

# Advantage of a quantum player over a classical one in $2 \times 2$ quantum games

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We study a general  $2 \times 2$  symmetric entangled quantum game. When one player has access only to classical strategies, while the other can use the full range of quantum strategies, there are ‘miracle’ moves available to the quantum player that can direct the game towards the quantum player’s preferred result regardless of the classical player’s strategy. The advantage pertaining to the quantum player is dependent on the degree of entanglement. Below a critical level, dependent on the pay-offs in the game, the miracle move is of no advantage.

**Keywords:** quantum games; game theory;  
degree of entanglement; entanglement thresholds

## 1. Introduction

Quantum game theory is an interesting new development in the fields of game theory and quantum information. First initiated by Meyer (1999), a protocol for two-player two-strategy ( $2 \times 2$ ) games was developed by Eisert *et al.* (1999) and extended to multi-player games by Benjamin & Hayden (2001*a*). Where both players have access to the full set of quantum strategies, there is no Nash equilibrium (NE) amongst pure strategies (Benjamin & Hayden 2001*b*), although there is an infinite set of equilibria among mixed quantum strategies (Eisert & Wilkens 2000). A pure quantum strategy specifies a particular quantum operator to apply contingent on the game situation, whereas a mixed quantum strategy specifies a probabilistic mixture of operators. In a dynamical game, one generally would not expect convergence to an NE. In an entangled quantum game, if the (pure) strategy of one player is known, the other player can produce any desired final state by a suitable (pure) counter strategy, assuring them of the maximum pay-off. Hence it is always possible for one of the players to improve his/her pay-off by a unilateral change in strategy. For a discussion, see the recent review of quantum games by Flitney & Abbott (2002*a*).

When one player is restricted to classical moves and the other is permitted the full quantum domain, the quantum player has a clear advantage. Eisert found that in a two-player prisoners’ dilemma, the quantum player could guarantee an expected pay-off not less than that of mutual cooperation, while the classical player’s reward was substantially smaller. The advantage gained by the quantum player was found to be dependent on the level of entanglement. Below a critical level, the quantum player

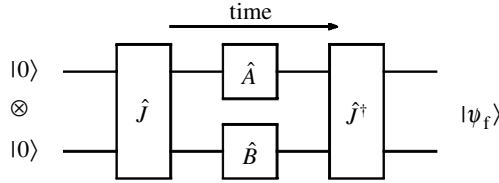


Figure 1. The flow of information or qubits (solid lines) in a general two-person quantum game.  $\hat{A}$  is Alice's move,  $\hat{B}$  is Bob's,  $\hat{J}$  is an entangling gate and  $\hat{J}^\dagger$  is a disentangling gate.

could do no better than adopting the classical dominant strategy. It is interesting to speculate on the relationship between the advantage obtainable by a quantum player over their classical rival and the advantage a quantum algorithm has over a classical one.

In this work we extend the result of Eisert *et al.* (1999) and a later generalization by Du *et al.* (2001, 2003) for the prisoners' dilemma to a general  $2 \times 2$  quantum game. Section 2 will summarize the protocol for  $2 \times 2$  entangled quantum games. In § 3 we determine the four different miracle moves, depending on the game result most desired by the quantum player, and consider the pay-offs as a function of the degree of entanglement. Section 4 presents threshold values of the entanglement for various game situations and § 5 briefly considers extensions to larger strategic spaces.

## 2. Quantum $2 \times 2$ games

Figure 1 is a protocol for a quantum game between Alice and Bob. The players' actions are encoded by qubits that are initialized in the  $|0\rangle$  state. An entangling operator  $\hat{J}$  is selected that commutes with the direct product of any pair of classical strategies used by the players. Alice and Bob carry out local manipulations on their qubit by unitary operators  $\hat{A}$  and  $\hat{B}$ , respectively, drawn from corresponding strategic spaces  $S_A$  and  $S_B$ . A projective measurement in the basis  $\{|0\rangle, |1\rangle\}$  is carried out on the final state and the pay-offs are determined from the standard pay-off matrix. The final quantum state  $|\psi_f\rangle$  is calculated by

$$|\psi_f\rangle = \hat{J}^\dagger(\hat{A} \otimes \hat{B})\hat{J}|\psi_i\rangle, \quad (2.1)$$

where  $|\psi_i\rangle = |00\rangle$  represents the initial state of the qubits. The quantum game protocol contains the classical variant as a subset, since when  $\hat{A}$  and  $\hat{B}$  are classical operations  $\hat{J}^\dagger$  exactly cancels the effect of  $\hat{J}$ . In the quantum game it is only the expectation values of the players' pay-offs that are important. For Alice (Bob), we can write

$$\langle \$ \rangle = P_{00}|\langle \psi_f | 00 \rangle|^2 + P_{01}|\langle \psi_f | 01 \rangle|^2 + P_{10}|\langle \psi_f | 10 \rangle|^2 + P_{11}|\langle \psi_f | 11 \rangle|^2, \quad (2.2)$$

where  $P_{ij}$  is the pay-off for Alice (Bob) associated with the game outcome  $ij$ ;  $i, j \in \{0, 1\}$ .

The classical pure strategies correspond to the identity operator  $\hat{I}$  and the bit-flip operator

$$\hat{F} \equiv i\hat{\sigma}_x = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \quad (2.3)$$

Without loss of generality, an entangling operator  $\hat{J}(\gamma)$  for an  $N$ -player game with two pure classical strategies (an  $N \times 2$  game) may be written as (Benjamin & Hayden 2001a; Du *et al.* 2001)<sup>†</sup>

$$\hat{J}(\gamma) = \exp(i\frac{1}{2}\gamma\hat{\sigma}_x^{\otimes N}) = \hat{I}^{\otimes N} \cos \frac{1}{2}\gamma + i\hat{\sigma}_x^{\otimes N} \sin \frac{1}{2}\gamma, \quad (2.4)$$

where  $\gamma \in [0, \frac{1}{2}\pi]$ ,  $\gamma = \frac{1}{2}\pi$  corresponding to maximum entanglement. That is,

$$\hat{J}(\frac{1}{2}\pi)|00 \cdots 0\rangle = \frac{1}{\sqrt{2}}(|00 \cdots 0\rangle + i|11 \cdots 1\rangle). \quad (2.5)$$

A pure quantum strategy  $\hat{U}(\theta, \alpha, \beta)$  is an SU(2) operator and may be written as

$$\hat{U}(\theta, \alpha, \beta) = \begin{pmatrix} e^{i\alpha} \cos(\frac{1}{2}\theta) & ie^{i\beta} \sin(\frac{1}{2}\theta) \\ ie^{-i\beta} \sin(\frac{1}{2}\theta) & e^{-i\alpha} \cos(\frac{1}{2}\theta) \end{pmatrix}, \quad (2.6)$$

where  $\theta \in [0, \pi]$  and  $\alpha, \beta \in [-\pi, \pi]$ . A classical mixed strategy can be simulated in the quantum game protocol by an operator in the set  $\tilde{U}(\theta) = \hat{U}(\theta, 0, 0)$ . Such a strategy corresponds to playing  $\hat{I}$  with probability  $\cos^2(\frac{1}{2}\theta)$  and  $\hat{F}$  with probability  $\sin^2(\frac{1}{2}\theta)$ . Where both players use such strategies, the game is equivalent to the classical game.

### 3. Miracle moves

When both players have access to the full set of quantum operators, for any  $\hat{A} = \hat{U}(\theta, \alpha, \beta)$ , there exists  $\hat{B} = \hat{U}(\theta, \alpha, -\frac{1}{2}\pi - \beta)$  such that

$$(\hat{A} \otimes \hat{I}) \frac{1}{\sqrt{2}}(|00\rangle + i|11\rangle) = (\hat{I} \otimes \hat{B}) \frac{1}{\sqrt{2}}(|00\rangle + i|11\rangle). \quad (3.1)$$

That is, on the maximally entangled state, any local unitary operation that Alice carries out on her qubit is equivalent to a local unitary operation that Bob carries out on his (Benjamin & Hayden 2001b). Hence either player can undo his/her opponent's move (assuming it is known) by choosing  $\hat{U}(\theta, -\alpha, \frac{1}{2}\pi - \beta)$  in response to  $\hat{U}(\theta, \alpha, \beta)$ . Indeed, knowing the opponent's move, either player can produce any desired final state.

We are interested in the classical-quantum game, where one player, say Alice, is restricted to  $S_{\text{cl}} \equiv \{\tilde{U}(\theta) : \theta \in [0, \pi]\}$ , while the other, Bob, has access to  $S_{\text{q}} \equiv \{\hat{U}(\theta, \alpha, \beta) : \theta \in [0, \pi]; \alpha, \beta \in [-\pi, \pi]\}$ . We shall refer to strategies in  $S_{\text{cl}}$  as 'classical' in the sense that the player simply executes his/her two classical moves with fixed probabilities and does not manipulate the phase of their qubit. However,  $\tilde{U}(\theta)$  only gives the same results as a classical mixed strategy when both players employ these strategies. If Bob employs a quantum strategy, he can exploit the entanglement to his advantage. In this situation, Bob has a distinct advantage since only he can produce any desired final state by local operations on his qubit. Without knowing Alice's move, Bob's best plan is to play the 'miracle' quantum move

<sup>†</sup> Any other choice of  $\hat{J}$  would be equivalent, via local unitary operations, and would result only in a rotation of  $|\psi_{\text{f}}\rangle$  in the complex plane, consequently leading to the same game equilibria.

consisting of assuming that Alice has played  $\tilde{U}(\frac{1}{2}\pi)$ , the ‘average’ move from  $S_{cl}$ , undoing this move by

$$\hat{V} = \hat{U}(\tfrac{1}{2}\pi, 0, \tfrac{1}{2}\pi) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad (3.2)$$

and then preparing his desired final state. The operator

$$\hat{f} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (3.3)$$

has the property

$$(\hat{I} \otimes \hat{f}) \frac{1}{\sqrt{2}}(|00\rangle + i|11\rangle) = (\hat{F} \otimes \hat{I}) \frac{1}{\sqrt{2}}(|00\rangle + i|11\rangle), \quad (3.4)$$

so Bob can effectively flip Alice’s qubit as well as adjusting his own.

Suppose we have a general  $2 \times 2$  game with pay-offs

	Bob:0	Bob:1
Alice:0	$(p, p')$	$(q, q')$
Alice:1	$(r, r')$	$(s, s')$

(3.5)

where the unprimed values refer to Alice’s pay-offs and the primed to Bob’s. Bob has four possible miracle moves depending on the final state that he prefers,

$$\left. \begin{aligned} \hat{M}_{00} &= \hat{V}, \\ \hat{M}_{01} &= \hat{F}\hat{V} = \frac{i}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \\ \hat{M}_{10} &= \hat{f}\hat{V} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \\ \hat{M}_{11} &= \hat{F}\hat{f}\hat{V} = \frac{i}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}, \end{aligned} \right\} \quad (3.6)$$

given a preference for  $|00\rangle$ ,  $|01\rangle$ ,  $|10\rangle$  or  $|11\rangle$ , respectively. In the absence of entanglement, any  $\hat{M}_{ij}$  is equivalent to  $\tilde{U}(\frac{1}{2}\pi)$ , that is, the mixed classical strategy of flipping or not-flipping with equal probability.

When we use an entangling operator  $\hat{J}(\gamma)$  for an arbitrary  $\gamma \in [0, \frac{1}{2}\pi]$ , the expectation value of Alice’s pay-off if she plays  $\tilde{U}(\theta)$  against Bob’s miracle moves are, respectively,

$$\left. \begin{aligned} \langle \$_{00} \rangle &= \tfrac{1}{2}p(\cos \tfrac{1}{2}\theta + \sin \tfrac{1}{2}\theta \sin \gamma)^2 + \tfrac{1}{2}q \cos^2 \tfrac{1}{2}\theta \cos^2 \gamma \\ &\quad + \tfrac{1}{2}r(\sin \tfrac{1}{2}\theta - \cos \tfrac{1}{2}\theta \sin \gamma)^2 + \tfrac{1}{2}s \sin^2 \tfrac{1}{2}\theta \cos^2 \gamma, \\ \langle \$_{01} \rangle &= \tfrac{1}{2}p \cos^2 \tfrac{1}{2}\theta \cos^2 \gamma + \tfrac{1}{2}q(\cos \tfrac{1}{2}\theta + \sin \tfrac{1}{2}\theta \sin \gamma)^2 \\ &\quad + \tfrac{1}{2}r \sin^2 \tfrac{1}{2}\theta \cos^2 \gamma + \tfrac{1}{2}s(\sin \tfrac{1}{2}\theta - \cos \tfrac{1}{2}\theta \sin \gamma)^2, \\ \langle \$_{10} \rangle &= \tfrac{1}{2}p(\cos \tfrac{1}{2}\theta - \sin \tfrac{1}{2}\theta \sin \gamma)^2 + \tfrac{1}{2}q \cos^2 \tfrac{1}{2}\theta \cos^2 \gamma \\ &\quad + \tfrac{1}{2}r(\sin \tfrac{1}{2}\theta + \cos \tfrac{1}{2}\theta \sin \gamma)^2 + \tfrac{1}{2}s \sin^2 \tfrac{1}{2}\theta \cos^2 \gamma, \\ \langle \$_{11} \rangle &= \tfrac{1}{2}p \cos^2 \tfrac{1}{2}\theta \cos^2 \gamma + \tfrac{1}{2}q(\cos \tfrac{1}{2}\theta - \sin \tfrac{1}{2}\theta \sin \gamma)^2 \\ &\quad + \tfrac{1}{2}r \sin^2 \tfrac{1}{2}\theta \cos^2 \gamma + \tfrac{1}{2}s(\sin \tfrac{1}{2}\theta + \cos \tfrac{1}{2}\theta \sin \gamma)^2. \end{aligned} \right\} \quad (3.7)$$

Table 1. *Pay-off matrices for some  $2 \times 2$  games*

(A summary of pay-off matrices with NE and PO results for various classical games. PD is the prisoners' dilemma and BoS is the battle of the sexes. In the matrices, the rows correspond to Alice's options of cooperation ( $C$ ) and defection ( $D$ ), respectively, while the columns are the options for Bob. In the parentheses, the first pay-off is Alice's, the second Bob's and  $a > b > c > d$ . The condition specifies a constraint on the values of  $a$ ,  $b$ ,  $c$  and  $d$  necessary to create the dilemma. The final column gives standard values for the pay-offs.)

game	pay-off matrix		NE pay-offs	PO pay-offs	condition	$(a, b, c, d)$
chicken	$(b, b)$ $(a, c)$	$(c, a)$ $(d, d)$	$(a, c)$ or $(c, a)$	$(b, b)$	$2b > a + c$	$(4, 3, 1, 0)$
PD	$(b, b)$ $(a, d)$	$(d, a)$ $(c, c)$	$(c, c)$	$(b, b)$	$2b > a + d$	$(5, 3, 2, 1)$
deadlock	$(c, c)$ $(a, d)$	$(d, a)$ $(b, b)$	$(b, b)$	$(b, b)$	$2b > a + d$	$(3, 2, 1, 0)$
stag hunt	$(a, a)$ $(b, d)$	$(d, b)$ $(c, c)$	$(a, a)$ or $(c, c)$	$(a, a)$		$(3, 2, 1, 0)$
BoS	$(a, b)$ $(c, c)$	$(c, c)$ $(b, a)$	$(a, b)$ or $(b, a)$	$(a, b)$ or $(b, a)$		$(2, 1, 0)$

We add primes to  $p$ ,  $q$ ,  $r$  and  $s$  to get Bob's pay-offs. Although the miracle moves are in some sense best for Bob, in that they guarantee a certain minimum pay-off against any classical strategy from Alice, there is not necessarily any NE among pure strategies in the classical-quantum game.

#### 4. Critical entanglements

In each of the four cases of equation (3.7), there can be critical values of the entanglement parameter  $\gamma$  below which the quantum player no longer has an advantage. We will consider some examples. The most interesting games are those that pose some sort of dilemma for the players. A non-technical discussion of various dilemmas in  $2 \times 2$  game theory is given in Poundstone (1992), from which we have taken the names of the following games. The results for the prisoners' dilemma, using the standard pay-offs for that game, were found by Eisert *et al.* (1999) and, for generalized pay-offs, by Du *et al.* (2003).

Below we introduce a number of games and discuss the dilemma faced by the players and their possible strategies. The games, along with some important equilibria, are summarized in table 1. Detailed results for the various threshold values of the entanglement parameter are given for the game of chicken. A summary of the thresholds for the collection of games is given in table 2. In the following, the pay-offs shall be designated  $a$ ,  $b$ ,  $c$  and  $d$ , with  $a > b > c > d$ . The two pure classical strategies for the players are referred to as cooperation ( $C$ ) and defection ( $D$ ), for reasons that shall soon become apparent. The NE's referred to are in classical pure strategies unless otherwise indicated.

Table 2. Critical entanglements for  $2 \times 2$  quantum games

(Values of  $\sin^2 \gamma$  above which (or below which where indicated by ‘<’) the expected value of Bob’s pay-off exceeds, respectively, Alice’s pay-off, Bob’s classical NE pay-off and Bob’s pay-off for the PO outcome. Where there are two NE (or PO) results, the one where Bob’s pay-off is smallest is used. The strategies are Alice’s and Bob’s, respectively. In the last line, ‘if  $a + c > 2b$ ’ refers to a condition on the numerical values of the pay-offs and not to a condition on  $\gamma$ .)

game	strategies	$\langle \$_B \rangle > \langle \$_A \rangle$	$\langle \$_B \rangle > \$_B$ (NE)	$\langle \$_B \rangle > \$_B$ (PO)
chicken	$\hat{C}, \hat{M}_{01}$	always	$< (a + b - 2c)/(b - d)$	$< (a - b)/(b - d)$
	$\hat{D}, \hat{M}_{01}$	$\frac{1}{2}$	$(c - d)/(a - c)$	$(2b - c - d)/(a - c)$
PD	$\hat{C}, \hat{M}_{01}$	always	always	$(a - b)/(c - d)$
	$\hat{D}, \hat{M}_{01}$	$d/(2(a - d))$	$(c - d)/(a - d)$	$(2b - c - d)/(a - d)$
deadlock	$\hat{C}, \hat{M}_{01}$	always	$(2b - a - c)/(b - c)$	$(2b - a - c)/(b - c)$
	$\hat{D}, \hat{M}_{01}$	$\frac{1}{2}$	$(b - d)/(a - d)$	$(b - d)/(a - d)$
stag hunt	$\hat{C}, \hat{M}_{00}$	$< \frac{1}{2}$	$(c - d)/(a - c)$	never
	$\hat{D}, \hat{M}_{00}$	never	$< (a + b - 2c)/(b - d)$	never
BoS	$\hat{O}, \hat{M}_{11}$	$\frac{1}{2}$	$(b - c)/(a - b)$	$(b - c)/(a - b)$
	$\hat{T}, \hat{M}_{11}$	always	if $a + c > 2b$	if $a + c > 2b$

(a) *Chicken*

The archetypal version of chicken is described as follows.

The two players are driving towards each other along the centre of an empty road. Their possible actions are to swerve at the last minute (co-operate) or not to swerve (defect). If only one player swerves, he/she is the ‘chicken’ and gets a poor pay-off, while the other player is the ‘hero’ and scores best. If both swerve, they get an intermediate result, but clearly the worst possible scenario is for neither player to swerve.

Such a situation often arises in the military/diplomatic posturing amongst nations. Each does best if the other backs down against their strong stance, but the mutual worst result is to go to war! The situation is described by the pay-off matrix

	Bob:C	Bob:D
Alice:C	(b, b)	(c, a)
Alice:D	(a, c)	(d, d)

(4.1)

Provided  $2b > a + c$ , the Pareto optimal (PO) result, the one for which it is not possible to improve the pay-off of one player without reducing the pay-off of the other, is mutual cooperation. In the discussion below, we shall choose  $(a, b, c, d) = (4, 3, 1, 0)$ , satisfying this condition, whenever we want a numerical example of the pay-offs. There are two NE in the classical game,  $CD$  and  $DC$ , from which neither player can improve their result by a unilateral change in strategy. Hence the rational player hypothesized by game theory is faced with a dilemma for which there is no solution: the game is symmetric yet both players want to do the opposite of the other. For the chosen set of numerical pay-offs, there is a unique NE in mixed classical strategies:

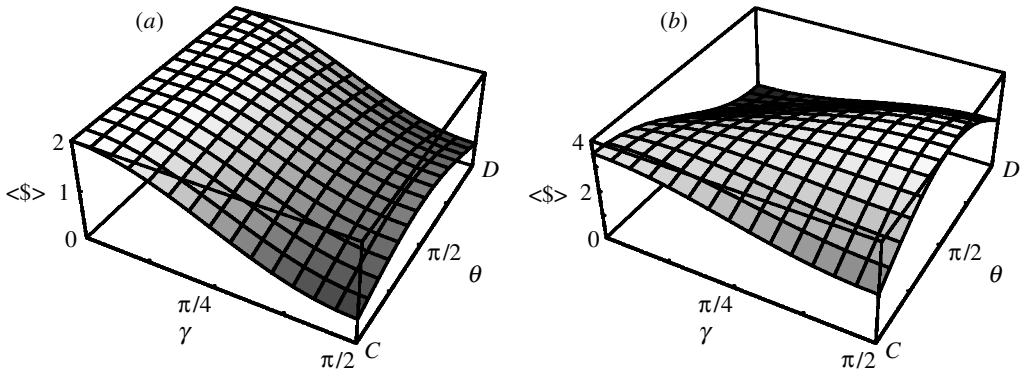


Figure 2. In the game of chicken, the expected pay-offs for (a) Alice and (b) Bob when Bob plays the quantum miracle move  $\hat{M}_{01}$ , as a function of Alice's strategy ( $\theta = 0$  corresponds to cooperation and  $\theta = \pi$  corresponds to defection) and the degree of entanglement  $\gamma$ . The surfaces are drawn for pay-offs  $(a, b, c, d) = (4, 3, 1, 0)$ . If Alice knows that Bob is going to play the quantum miracle move, she does best by choosing the crest of the curve,  $\theta = \frac{1}{2}\pi$ , irrespective of the level of entanglement. Against this strategy, Bob scores between two and four, an improvement for all  $\gamma > 0$  over the pay-off he could expect playing a classical strategy.

each player cooperates or defects with probability  $\frac{1}{2}$ . In our protocol, this corresponds to both players selecting  $\hat{U}(\frac{1}{2}\pi)$ .

A quantum version of chicken has been discussed in the literature (Marinatto & Weber 2000*a, b*; Benjamin 2000). In this version, a final state of a player's qubit being  $|0\rangle$  corresponds to the player having cooperated, while  $|1\rangle$  corresponds to having defected.

The preferred outcome for Bob is  $CD$  or  $|01\rangle$ , so he will play  $\hat{M}_{01}$ . If Alice cooperates, the pay-offs are

$$\left. \begin{aligned} \langle \$_A \rangle &= \frac{1}{2}(b-d)\cos^2\gamma + \frac{1}{2}(c+d), \\ \langle \$_B \rangle &= \frac{1}{2}(b-d)\cos^2\gamma + \frac{1}{2}(a+d). \end{aligned} \right\} \quad (4.2)$$

Increasing entanglement is bad for both players. However, Bob outscores Alice by  $\frac{1}{2}(a-c)$  for all entanglements and does better than the poorer of his two NE results (c) provided that

$$\sin\gamma < \sqrt{\frac{a+b-2c}{b-d}}, \quad (4.3)$$

which, for the pay-offs  $(4, 3, 1, 0)$ , means that  $\gamma$  can take any value. He performs better than the mutual cooperation result (b) provided that

$$\sin\gamma < \sqrt{\frac{a-b}{b-d}}, \quad (4.4)$$

which yields a value of  $1/\sqrt{3}$  for the chosen pay-offs.

Suppose instead that Alice defects. The pay-offs are now

$$\left. \begin{aligned} \langle \$_A \rangle &= \frac{1}{2}(a-c)\cos^2\gamma + \frac{1}{2}(c+d), \\ \langle \$_B \rangle &= \frac{1}{2}(a-c)\sin^2\gamma + \frac{1}{2}(c+d). \end{aligned} \right\} \quad (4.5)$$

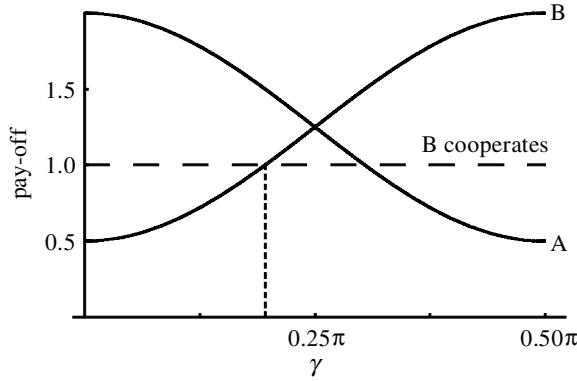


Figure 3. The pay-offs for Alice and Bob versus the level of entanglement ( $\gamma$ ) in the game of chicken when Alice defects. The standard pay-offs  $(a, b, c, d) = (4, 3, 1, 0)$  are chosen. The solid lines correspond to the results when Bob plays the quantum move  $\hat{M}_{01}$  and the dashed line gives Bob's pay-off when he cooperates. Below an entanglement of  $\arcsin(1/\sqrt{3})$  (short dashes), Bob does best, against a defecting Alice, by switching to the strategy 'always cooperate'.

Increasing entanglement improves Bob's result and worsens Alice's. Bob scores better than Alice provided  $\gamma > \frac{1}{4}\pi$ , regardless of the numerical value of the pay-offs. Bob does better than his worst NE result ( $c$ ) when

$$\sin \gamma > \sqrt{\frac{c-d}{a-c}}, \quad (4.6)$$

which yields a value of  $1/\sqrt{3}$  for the default pay-offs, and better than his PO result ( $b$ ) when

$$\sin \gamma > \sqrt{\frac{2b-c-d}{a-c}}, \quad (4.7)$$

which has no solution for the default values. Thus, except for specially adjusted values of the pay-offs, Bob cannot assure himself of a pay-off at least as good as that achievable by mutual cooperation. However, Bob escapes from his dilemma for a sufficient degree of entanglement as follows. Against  $\hat{M}_{01}$ , Alice's optimal strategy from the set  $S_{cl}$  is given by

$$\tan \theta = \frac{2(c-d)}{b+c-a-d} \frac{\sin \gamma}{\cos^2 \gamma}. \quad (4.8)$$

For  $(a, b, c, d) = (4, 3, 1, 0)$ , this gives  $\theta = \frac{1}{2}\pi$ . Since  $\hat{M}_{01}$  is Bob's best counter to  $\tilde{U}(\frac{1}{2}\pi)$ , these strategies form an NE in a classical-quantum game of chicken and are the preferred strategies of the players. For this choice, above an entanglement of  $\gamma = \frac{1}{6}\pi$ , Bob performs better than his mutual-cooperation result.

The expected pay-offs for Alice and Bob as a function of Alice's strategy and the degree of entanglement are shown in figure 2. In figure 3 we can see that if Bob wishes to maximize the minimum pay-off he receives, he should alter his strategy from the quantum move  $\hat{M}_{01}$  to cooperation, once the entanglement drops below  $\arcsin(1/\sqrt{3})$ .

(b) Prisoners' dilemma

The most famous dilemma is the prisoners' dilemma. This may be specified in general as

	Bob:C	Bob:D
Alice:C	(b, b)	(d, a)
Alice:D	(a, d)	(c, c)

(4.9)

In the classical game, the strategy 'always defect' dominates since it gives a better pay-off than cooperation against any strategy by the opponent. Hence, the NE for the prisoners' dilemma is mutual defection, resulting in a pay-off of  $c$  to both players. However, both players would have done better with mutual cooperation, resulting in a pay-off of  $b$ , giving rise to a dilemma that occurs in many social and political situations. The sizes of the pay-offs are generally adjusted so that  $2b > a + d$  making mutual cooperation the PO outcome. The most common set of pay-offs is  $(a, b, c, d) = (5, 3, 1, 0)$ .

In the classical-quantum game, Bob can help engineer his preferred result of  $|01\rangle$  ( $CD$ ) by adopting the strategy  $\hat{M}_{01}$ . The most important critical value of the entanglement parameter  $\gamma$  is the threshold below which Bob performs worse with his miracle move than he would if he chose the classical dominant strategy of 'always defect'. This occurs for

$$\sin \gamma = \sqrt{\frac{c-d}{a-d}}, \quad (4.10)$$

which yields the value  $\sqrt{1/5}$  for the usual pay-offs. As noted in Du *et al.* (2001), below this level of entanglement the quantum version of prisoners' dilemma behaves classically with an NE of mutual defection.

(c) Deadlock

Deadlock is characterized by reversing the pay-offs for mutual cooperation and defection in the prisoners' dilemma,

	Bob:C	Bob:D
Alice:C	(c, c)	(d, a)
Alice:D	(a, d)	(b, b)

(4.11)

Defection is again the dominant strategy, and there is even less incentive for the players to cooperate in this game than in the prisoners' dilemma since the PO result is mutual defection. However, both players would prefer if their opponent cooperated so they could stab them in the back by defecting and achieve the maximum pay-off of  $a$ . There is no advantage to cooperating, so there is no real dilemma in the classical game. In the classical-quantum game, Bob can again use his quantum skills to engineer at least partial cooperation from Alice, against any possible strategy from her, by playing  $\hat{M}_{01}$ .

(d) Stag hunt

In stag hunt, both players prefer the outcome of mutual cooperation, since it gives a pay-off superior to all other outcomes. However, each are afraid of defection

by the other, which, although it reduces the defecting player’s pay-off, has a more detrimental effect on the cooperator’s pay-off, as indicated in the pay-off matrix below,

	Bob: <i>C</i>	Bob: <i>D</i>
Alice: <i>C</i>	( <i>a</i> , <i>a</i> )	( <i>d</i> , <i>b</i> )
Alice: <i>D</i>	( <i>b</i> , <i>d</i> )	( <i>c</i> , <i>c</i> )

(4.12)

Both mutual cooperation and mutual defection are NE, but the former is the PO result. There is no dilemma when two rational players meet. Both recognize the preferred result and have no reason, given their recognition of the rationality of the other player, to defect. Mutual defection will result only if both players allow fear to dominate over rationality. This situation is not changed in the classical–quantum game. However, having the ability to play quantum moves may be of advantage when the classical player is irrational in the sense that they do not try to maximize their own pay-off. In that case, the quantum player should choose to play the strategy  $\hat{M}_{00}$  to steer the result towards the mutual cooperation outcome.

(*e*) *Battle of the sexes*

In this game, Alice and Bob each want the company of the other in some activity, but their preferred activity differs: opera (*O*) for Alice and television (*T*) for Bob. If the players end up doing different activities, both are punished by a poor pay-off. In matrix form this game can be represented as

	Bob: <i>O</i>	Bob: <i>T</i>
Alice: <i>O</i>	( <i>a</i> , <i>b</i> )	( <i>c</i> , <i>c</i> )
Alice: <i>T</i>	( <i>c</i> , <i>c</i> )	( <i>b</i> , <i>a</i> )

(4.13)

The options on the main diagonal are both PO and are NE, but there is no clear way of deciding between them. Bob’s quantum strategy will be to choose  $\hat{M}_{11}$  to steer the game towards his preferred result of *TT*. Marinatto & Weber (2000*a*, *b*; Benjamin 2000) discuss a quantum version of battle of the sexes using a slightly different protocol for a quantum game from the one used in the current work.

With  $\hat{M}_{11}$ , Bob outscores Alice provided  $\gamma > \frac{1}{4}\pi$ , but is only assured of scoring at least as well as the poorer of his two NE results (*b*) for full entanglement, and is never certain of bettering it. The quantum move, however, is better than using a fair coin to decide between  $\hat{O}$  and  $\hat{T}$  for  $\gamma > 0$ , and equivalent to it for  $\gamma = 0$ . Hence, even though Bob cannot be assured of scoring greater than *b*, he can improve his worst case pay-off for any  $\gamma > 0$ . Figure 4 shows Alice and Bob’s pay-offs as a function of the degree of entanglement and Alice’s strategy.

5. Extensions

The situation is more complex for multi-player games. No longer can a quantum player playing against classical players engineer any desired final state, even if the opponents’ moves are known. However, a player can never be worse off by having access to the quantum domain, since this includes the classical possibilities as a subset.

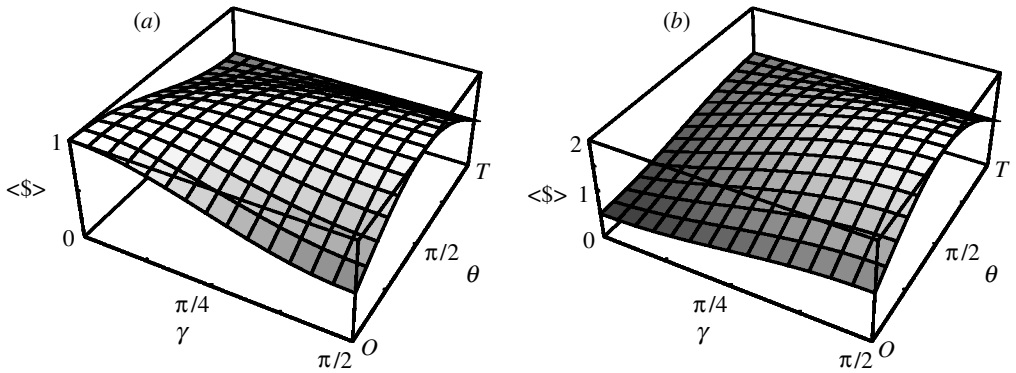


Figure 4. In the battle of the sexes, the expected pay-offs for (a) Alice and (b) Bob when Bob plays the quantum miracle move  $\hat{M}_{11}$ , as a function of Alice's strategy ( $\theta = 0$  corresponds to opera and  $\theta = \pi$  corresponds to television) and the degree of entanglement  $\gamma$ . The surfaces are drawn for pay-offs  $(a, b, c) = (2, 1, 0)$ . If Alice knows that Bob is going to play the quantum miracle move, she does best by choosing the crest of the curve, so her optimal strategy changes from  $O$  for no entanglement, to  $\theta = \frac{1}{2}\pi$  for full entanglement. Against this strategy, Bob starts to score better than one for an entanglement exceeding approximately  $\frac{1}{5}\pi$ .

In two-player games with more than two pure classical strategies, the prospects for the quantum player are better. Some entangled quantum  $2 \times 3$  games have been considered in the literature (Iqbal & Toor 2001; Flitney & Abbott 2002b). Here, the full set of quantum strategies is  $SU(3)$  and there are nine possible miracle moves (before considering symmetries).  $S_{cl}$ , the strategies that do not manipulate the phase of the player's qutrit (a qutrit is the three-state equivalent of a qubit) can be written as the product of three rotations, each parametrized by a rotation angle. Since the form is not unique, it is much more difficult to say what constitutes the average move from this set, so the expressions for the miracle moves are open to debate. Also, an entangling operator for a general  $2 \times 3$  quantum game has not been given in the literature. Nevertheless, the quantum player will still be able to manipulate the result of the game to increase the probability of his/her favoured result.

## 6. Conclusion

With a sufficient degree of entanglement, the quantum player in a classical–quantum two-player game can use the extra possibilities available to help steer the game towards their most desired result, giving a pay-off above that achievable by classical strategies alone. We have given the four miracle moves in quantum  $2 \times 2$  game theory and show when they can be of use in several game theory dilemmas. There are critical values of the entanglement parameter  $\gamma$  below (or occasionally above) which it is no longer an advantage to have access to quantum moves. That is, where the quantum player can no longer outscore his/her classical Nash equilibrium result. These represent a phase change in the classical–quantum game where a switch between the quantum miracle move and the dominant classical strategy is warranted. With typical values for the pay-offs and a classical player opting for his/her best strategy, the critical value for  $\sin \gamma$  is  $\sqrt{1/3}$  for chicken,  $\sqrt{1/5}$  for the prisoners' dilemma and  $\sqrt{2/3}$  for deadlock, while for the stag hunt there is no particular advantage to the

quantum player. In the battle of the sexes, there is no clear threshold, but for any non-zero entanglement Bob can improve his worst-case result.

The quantum player's advantage is not as strong in classical–quantum multi-player games but, in multi-strategy two-player games, depending on the level of entanglement, the quantum player would again have access to moves that improve his/her result. The calculation of these moves is problematic because of the larger number of degrees of freedom and it has not been attempted here.

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