probabilities that are given by Eqs. (13). Cereceda expresses probabilities p_2, p_3, p_6, p_7, p_{10}, p_{11}, p_{13}, p_{16} in terms of probabilities p_1, p_4, p_5, p_8, p_9, p_{12}, p_{14}, p_{15} as

$$\begin{aligned} &p_2 = (1 - p_1 - p_4 + p_5 - p_8 - p_9 + p_{12} + p_{14} - p_{15})/2, \\ &p_3 = (1 - p_1 - p_4 - p_5 + p_8 + p_9 - p_{12} - p_{14} + p_{15})/2, \\ &p_6 = (1 + p_1 - p_4 - p_5 - p_8 - p_9 + p_{12} + p_{14} - p_{15})/2, \\ &p_7 = (1 - p_1 + p_4 - p_5 - p_8 + p_9 - p_{12} - p_{14} + p_{15})/2, \\ &p_{10} = (1 - p_1 + p_4 + p_5 - p_8 - p_9 - p_{12} + p_{14} - p_{15})/2, \\ &p_{11} = (1 + p_1 - p_4 - p_5 + p_8 - p_9 - p_{12} - p_{14} + p_{15})/2, \\ &p_{13} = (1 - p_1 + p_4 + p_5 - p_8 + p_9 - p_{12} - p_{14} - p_{15})/2, \\ &p_{16} = (1 + p_1 - p_4 - p_5 + p_8 - p_9 + p_{12} - p_{14} - p_{15})/2, \end{aligned} \tag{25}$$

and the payoff relations (8) now involve only eight 'independent' probabilities.

5. ESS in quantum prisoner's dilemma

For the strategy of defection ($x^\star = 0$) in the quantum game with payoff relations (8), the ESS definition (6) and Eqs. (17) give

$$\Pi(0,0) - \Pi(x,0) = x\left\{\Pi(S',S') - \Pi(S,S')\right\}, \tag{26}$$

which is equated to zero so that the strategy $x^\star = 0$ remains a symmetric NE in the quantum game, as it is the case in the game when joint probabilities are factorizable, and which is described by Eqs. (20). With setting $\Pi(S',S') = \Pi(S,S')$ the second part of the ESS definition (6), which is evaluated in (17), reduces itself to

$$\Pi(0,x) - \Pi(x,x) = x^2\left\{\Pi(S',S) - \Pi(S,S)\right\}. \tag{27}$$

With Cereceda's analysis and using Eqs. (25), setting $\Pi(S,S') - \Pi(S',S') = 0$ results in

$$\begin{aligned} &p_1 + p_5 + p_8 + p_{12} + p_{14} + p_{15} = 1 + p_4 + p_9, \\ &p_4 + p_5 + p_8 + p_9 + p_{14} + p_{15} = 1 + p_1 + p_{12}, \end{aligned} \tag{28}$$

and under the constraints (28) the strategy $x^\star = 0$ then remains a symmetric NE even for non-factorizable joint probabilities. Also, using Eqs. (25) the constraints (23) can be re-expressed in term of 'independent probabilities' as

$$\begin{aligned} &1 - p_1 + p_4 + p_5 - p_8 + p_9 - p_{12} - p_{14} + p_{15} = \Omega_2/\Omega_1, \\ &p_5 + p_{12} = p_8 + p_9, \end{aligned} \tag{29}$$

which allows us to arbitrarily eliminate probabilities p_1 and p_{12} from the constraints (28) to re-express them as

$$p_5 + p_{15} = \Omega_2/\Omega_1, \qquad p_8 + p_{14} = 1 - \Omega_2/\Omega_1. \tag{30}$$

Using Eqs. (25), while considering the strategy $x^\star = 0$ for the second part of the ESS definition (27) becomes

$$\Pi(0,x) - \Pi(x,x) = x^2\left\{\Omega_3(p_1 - p_9) + \Omega_2(p_{12} - p_4)\right\}, \tag{31}$$

which simplifies further when we eliminate p_1 and p_{12} using (29) and afterwards eliminate p_{14} and p_{15} using (28) to obtain

$$\Pi(0,x) - \Pi(x,x) = x^2(p_8 + p_9 - p_4 - p_5)\Omega_1. \tag{32}$$

As $\Omega_1 > 0$, the strategy $x^\star = 0$ thus becomes an ESS if

$$p_8 + p_9 > p_4 + p_5, \tag{33}$$

and when joint probabilities p_i satisfy constraints (29), (28), along with the constraints given by normalization and causal communication.

6. Discussion

The game-theoretic solution concept of an ESS is investigated within a quantization scheme that constructs quantum games from the non-factorizable property of quantum-mechanical joint probabilities. Neither entanglement nor violation of Bell's inequality [41,43] is used explicitly in this construction.[2]

Eq. (32) shows that probabilities p_4, p_5, p_8, p_9 can be taken to be 'independent' as, out of the remaining four probabilities, the

[2] As a set of joint probabilities that violates Bell's inequality must be non-factorizable, a non-classical solution of a game played in this scheme can emerge even when there is no entanglement and the quantum state under consideration is separable. This is understandable as a direct link between violation of Bell's inequality and separability of a quantum state is established only for pure states via Gisin's theorem [51]. A separable mixed state may still violate a Bell's inequality, which will correspond to a set of non-factorizable joint probabilities. This is also consistent with reported results [52] showing, for example, that a quantum game can have a solution in the so-called "pseudo-classical domain", in which Bell's inequality is not violated. These domains exist between fully classical and fully quantum domains — where Bell's inequality is violated.

probabilities p_{14} and p_{15} are obtained from (30) and probabilities p_1 and p_{12} are obtained from (29). The remaining eight probabilities p_2, p_3, p_6, p_7, p_{10}, p_{11}, p_{13}, p_{16} are then obtained from (25). The scheme used to obtain a quantum game assumes that a set of non-factorizable joint probabilities, which satisfies normalization (10) and the causal communication constraint (24) can always be generated by some bipartite quantum state (pure or mixed) provided that the set does not violate CHSH form of Bell's inequality beyond Cirel'son's limit [53].

A natural question here is to ask if Bell's inequality is violated by requiring $p_8 + p_9 > p_4 + p_5$, which makes the strategy of defection ($x^\star = 0$) an ESS. To answer this we consider probabilistic form [44] of CHSH version of Bell's inequality [43] expressed as $-2 \leqslant \Delta \leqslant 2$ where $\Delta = 2(p_1 + p_4 + p_5 + p_8 + p_9 + p_{12} + p_{14} + p_{15} - 2)$. We insert values for p_1, p_{12}, p_{14}, p_{15} using (29), (30) to obtain

$$\Delta = 2(2p_4 + p_9 - 1). \tag{34}$$

Now, comparing (34) to (33) shows that the violation of the CHSH inequality is not essential for the strategy of defection to be an ESS for a set of non-factorizable probabilities, when for a factorizable set of probabilities this strategy is non-ESS and a symmetric NE only.

To summarize, a non-classical solution for an ESS in the quantum PD game has been shown to emerge due to joint probabilities that are non-factorizable. An ESS offers a stronger solution concept than a NE and we consider the situation in which the same NE, consisting of the strategy of defection on behalf of both players, continues to exist in both the classical and the quantum versions of the PD game, which correspond to situations of joint probabilities being factorizable and non-factorizable, respectively. It is shown that non-factorizable quantum joint probabilities can bring evolutionary stability to the strategy of defection via the 2nd part of the ESS definition (6).

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