Minimal Brownian Ratchet: An Exactly Solvable Model

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We develop an exactly-solvable three-state discrete-time minimal Brownian ratchet (MBR), where the transition probabilities between states are asymmetric. By solving the master equations we obtain the steady-state probabilities. Generally the steady-state solution does not display detailed balance, giving rise to an induced directional motion in the MBR. For a reduced two-dimensional parameter space we find the null-curve on which the net current vanishes and detailed balance holds. A system on this curve is said to be balanced. On the null-curve, an additional source of external random noise is introduced to show that a directional motion can be induced under the zero overall driving force. We also indicate the off-balance behavior with biased random noise.

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The Brownian ratchet and pawl system was first correctly explained by Smoluchowski [1] and later revisited by Feynman [2] – this has inspired much activity in the area of Brownian ratchets, despite flaws in Feynman's analysis of the thermal efficiency of the ratchet engine [3] and detailed balance [4].

Interest has revived because molecular motors [5] have been described in terms of Brownian ratchet [6, 7] models. Another area of interest has been in Parrondo's paradox [8] where losing strategies cooperate to win. This can be illustrated in terms of games which lose when played individually, but win when alternated – this has been shown to be a discrete-time Brownian ratchet [12], otherwise known as a "Parrondian ratchet."

Jarzynski *et al.* [13] developed an exactly solvable Brownian ratchet that can be operated as heating system or refrigerator, depending on the parameters between two heat reservoirs of different temperatures. However this is treated as a six state system and solution is via matrix inversion of coupled linear equations. The derivation is somewhat complex, so the physical picture and key ingredients of the observed properties are obscured.

Westerhoff *et al.* [14] have analyzed enzyme transport using a four-state model. In this paper, for the first time, we develop a three-state discrete-time Brownian ratchet model that can be solved analytically. We call it the *Minimal Brownian Ratchet* (MBR) [9]. By setting up and solving the steady-state solution of the corresponding master equations we obtain the null-surface, in the parameter space, of the noise-free system. The obtained solution does not show any critical behavior and can be suitably explained in terms of non-singular behaviors.

The minimal ingredients of the ratchet are an asymmetric potential and noise. In Fig. 1 we show the state diagram of the MBR. The MBR has three states, $\{S_0, S_1, S_2\}$, where the transition probabilities between states are asymmetric. The transition probability that a random walker in state S_k steps in the positive direction is p_k. The probability of a shift in the negative direction is $(1-p_k)$. This is true of $k \in \{0, 2, 3\}$. We define the

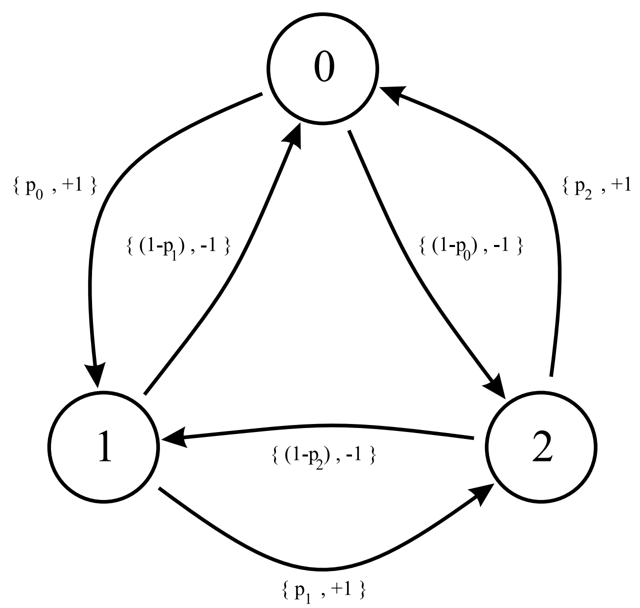

FIG. 1: State-transition diagram of 3-state discrete-time Brownian ratchet with asymmetric transition probabilities p_0, p_1 and p_2 in the positive direction (counter-clockwise) and $(1-p_0)$, $(1-p_1)$ and $(1-p_2)$ in the negative direction (clockwise). Each transition has two numbers associated with it, $\{p_k, R_k\}$. The first number in the brackets, p_k, is the conditional probability of that transition (given the initial state). The second number, R_k, is the reward associated with that transition. All "winning" transitions have a reward of $+1$ and all "losing" transitions have a reward of -1.

positive direction as counterclockwise. The condition of total probability, $p_1 + (1-p_1) = p_0 + (1-p_0) = 1$, is automatically enforced by our choice of symbols.

It is straight forward to set up the following difference equations for the probability distributions of the system:

$$\begin{aligned} P_0(t+1) &= P_1(t)\cdot(1-p_1) + P_2(t)\cdot p_2 \qquad (1)\\ P_1(t+1) &= P_0(t)\cdot p_0 + P_2(t)\cdot(1-p_2)\\ P_2(t+1) &= P_0(t)\cdot(1-p_0) + P_1(t)\cdot p_1 \end{aligned}$$

where $P_k(t)$ is the probability for the random walker at time t to be on the state of S_k. This can be written in matrix form as $\mathbf{P}_{t+1} = \mathbf{P}_t B$ where $\mathbf{P}_t$ is the time varying probability (row) vector at time t and B is the probability transition matrix. We can write:

$$[B_{i,j}] = \begin{bmatrix} 0 & p_0 & (1-p_0) \\ (1-p_1) & 0 & p_1 \\ p_2 & (1-p_2) & 0 \end{bmatrix}. \tag{2}$$

The steady-state probability, after a sufficiently long time, $\lim_{t\to\infty} \mathbf{P_t} = \mathbf{P}_\infty$ is simply given as

$$\mathbf{P}_\infty = \mathbf{P}_\infty B \tag{3}$$

which is a characteristic value problem. A partial probability current, I, can be defined as

$$I = P_0 p_0 - P_1(1-p_1) = P_1 p_1 - P_2(1-p_2) = P_2 p_2 - P_0(1-p_0). \tag{4}$$

This can be written in matrix form as $\mathbf{I}_t = \mathbf{P}_t F$ where $\mathbf{I}_t$ represents the current of probability between the various states at time t. In steady-state, all the currents have the same value, I. The matrix F relates the time varying probability vector, $\mathbf{P}_t$ to current, $\mathbf{I}_t$. It has the value:

$$[F_{i,j}] = \begin{bmatrix} p_0 & 0 & -(1-p_0) \\ -(1-p_1) & p_1 & 0 \\ 0 & -(1-p_2) & p_2 \end{bmatrix}. \tag{5}$$

If $I = 0$ there is no net current and detailed balance is satisfied, otherwise there exists a net current and the system will assume a non-equilibrium steady-state. If we consider the special case when $p_0 = p_1 = p_2 = 1$ then it is clear that the mean rate of change of state will always be $+1$ per time tick. This is three times the current as defined in Eq. 4. This apparent paradox is resolved by the fact that the mean rate of change of state must be the sum of all currents over all states. If we regard the state, k, as being a function of time then we could define the slope of the graph as $Y = E\left[k(t+1) - k(t)\right]$, where $E\left[\cdot\right]$ is the expected value operator. Using this definition, in the steady-state limit, we obtain $Y = 3I$ in agreement with Taylor *et al.* [10].

Solving the Eqs. 3 together with the normalization condition,

$$P_0 + P_1 + P_2 = 1\ , \tag{6}$$

is again straightforward. We present the closed form of the solutions for steady-state probability and current in the Appendix. If we impose the constraint that $Y = 0$ then it follows that the condition for detailed balance [11] is

$$p_0 p_1 p_2 = (1-p_0)(1-p_1)(1-p_2)\,, \tag{7}$$

which is the equation of a two-dimensional surface in the three dimensional parameter space, $\{p_0, p_1, p_2\}$.

It is possible to further restrict the choices of $\{p_0, p_1, p_2\}$ without losing the important properties of the ratchet. Parrondo's original definition imposed the further constraints $p_0 = q$ and $p_1 = p_2 = p$. This reduced the parameter space to a two dimensional space with parameters $\{q, p\}$. In two dimensional $\{q, p\}$ parameter space, the condition of detailed balance, i.e., $I = 0$, gives the equation for a curve which we call the null-curve:

$$q = \frac{1}{1 + \left(\frac{p}{(1-p)}\right)^2}. \tag{8}$$

This is a special restricted case of the more general equation, Eq 7. The null-curve is a special case of the more general null-surface or null-hypersurface, in higher dimensions. Fig. 2 shows the "winning" and "losing" regions of the MBR. Note that as expected from the symmetry of the system the curve is invariant under the transformations $q \to (1-q)$ and $p \to (1-p)$. This also apparent from a consideration of Eq. 7.

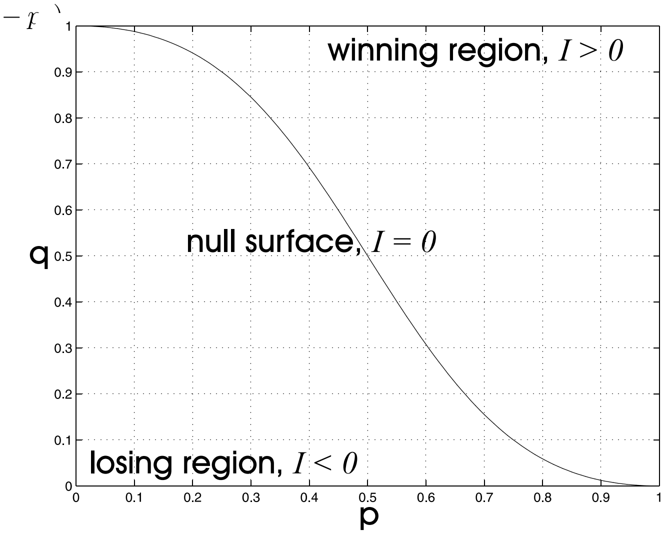

FIG. 2: Null-surface of 3-state discrete-time Brownian ratchet. On the null-surface, $q = 1/\left(1 + (p/(1-p))^2\right)$, the current vanishes. Above the curve, the system has positive net current. Below the curve, the system has negative net current.

On the null-surface, we add more noise controlled by parameter γ to the MBR as follows. With a probability of γ, a random walker follows the dynamics of the MBR otherwise, with the probability of $(1-\gamma)$, the walker randomly takes a right or left step. For $\gamma = 0$ the model is exactly same as the original MBR and the net current remains zero since we are on the null-surface. In the other limit, for $\gamma = 1$, the randomizing process dominates and the system reduces to a simple unbiased random walk where the net current is also zero. However, counter-intuitively, for $0 < \gamma < 1$ non-zero current is induced by introducing the noise parameter. It is important to note that γ *influences* the level of noise in the ratchet but is not *identical* with the noise. We refer to γ as a noise "parameter."

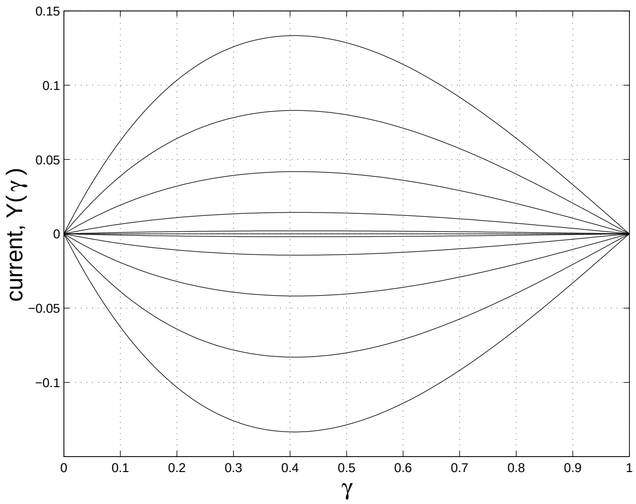

FIG. 3: Total probability current versus noise parameter, γ, on the null-surface. For values of $p \neq \frac{1}{2}$, additional noise induces a net current which increases, in magnitude, with increasing γ and then decreases, in magnitude, to zero after γ exceeds an optimum value. The bottom curve corresponds to $p = 0$. All the other curves represent increments of $\Delta p = \frac{1}{10}$. The top curve corresponds to $p = \frac{1}{2}$. The middle curve corresponds to $p = 1$. Parrondo's original games had $p = \frac{3}{4}$.

In Fig. 3 we show the current versus noise parameter, γ, for different values of parameters p and $q = 1/1 + p^2/(1-p)^2$. The exact expression for the current is given in the Appendix. The current has an extremal value at $\gamma \approx 0.382$. The position of this extremum appears to be independent of p.

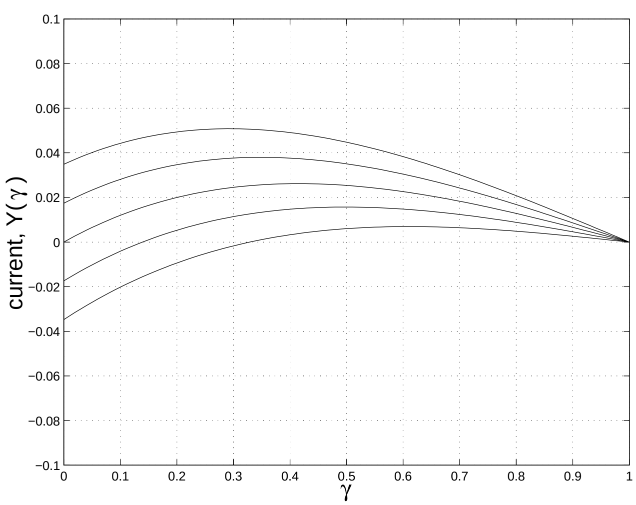

FIG. 4: Total probability current versus noise parameter, γ, off the null-surface.

Fig. 4 shows that the dependence of the maximum of current is on p. In the off-balance region, $p \neq 1/\left(1 + \left(p/\left(1-p\right)\right)^2\right)$, the net current is not zero for $\gamma = 0$ but still should be zero for $\gamma = 1$ and the intermediate behavior is qualitatively the same as the balanced behavior. The actual values of p and q for the various curves in Fig. 4 are in linear increments of 0.01 for p and 0.01 for q. The top curve has parameters $p = 0.77$ and $q = 0.12$. The bottom curve has parameters $p = 0.73$ and $q = 0.08$. As γ is increased from zero, the current increases to a maximum and then falls off, which has the form of stochastic resonance[15].

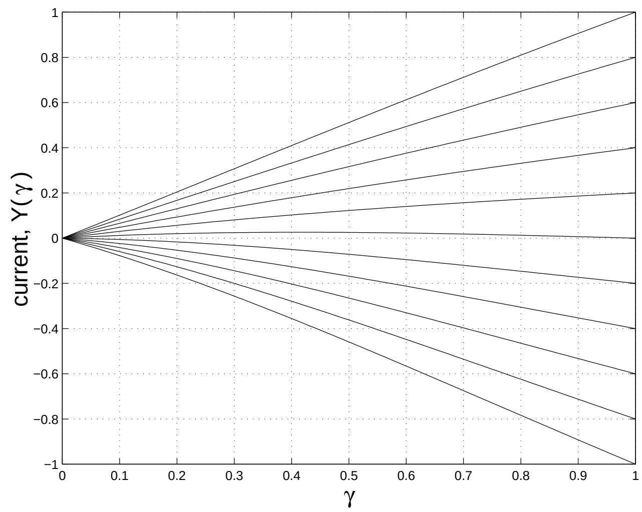

FIG. 5: Total probability current versus noise parameter, γ, off the null-surface with biased noise. The bottom curve corresponds to $s = 0$. All the other curves represent increments of $\Delta s = \frac{1}{10}$. The top curve corresponds to $s = 1$.

In Fig. 5 we show the off-balanced behavior of the current on the straight line which intersects the null-surface at the point $(p = 0.75, q = 0.10)$. These also correspond to Parrondo's original choices for the games.

We generalize the MBR by introducing a bias into the added noise, with a new parameter ϵ. The walker takes a right step with probability of $0.5 - \epsilon$ and a left step with probability of $0.5 + \epsilon$. For $\epsilon \neq 0$ this noise introduces nonzero net current. The new parameter, ϵ, is essentially a measure of the degree bias in the added noise. The current versus noise curves, as a function of ϵ, are not shown since they coincide with those of Fig. 5 under the transform of $\gamma \to (1-\gamma)$. The biased noise essentially produces the same un-balanced behavior.

We can generalize this model to a system of size N by repeating the unit cell of modulo-3 N times with a periodic boundary condition. In this case, the periodic potential ensures

$$p_k(t) = p_{k+3n}(t) \ \forall \ n = 0, \pm 1, \pm 2, \cdots . \tag{9}$$

Because of the normalization condition, $\sum_{k=1}^{N} P_k(t) = 1$, the current will be reduced by factor of N. The corresponding master equations and solutions are exactly same as the minimal model.

For different moduli, in principle, we can also set up the master equations and solve them exactly by matrix inversion for the set of linear equations. It can be shown that these results have qualitatively the same statistical behavior as the 3-state MBR.

In summary, we have developed a minimal discrete-time Brownian ratchet and study the statistical properties of the model by solving the master equations. We obtain the steady-state solution of the MBR, which is independent of the initial conditions of the model, showing the minimal features of the discrete-time Brownian ratchet. We obtained the null-surface with and without a noise term.

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I. APPENDIX, CALCULATION OF THE TOTAL PROBABILITY CURRENT IN THE MBR

It is possible to solve Eq. 3 to obtain the steady-state value of the time-varying probability vector, $\pi = \lim_{t\to\infty} \mathbf{P}$, using the standard methods for characteristic value, or eigenvalue, problems. We obtain $\pi = [\pi_0, \pi_1, \pi_2]$ where

$$\pi_0 = \frac{1 - p_1 + p_1 p_2}{2 + (1-p_0)(1-p_1)(1-p_0) + p_0 p_1 p_2} \tag{10}$$

and

$$\pi_1 = \frac{1 - p_2 + p_0 p_2}{2 + (1-p_0)(1-p_1)(1-p_0) + p_0 p_1 p_2} \tag{11}$$

and

$$\pi_2 = \frac{1 - p_0 + p_0 p_1}{2 + (1-p_0)(1-p_1)(1-p_0) + p_0 p_1 p_2}. \tag{12}$$

These expressions are consistent with the results of Pearce [16]. It is easy to check that they are the solution to Eq. 3 by direct substitution.

We can substitute the results from Equations 10, 11 and 12 into Eq. 4 to solve for the partial current, I.

$$I = \frac{(p_0 + p_1 + p_2) - (p_0 p_1 + p_0 p_2 + p_1 p_2) + 2 p_0 p_1 p_2 - 1}{3 - (p_0 + p_1 + p_2) + (p_0 p_1 + p_0 p_2 + p_1 p_2)}. \tag{13}$$

The terms in this equation are very similar to those in Newton's relations for polynomial expansions. This might lead us to suspect that further simplification is possible. We can write:

$$I = \frac{(1 + p_0 p_1 p_2) - (1 + (1-p_0)(1-p_1)(1-p_2))}{(1 + p_0 p_1 p_2) + (1 + (1-p_0)(1-p_1)(1-p_2))}. \tag{14}$$

This can be further simplified to yield:

$$I = \frac{1-\Gamma}{1+\Gamma} \tag{15}$$

where

$$\Gamma = \frac{1 + (1-p_0)(1-p_1)(1-p_2)}{1 + p_0 p_1 p_2}. \tag{16}$$

Eq. 15 has the same form as the equation for the admittance of a transmission line with a reflection coefficient Γ. We note that Γ lies in the range $(1/2) \le \Gamma \le 2$ which is different to the admissible range of values for a physical transmission line. We note that the expected value of the rate of change of state is $Y = 3 \cdot I$.

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