THE ROYAL SOCIETY

The paradox of Parrondo's games

By Gregory P. Harmer[1], Derek Abbott[1] and Peter G. Taylor[2]

[1] *Centre for Biomedical Engineering (CBME) and Department of Electrical and Electronic Engineering, University of Adelaide, SA 5005, Australia*

[2] *Department of Applied Mathematics, University of Adelaide, SA 5005, Australia*

Received 25 February 1999; accepted 4 June 1999

We introduce Parrondo's paradox that involves games of chance. We consider two fair games, A and B, both of which can be made to lose by changing a biasing parameter. An apparently paradoxical situation arises when the two games are played in any alternating order. A winning expectation is produced, even though both games A and B are losing when we play them individually. We develop an explanation of the phenomenon in terms of a Brownian ratchet model, and also develop a mathematical analysis using discrete-time Markov chains. From the analysis we investigate the range of parameter values for which Parrondo's paradox exists.

Keywords: gambling paradox; Brownian ratchet; discrete-time Markov chains

1. Introduction

The study of probability dates back to the 17th century. It arises from games of chance, originating from the ancient game of throwing bones—the forerunners of dice. Strongly associated with probability is gambling, from dice to actuarial tables and risk–benefit analysis, gambling has historically been at the forefront of expanding probability theory (Shlesinger 1996). This dates back to correspondence between Pascal and Fermat in 1654, when a problem was posed to Pascal by a French gambler. 'Games of chance' can be considered processes that consist of random events or random variables. The erratic Brownian motion of dust particles or pollen grains in a liquid, due to collisions with the liquid molecules, is the classic example (Hughes 1995). The motion of each grain is sufficiently erratic that it can be considered to be random, the simplest model being that of a random walk.

The apparent paradox that two losing games A and B can produce a winning outcome when played in an alternating sequence was devised by Parrondo as a pedagogical illustration of the Brownian ratchet (Harmer & Abbott 1999). However, as Parrondo's games are remarkable and may have important applications in areas such as electronics, biology and economics, they require analysis in their own right.

In this paper, we first introduce the concept of the Brownian ratchet and then illustrate Parrondo's games. Graphical simulations of the expectations of Parrondo's games are then explained, in terms of the Brownian ratchet model. An analysis is then presented to explain the simulations using discrete-time Markov chains.

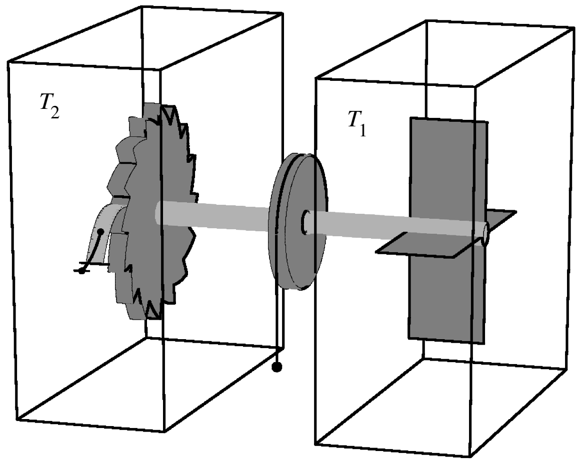

Figure 1. The ratchet and pawl machine. There are two boxes with a vane in one and a wheel that can only turn one way, a ratchet and pawl, in the other. Each box is in a thermal bath of gas molecules. The two boxes are connected mechanically by a thermally insulated axle. The whole device is considered to be reduced to microscopic size so that gas molecules can randomly bombard the vane, to produce motion. At equilibrium, when $T_1 = T_2$, there is no net motion of the ratchet wheel. For the non-equilibrium case, when $T_1 > T_2$, external energy is introduced and directed motion is possible.

(*a*) *Brownian ratchets*

The ratchet and pawl device, shown in figure 1, was introduced in the last century as a proposed perpetual motion machine. Given that the two compartments contained gas, the implication was that the device should be able to harness the thermal Brownian fluctuations of the gas molecules, by a process of rectification. Smoluchowski was the first to find the correct explanation as to why net motion is not possible at equilibrium, for this ratchet and pawl device; which he called *Zahnrad mit einer Sperrklinke* in German (von Smoluchowski 1912). This device was then later revisited by Feynman *et al.* (1963). However, Feynman's work was deeply flawed on two counts. In equilibrium, his equations of detailed balance made incorrect use of energy probabilities rather than using crossing rate probabilities (Abbott *et al.* 2000) and, secondly, for non-equilibrium conditions, he incorrectly assumed quasi-static conditions in his calculations of thermal efficiency (Parrondo & Español 1996). Despite these shortcomings, Feynman's work was greatly influential and was the source of inspiration for the 'Brownian ratchet' concept described in the seminal paper by Magnasco (1993). It should be noted that although these two concepts have distinct differences, the important similarity is that they both give rise to directed motion in exchange for external energy—but in the case of equilibrium, both systems maintain detailed balance and no net motion is possible. In both cases, net motion is strictly a non-equilibrium phenomenon.

The focus of recent research is to harness Brownian motion and convert it to directed motion, or more generally, a Brownian motor, without the use of macroscopic forces or gradients. This research was inspired by considering molecules in chemical reactions, termed molecular motors (Astumian & Bier 1994). The roots of these Brownian devices trace back to Feynman's exposition of the ratchet and pawl system.

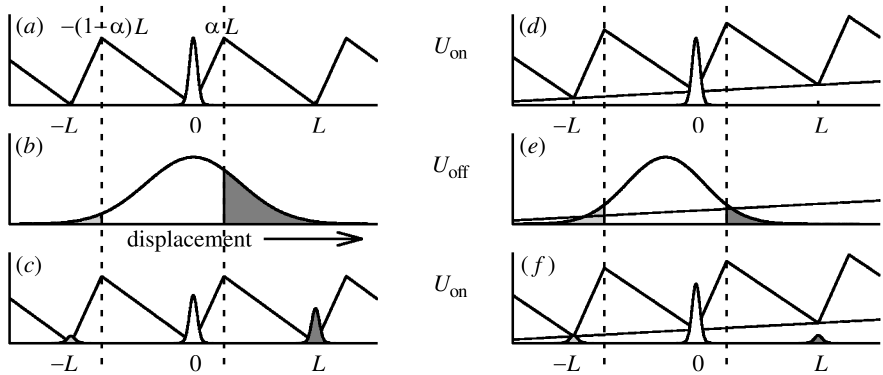

Figure 2. This shows how the mechanism of the ratchet potential works. The diagrams on the left, (*a*)–(*c*), show when there is no macroscopic gradient present and the net movement of particles is in the forward direction (defined by arrow). The diagrams on the right, (*d*)–(*f*), have a slight gradient present; this causes the particles to drift backwards while U_{off} is acting. Hence the net flow of particles in the forward direction is reduced.

By supplying energy from external fluctuations or non-equilibrium chemical reactions in the form of a thermal or chemical gradient, for example, directed motion is possible even in an isothermal system (Astumian 1997; Bier 1997*b*). These types of devices have been shown to work theoretically (Astumian & Bier 1994; Magnasco 1993), even against a small macroscopic gradient (Hänggi & Bartussek 1996). Recently, with the technology available to build micrometre-scale structures, many man-made Brownian ratchet devices have been constructed, and actually work (Astumian 1997; Bier 1997*a*).

There are several mechanisms by which directed Brownian motion can be achieved (Faucheux *et al.* 1995; Rousselet 1994). We will consider one of the mechanisms, termed the *flashing ratchet* (Doering 1995; Hänggi & Bartussek 1996), that may prove fruitful when considering Parrondo's games. Consider a system where there exists two one-dimensional potentials, U_{on} and U_{off}, as shown in figure 2. The asymmetry of the potential U_{on} is determined by α, where $0 \leqslant \alpha \leqslant 1$. Having $\alpha = \frac{1}{2}$ creates a triangular symmetric potential, otherwise the potential is asymmetrical like U_{on} in figure 2 where $\alpha < \frac{1}{2}$. Let there be Brownian particles existing in the potential diffusing to a position of least energy. In equilibrium, if the potential height is larger than the thermal noise, the particles are localized in a potential minima. However, time modulating the potential U_{on} and U_{off} can induce motion, hence the term *flashing ratchets*. When the U_{on} is applied, the particles are trapped in the minima of the potential so the concentration of the particles is peaked. Switching the potential off allows the particles to diffuse freely so the concentration is a set of normal curves centred around the minima. When U_{on} is switched on again there is a probability P_{fwd}, proportional to the darker shaded area of the curve, that some particles are to the right of αL. These particles move forwards to the minima located at L. Similarly, there is a probability P_{bck} (lightly shaded) that some particles are to the left of $-(1-\alpha)L$, and move to the left minima located at $-L$. Since $\alpha < \frac{1}{2}$ then $P_{\mathrm{fwd}} > P_{\mathrm{bck}}$ and the net motion of the particles is to the right. We can define the probability current as $J = P_{\mathrm{fwd}} - P_{\mathrm{bck}}$ for a particle diffusing forward one step in the potential.

When a tilted periodic potential is toggled 'on' and 'off'—by solving the Fokker–Planck equation for this system—Brownian particles are shown to move 'uphill' (Doering 1995). If the potential is held in either the 'on' state or the 'off' state, the particles move 'downhill'. This is the inspiration for Parrondo's paradox: the individual states are said to be like 'losing' games and when they are alternated we get uphill motion or 'winning' expectations.

(b) *Parrondo's games*

Game A, which is described by (1.1), is straightforward and can be thought of as tossing a weighted coin, or going on a biased random walk:

$$\text{Game A:} \quad \begin{cases} \boldsymbol{P}[\text{winning}] = p = p^* - \varepsilon, \\ \boldsymbol{P}[\text{losing}] = 1 - p = (1 - p^*) + \varepsilon. \end{cases} \tag{1.1}$$

Game B is a little more complex and can be generally described by the following statement. If the present capital is a multiple of M, then the chance of winning is p_1; if it is not a multiple of M, the chance of winning is p_2. It can be described mathematically by (1.2)

$$\text{Game B:} \quad \begin{cases} \boldsymbol{P}[\text{winning} \mid \text{capital mod } M = 0] = p_1 = p_1^* - \varepsilon, \\ \boldsymbol{P}[\text{losing} \mid \text{capital mod } M = 0] = 1 - p_1 = (1 - p_1^*) + \varepsilon, \\ \boldsymbol{P}[\text{winning} \mid \text{capital mod } M \neq 0] = p_2 = p_2^* - \varepsilon, \\ \boldsymbol{P}[\text{losing} \mid \text{capital mod } M \neq 0] = 1 - p_2 = (1 - p_2^*) + \varepsilon, \end{cases} \tag{1.2}$$

where the star denotes that the game is fair when $\varepsilon = 0$. Substituting Parrondo's original numbers for p^*, p_1^*, p_2^* and M into games A and B gives $p = \frac{1}{2} - \varepsilon$, $p_1 = \frac{1}{10} - \varepsilon$, $p_2 = \frac{3}{4} - \varepsilon$ and $M = 3$ (Harmer & Abbott 1999). We refer to *capital* and *gain* as if anyone playing these games is against a common opponent, the bank for example. The gain is based upon a one-unit capital where negative gains indicate a loss, thus a gain of five is equivalent to five units of capital.

We will digress for a moment to discuss what constitutes a fair game. The behaviour of game B differs from game A in that the starting capital affects whether we are likely to win or not. If the starting capital is a multiple of three, then we will lose a little, and vice versa. The concept of what it means for a game to be winning, losing or fair can be defined precisely in terms of hitting probabilities and expected hitting times of discrete-time Markov chains, as is done in our analysis section. Before then we shall be a little looser with this terminology. We shall consider a game to be winning, losing or fair according to whether the probability of moving up n states is greater than, less than, or equal to the probability of moving down n states as n becomes large.

Using the above criterion, both game A and game B are fair when ε is set to zero. This is true of game A because the probabilities of moving up and down n states are equal for all n. It is also true of game B even though the value of the starting capital influences the probability of going up and down n states for small values of n. Using this criterion, both games A and B lose when $\varepsilon > 0$.

2. Simulation results

It is clear now that both game A and game B lose when $\varepsilon > 0$. Consider the scenario if we start switching between the two losing games, play two games of A, two games

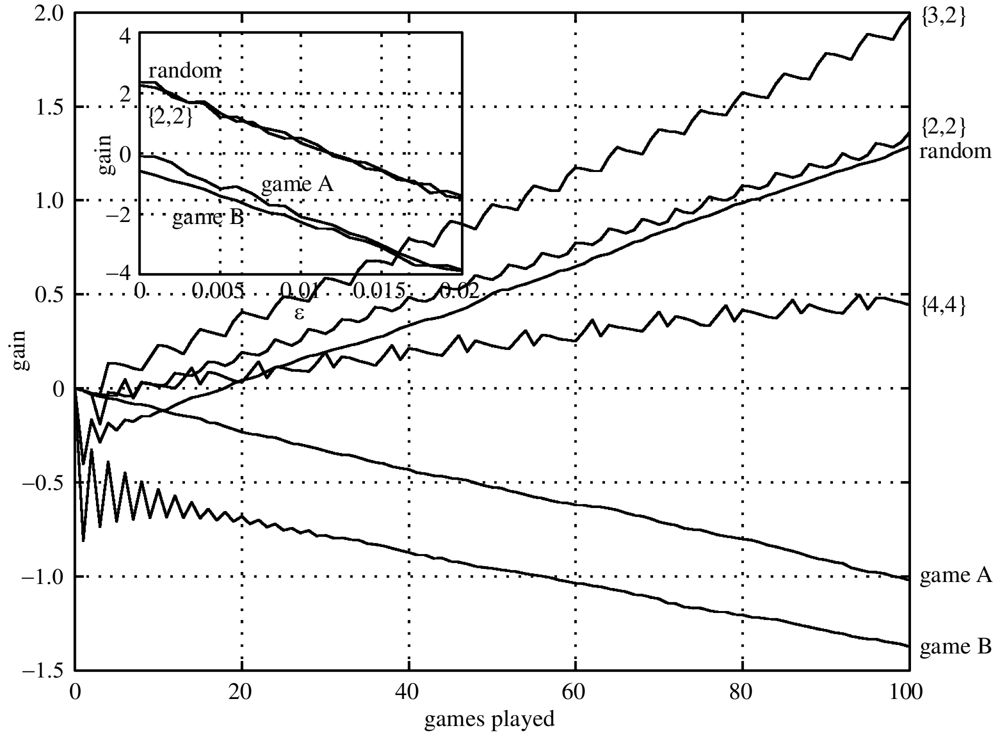

Figure 3. The main plot shows the effect of playing A and B individually and the effect of switching between games A and B. The simulation was performed by playing $(A^a_{0.005}B^b_{0.005})^{100/(a+b)}(0)$ and averaged over 50 000 trials ($p^* = \frac{1}{2}$, $p^*_1 = \frac{1}{10}$, $p^*_2 = \frac{3}{4}$). The values of a and b are shown by the vectors $[a, b]$. The inset shows the effect of the games' performance when varying ε by playing the games individually and alternately. That is, the inset shows the outcome after the hundredth game is played.

of B, two of A, and so on. The result, which is quite counter-intuitive, is that we start winning. That is, we can play the two losing games A and B in such a way as to make a winning outcome. Furthermore, deciding which game to play by tossing a fair coin also yields a winning outcome. Figure 3 shows the progress when playing games A and B, as well is the affect of switching periodically and randomly between the games. We will use the notation of Harmer & Abbott (1999): $(A^a_\varepsilon B^b_\varepsilon)^n(x)$ denotes starting with a capital of x, playing game A a times, game B b times, which is repeated n times.

How well-behaved is the randomized game? We want to determine how erratic the final capital is after a number of games have been played. We have evaluated this by calculating the standard deviation of the final capital over the 10 000 trials. The thick lines in figure 4*a* show the games played individually and randomized. The thin lines show the games plus or minus one standard deviation. Let us first consider game A as its characteristics are well known. The final capital of game A after playing $n = 100$ games is approximately a normal distribution (see figure 4*b*), and has a standard deviation of $2\sqrt{npq}$, which is proportional to $\sqrt{n}$ (Harmer & Abbott 1999). From figure 4*d*, the distribution of the randomized game is also approximately a normal distribution, hence the standard deviation of the randomized game is of the same

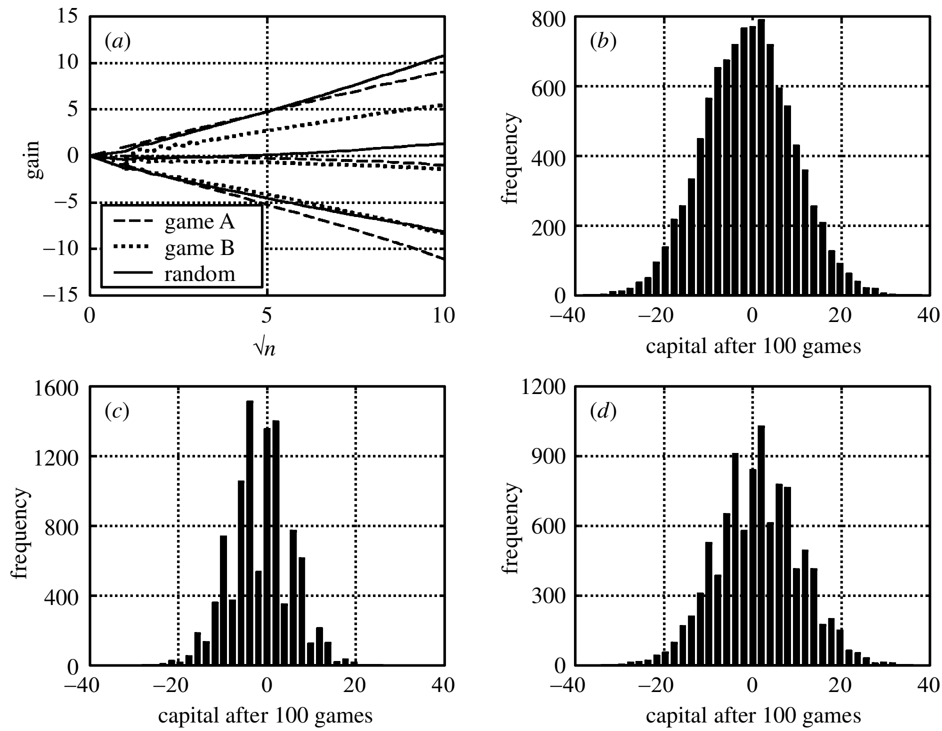

Figure 4. (a) The solid lines show the result as the games are played with $\varepsilon = 0.005$ averaged over 10 000 trials. The thin lines show one standard deviation above and below each of the games. (b)–(d) Histograms of the capital after the hundredth game of game A, game B and the randomized game, respectively, all of which are approximately normal distributions. Note that the gaps in the histograms reflect that the final capital is never odd as we are playing an even number of games.

order as game A. This is shown in figure 4a where the standard deviation for the randomized game is also proportional to $\sqrt{n}$. It may be written as $k\sqrt{n}$ where it is observed from figure 4a that $k < 2\sqrt{pq}$. Thus, we can conclude that the behaviour of the randomized game is approximately the same, if not better than that of game A.

(a) *Observations*

We have two similar systems: the Brownian ratchet requires that the energy profile be flashed on and off to get directed movement of particles; and Parrondo's games that require switching between games in order to win. We can use the mechanics of the Brownian ratchet to heuristically explain how Parrondo's games work. Game A is well known, and after playing a number of times, the capital has approximately a normal distribution. This is equivalent to when the potential is off in Brownian ratchets, seen by the particle distribution in figure 2b. Thus, an appropriate assumption would be that game B has a potential associated with it like that of the ratchet. With a little more investigation it is possible to find the potential associated with game B (Harmer & Abbott 1999). Although the potential is a little more complicated, it works in a very similar fashion to the energy profiles shown in figure 2. The analogy of quantities between the Brownian ratchet and Parrondo's games are shown in table 1.

Table 1. *Relationship between quantities used for the Brownian ratchet and Parrondo's paradox*

quantity	Brownian ratchet	Parrondo's paradox
source of potential	electrostatic, gravity	rules of games
duration	time	number of games played
potential	potential field gradient	parameter ε
switching	U_{on} and U_{off} applied	games A and B played
switching durations	for 'time on' and 'time off'	a and b
measurement/output	displacement x	capital or gain
external energy	switching U_{on} and U_{off}	alternating games
potential asymmetry	depends on α	branching of B to p_1 or p_2
mode of analysis	Fokker–Planck equation	discrete-time Markov chains

When we consider the ratchet and pawl machine, we can only get directed motion when energy is added to the system. Similarly, for a flashing Brownian ratchet, energy is used when switching between two states to produce 'uphill' motion of Brownian particles. In the simulations of Parrondo's games, from two losing games we can yield a winning expectation. This creates a paradox, 'money for free.' Where is the 'energy' coming from in Parrondo's games? This is an unsolved problem and remains an open question. Perhaps the answer lies in the context in which Parrondo's games are applied. For instance, assuming they can be applied to stock market models, the 'switching energy' can be thought of as the buying and selling transaction cost. However, in the case of two individuals gaming, the interpretation of switching energy becomes problematic as there is no apparent 'cost' in the process of switching—this appears truly paradoxical.

3. Analysis

The parameters of Parrondo's games can be chosen such that individually each game is losing but a randomization between the games is winning. In this section we present the mathematical analysis that establishes this. We do this by establishing conditions for recurrence of the corresponding discrete-time Markov chains. In order to simply the algebra of the analysis, it is carried out using $M = 3$ and the switching between the games is restricted to doing so randomly.

The analysis of game A is elementary and can be found in many textbooks (see, for example, Karlin & Taylor 1975), but we present it here in the interest of motivating our analysis of game B.

We win a single round of game A with probability p and lose with probability $1-p$. Assuming that we bet one unit on each round of the game, we wish to calculate the probability f_j that our capital ever reaches zero given that we start with a capital of j units. It is a consequence of Markov chain theory (see Karlin & Taylor 1975) that either

(1) $f_j = 1$ for all $j \geqslant 0$, in which case the game is either fair or losing; or

(2) $f_j < 1$ for all $j > 0$, in which case there is some probability that our capital will grow indefinitely and so the game is winning.

The set of numbers $\{f_j\}$ is the minimal non-negative solution to the set of equations

$$f_j = pf_{j+1} + (1-p)f_{j-1}, \quad j \geqslant 1, \tag{3.1}$$

subject to the boundary condition

$$f_0 = 1. \tag{3.2}$$

The general solution to equation (3.1) is of the form

$$f_j = A\left(\frac{1-p}{p}\right)^j + B, \tag{3.3}$$

where A and B are constants. Invoking the boundary condition (3.2), this becomes

$$f_j = A\left[\left(\frac{1-p}{p}\right)^j - 1\right] + 1. \tag{3.4}$$

If $(1-p)/p \geqslant 1$, the minimal non-negative solution to (3.1) and (3.2) occurs when $A = 0$ and so $f_j = 1$ for all $j \geqslant 0$. If $(1-p)/p < 1$, the minimal non-negative solution to (3.1) and (3.2) occurs when $A = 1$ and so

$$f_j = \left(\frac{1-p}{p}\right)^j \quad \text{for all } j > 0.$$

Thus we can write

$$f_j = \min\left(1, \left(\frac{1-p}{p}\right)^j\right) \tag{3.5}$$

and we observe that the game is winning if $(1-p)/p < 1$, that is if $p > \frac{1}{2}$. By symmetry, we can deduce that the game is losing if $p < \frac{1}{2}$ and is fair if $p = \frac{1}{2}$. This result, of course, accords with our intuition.

Now let us turn to game B. Here the probability that we win a single round depends on the value of our current capital. If the capital is a multiple of three, the probability of winning is p_1, whereas if the current capital is not a multiple of three, the probability of winning is p_2. The corresponding losing probabilities are $1-p_1$ and $1-p_2$, respectively. Let g_j be the probability that our capital ever reaches zero given that we start with a capital of j units. As with game A, Markov chain theory tells us that either

(1) $g_j = 1$ for all $j \geqslant 0$, in which case the game is either fair or losing; or

(2) $g_j < 1$ for all $j > 0$, in which case there is some probability that our capital will grow indefinitely and so the game is winning.

The set of numbers $\{g_j\}$ satisfies the equations

$$\left.\begin{aligned} g_{3i} &= p_1 g_{3i+1} + (1-p_1)g_{3i-1}, && i \geqslant 1, \\ g_{3i+1} &= p_2 g_{3i+2} + (1-p_2)g_{3i}, && i \geqslant 0, \\ g_{3i+2} &= p_2 g_{3i+3} + (1-p_2)g_{3i+1}, && i \geqslant 0, \end{aligned}\right\} \tag{3.6}$$

subject to the boundary condition

$$g_0 = 1. \tag{3.7}$$

Eliminating g_{3i+1}, g_{3i+2} and g_{3i-1} from equations (3.6), we find that, for $i \geqslant 1$,

$$g_{3i}[1 - p_1 - 2p_2 + p_2^2 + 2p_1p_2] = p_1p_2^2 g_{3i+3} + (1 - p_1)(1 - p_2)^2 g_{3i-3}, \tag{3.8}$$

which has a general solution of the form

$$g_{3i} = A\left(\frac{(1 - p_1)(1 - p_2)^2}{p_1p_2^2}\right)^i + B. \tag{3.9}$$

Use of (3.7) yields

$$g_{3i} = A\left[\left(\frac{(1 - p_1)(1 - p_2)^2}{p_1p_2^2}\right)^i - 1\right] + 1, \tag{3.10}$$

and, as in the case of game A, we deduce that

$$g_{3i} = \min\left(1, \left(\frac{(1 - p_1)(1 - p_2)^2}{p_1p_2^2}\right)^i\right). \tag{3.11}$$

Even though to do so would be elementary, it is not necessary to calculate expressions for g_{3i+1} and g_{3i+2}, because they will both be equal to one or less than one according to whether g_{3i} is equal to one or less than one. As for game A, we deduce that game B is winning if

$$\frac{(1 - p_1)(1 - p_2)^2}{p_1p_2^2} < 1, \tag{3.12}$$

losing if

$$\frac{(1 - p_1)(1 - p_2)^2}{p_1p_2^2} > 1 \tag{3.13}$$

and fair if

$$\frac{(1 - p_1)(1 - p_2)^2}{p_1p_2^2} = 1. \tag{3.14}$$

Note that the parameters $p_1 = \frac{1}{10}$ and $p_2 = \frac{3}{4}$, found by trial and error to give rise to a fair game in Harmer & Abbott (1999), satisfy (3.14).

Now consider the situation where we play game A with probability γ and game B with probability $1-\gamma$. If our capital is a multiple of three, the probability that we win the randomized game is $q_1 = \gamma p + (1 - \gamma)p_1$, whereas if our capital is not a multiple of three, the probability that we win is $q_2 = \gamma p + (1 - \gamma)p_2$. The probabilities of losing are $1 - q_1$ and $1 - q_2$, respectively. We observe that this is identical to game B except that the probabilities have changed. It follows from (3.12)–(3.14) that the randomized game is a winning player if

$$\frac{(1 - q_1)(1 - q_2)^2}{q_1q_2^2} < 1, \tag{3.15}$$

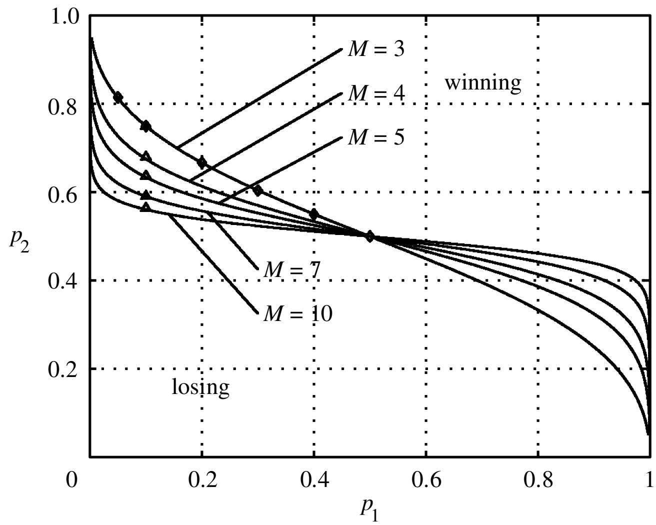

Figure 5. The solid line shows the values of p_1^* and p_2^* satisfying (3.14) to make game B fair. A continuous range of possibilities exist for each value M. The points shown by the diamonds and triangles are values found by trail and error using numerical simulations.

losing if

$$\frac{(1-q_1)(1-q_2)^2}{q_1q_2^2} > 1 \tag{3.16}$$

and fair if

$$\frac{(1-q_1)(1-q_2)^2}{q_1q_2^2} = 1. \tag{3.17}$$

The existence of the paradox of Parrondo's games will be established if we can find parameters p, p_1, p_2 and γ for which

$$\frac{1-p}{p} > 1, \quad \frac{(1-p_1)(1-p_2)^2}{p_1p_2^2} > 1 \quad \text{and} \quad \frac{(1-q_1)(1-q_2)^2}{q_1q_2^2} < 1.$$

If we take $p = \frac{5}{11}$, $p_1 = \frac{1}{121}$, $p_2 = \frac{10}{11}$ and $\gamma = \frac{1}{2}$, then

$$\frac{1-p}{p} = \tfrac{6}{5} > 1, \quad \frac{(1-p_1)(1-p_2)^2}{p_1p_2^2} = \tfrac{6}{5} > 1,$$

but

$$\frac{(1-q_1)(1-q_2)^2}{q_1q_2^2} = \tfrac{217}{300} < 1,$$

which shows that, with these parameters, games A and B are losing, but the randomized game in which games A and B are both played with probability $\frac{1}{2}$ is winning.

4. Parameter space

In this section we shall describe in more depth the regions of the parameter space in which Parrondo's paradox can occur and consider several generalizations of Parrondo's games.

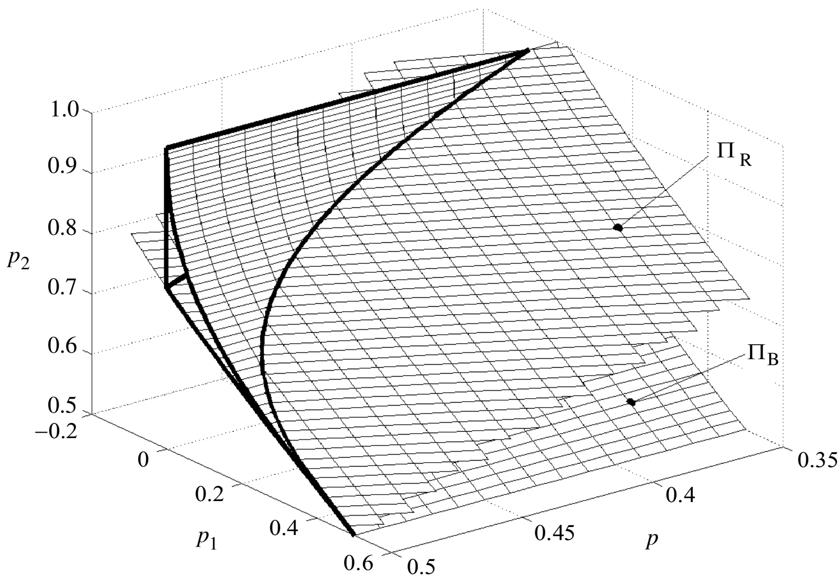

Figure 6. This shows the two planes Π_B and Π_R defined by (3.14) and (3.17), respectively. The third plane Π_A, $p = \frac{1}{2}$, which is not shown for clarity, is parallel to the p_1p_2-plane. The region enclosed by the three planes (and the boundary conditions $0 < p, p_1, p_2 < 1$) gives the parameter space for which the paradox exists. The parameter space shown has $M = 3$ and $\gamma = \frac{1}{2}$.

Although the analysis was performed using $M = 3$, it is easy to show that similar solutions can be found for the more general case of any integer value for $M \geqslant 3$. In this case the final result is to replace the exponent of 2 in equations relating to the randomized games by $M - 1$.

Satisfying $p < \frac{1}{2}$, (3.13) and (3.15) define the parameter space for which Parrondo's paradox exists. Game A is simple and only depends on p. The condition for winning or losing game B depends on parameters p_1 and p_2 (for a given value of M). This can be represented in two-dimensional space as shown in figure 5 for various values of M. The lines show combinations of p_1^* and p_2^* for game B. Using the definition of p_1^* and p_2^*, or equations (3.12) and (3.13), we deduce that the area above the lines creates winning games while the area below the lines creates losing games. The points shown by the triangles and diamonds where found by trial and error from numerical simulations. These values agree well with the analytical solution developed in this paper.

Now consider the randomized game. The winning or losing conditions are dependent on three parameters p, p_1 and p_2 (for given γ and M). This can be represented in three-dimensional space using axes $\{p, p_1, p_2\}$, as can games A and B. Figure 6 shows the plane for the randomized game, Π_R given by (3.17), and game B, Π_B given by (3.14) when $M = 3$ and $\gamma = \frac{1}{2}$. The plane for game A, Π_A given by $p = \frac{1}{2}$ and is parallel to the p_1p_2-plane. Extending the idea of winning and losing areas in figure 5 to volumes in figure 6 leads to an enclosed three-dimensional region where the paradox exists. This region of parameter space is shown by the thick outline in figure 6. The region is below both Π_{A} and Π_{B}, thus games A and B are losing, but above Π_{R}, so the randomized game is winning and the paradox exists. Hence, selecting any point $P(p, p_1, p_2)$ in the enclosed region gives rise to Parrondo's games becoming paradoxical.

5. Conclusion

By switching between two states a Brownian ratchet can move particles 'uphill' or up in potential—even if particles ordinarily move down in each of the states. This is the so-called flashing ratchet. Parrondo's inspiration was to recognize that the two states could be likened to two losing games A and B. When appropriate games are then alternated, a winning expectation is attained. This analogy was explored further by analysing a number of computer simulations. A number of characteristics of the resulting graphs were heuristically explained by using the Brownian ratchet model. By considering the games as a discrete-time process, we have used Markov chain theory to find an analytical solution to Parrondo's paradox. As a consequence of discrete-time Markov chain theory we are able to find a region in the parameter space for which the paradox exists.

Finally, we speculate that increased understanding of Parrondo's paradox may have applications in a wide range of physical processes, stochastic signal processing, biology and perhaps in economics.

This work was funded by the Australian Research Council.

References

Abbott, D., Davis, B. R. & Parrondo, J. M. R. 2000 The problem of detailed balance for the Feynman–Smoluchowski engine (FSE) and the multiple pawl paradox. In *Proc. 2nd Int. Conf. on Unsolved Problems of Noise and Fluctuations* (ed. D. Abbott & L. B. Kiss). American Institite of Physics. (In the press.)

Astumian, R. D. 1997 Thermodynamics and kinetics of a Brownian motor. *Science* **276**, 917–922.

Astumian, R. D. & Bier, M. 1994 Fluctuation driven ratchets: molecular motors. *Phys. Rev. Lett.* **72**, 1766–1769.

Bier, M. 1997*a* Brownian ratchets in physics and biology. *Contemp. Phys.* **38**, 371–379.

Bier, M. 1997*b* A motor protein model and how it relates to stochastic resonance, Feynman's ratchet, and Maxwell's demon. In *Stochastic dynamics*, vol. 386, pp. 81–87. Springer.

Doering, C. R. 1995 Randomly rattled ratchets. *Nuovo Cim.* D **17**, 685–697.

Faucheux, L. P., Bourdieu, L. S., Kaplan, P. D. & Libchaber, A. J. 1995 Optical thermal ratchet. *Phys. Rev. Lett.* **74**, 1504–1509.

Feynman, R. P., Leighton, R. B. & Sands, M. 1963 *The Feynman lectures on physics*, vol. 1, pp. 46.1–46.9. Reading, MA: Addison-Wesley.

Harmer, G. P. & Abbott, D. 1999 Parrondo's paradox. *Statistical Science* **14**, 206–213.

Hänggi, P. & Bartussek, R. 1996 Brownian rectifiers: how to convert Brownian motion into directed transport. In *Nonlinear physics of complex systems—current status and future trends*. Lecture Notes in Physics, vol. 476, pp. 294–308. Springer.

Hughes, B. D. 1995 *Random walks and random variables*, vol. 1. Oxford University Press.

Karlin, S. & Taylor, H. M. 1975 *A first course in stochastic processes*. Academic.

Magnasco, M. O. 1993 Forced thermal ratchets. *Phys. Rev. Lett.* **71**, 1477–1481.

Parrondo, J. M. R. & Español, P. 1996 Criticism of Feynman's analysis of the ratchet as an engine. *Am. Jl Phys.* **64**, 1125–1130.

Rousselet, J., Salome, L., Ajdari, A. & Prost, J. 1994 Directional motion of Brownian particles induced by a periodic asymmetric potential. *Nature* **370**, 446–448.

Shlesinger, M. F. 1996 A brief history of random processes. In *Proc. 1st Int. Conf. on Unsolved Problems of Noise* (ed. Ch. R. Doering, L. B. Kiss & M. F. Shlesinger), pp. 3–10. World Scientific.

von Smoluchowski, M. 1912 Experimentall nachweisbare, der üblichen Thermodynamic widersprechende Molekularphänomene. *Physikalische Zeitschrift*, **XIII**, 1069–1080.