Quantum Matching Pennies Game

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A quantum version of the matching pennies (MP) game is proposed that is played using an Einstein–Podolsky–Rosen–Bohm (EPR–Bohm) setting. We construct the quantum game without using state vectors, while considering only the quantum mechanical joint probabilities relevant to the EPR–Bohm setting. We embed the classical game within the quantum game such that the classical MP game results when the quantum mechanical joint probabilities become factorizable. We report new Nash equilibria in the quantum MP game that emerge when the quantum mechanical joint probabilities maximally violate the Clauser–Horne–Shimony–Holt form of Bell’s inequality.

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1. Introduction

A classical game$^{1,2}$ can be considered an abstract mathematical entity that is connected to the physical world$^3$ in at least three recognizable ways: a) it describes a strategic interaction among the participating players, b) it is implemented using a classical physical system that the players share to play the game, c) it is played in the presence of a referee who ensures that the participating players abide by its rules.

Quantum games$^{4–39}$ retain a) and c) but they are distinguished from the classical games in that the physical system used in the implementation of the game is quantum mechanical. This naturally gives rise to the central question for the area of quantum games: How quantum mechanical features of the shared physical system, used in the physical implementation of the game, express themselves in terms of the outcome/solution of the game?

For a faithful answer to this question it seems natural to establish, as the first step, a correspondence between the classical features, or classicality, of the shared physical system and the classical game and its particular outcome. Establishing this correspondence paves the way for the next step asking what impact it will have on the outcome/solution of the game as the classical features of the shared physical system are replaced by quantum features.

The physical system used in a two-party Einstein–Podolsky–Rosen–Bohm (EPR–Bohm) experiment$^{40–49}$ is known to have genuinely quantum features. This naturally motivates the use of a two-party EPR–Bohm physical system to play a two-player quantum game.

Motivated by developing this approach towards quantum games, we proposed in ref. 37 a scheme to play quantum games using EPR–Bohm experiment. We reported that this scheme is able to construct genuine quantum games from quantum mechanical probabilities only.$^{30}$ This is accomplished in the proposed scheme without referring to the quantum mechanical state vectors, and with little reliance on the mathematical tools of quantum mechanics.

We proposed this scheme for quantum games in view of Jarrett’s position$^{51}$ stating that the experimentally observed violations of Bell inequalities in EPR–Bohm experiments are due to violations of the conjunction of two probabilistic constraints — locality and completeness. Jarrett concluded$^{51}$ that “the predictions of quantum mechanics, in good agreement with the experimental results, satisfy locality, but violate completeness”. Winsberg and Fine$^{48,49}$ prefer the wording factorizability for Jarrett’s completeness. We adopted Winsberg and Fine’s terminology in ref. 37 as well as in this present paper. That is, the quantum features of EPR–Bohm experiments emerge for non-factorizable joint probabilities.

By constructing quantum games from unusual non-factorizable joint probabilities this scheme provides a unifying perspective for both quantum and classical games, and also presents a more easily accessible analysis of quantum games for researchers working outside the domain of quantum physics.

This scheme for quantum games$^{37}$ was applied to analyze the games$^3$ of Prisoner’s Dilemma (PD), Stag Hunt, and Chicken. For the PD game our analysis showed that, contrary to the widely held belief, no new solution that is different from the classical solution emerges when a quantum version of this game is constructed using an EPR–Bohm setting.

However, within the same setting, for three-player PD$^{38,39}$ a new solution indeed emerges that is also found to be Pareto-optimal.$^2$ Moreover, we showed that for the two-player quantum Chicken game, new solution(s) arise for two identified sets of quantum mechanical joint probabilities that maximally violate the Clauser–Horne–Shimony–Holt (CHSH) sum of correlations.$^{46}$

The classical game of PD has a unique Nash equilibrium (NE) consisting of a pair of identical pure strategies — and, in the two-player case, its quantum version in the scheme using the EPR–Bohm setting does not generate a new outcome. This motivates us, in the present paper, to study a quantum version of a two-player game, within the same scheme, that has a unique mixed NE. The well-known game of matching pennies (MP)$^{1,2}$ provides such an example.

Using the scheme based on EPR–Bohm experiments to play this game, we find the impact on the solution of this game when the factorizability condition on joint probabilities is dropped, while the conditions describing normalization and locality are retained.
2. Matching Pennies Game

In the game of MP each of the two players, henceforth labelled as Alice and Bob, have a penny that each secretly flips to heads \( H \) or tails \( T \). No communication takes place between Bob and Alice and they disclose their choices simultaneously to a referee, who organizes the game and ensures that its rules are respected by the participating players.

If the referee finds that the pennies match (both heads or both tails), he takes one dollar from Bob and gives it to Alice (+1 for Alice, −1 for Bob). If the pennies do not match (one heads and one tails), the referee takes one dollar from Alice and gives it to Bob (−1 for Alice, +1 for Bob).

As one player’s gain is exactly equal to the other player’s loss, the game is zero-sum and is represented with the payoff matrix:

\[
\begin{pmatrix}
\text{Alice} & \mathcal{H} & \mathcal{T} \\
(1,-1) & (a_1, b_1) & (a_2, b_2) \\
(a_3, b_3) & (a_4, b_4)
\end{pmatrix}
\]

where we take \( a_1 = +1, b_1 = -1 \); \( a_2 = -1, b_2 = +1 \); \( a_3 = -1, b_3 = +1 \); and \( a_4 = +1, b_4 = -1 \).

2.1 Nash equilibrium

It is well known that MP has no pure strategy Nash equilibrium\(^2\) and instead has a unique mixed strategy NE. For completeness of this paper we describe here how this is found. Consider repeated play of the game in which the outcome of the game.

\[
\Pi_A(x, y) = \begin{pmatrix} x \\ 1-x \end{pmatrix}^T \begin{pmatrix} (a_1, b_1) & (a_2, b_2) \\ (a_3, b_3) & (a_4, b_4) \end{pmatrix} \begin{pmatrix} y \\ 1-y \end{pmatrix}.
\]

A strategy pair \( (x^*, y^*) \) is a NE when

\[
\Pi_A(x^*, y^*) - \Pi_A(x, y^*) \geq 0, \\
\Pi_B(x^*, y^*) - \Pi_B(x, y^*) \geq 0.
\]

For the matrix (1) these inequalities read \( 2(x^* - x)(2y^* - 1) \geq 0 \) and \( 2(y^* - y)(-2x^* + 1) \geq 0 \) and generate the strategy pair \( (x^*, y^*) = (1/2, 1/2) \) as the unique NE of the game. At this NE the players’ payoffs work out as

\[
\Pi_A(1/2, 1/2) = 0 = \Pi_B(1/2, 1/2).
\]

2.2 Playing the game with four biased coins

The first step in our quantization scheme for the MP game consists of translating the game into a classical arrangement using a physical system that involves 16 joint probabilities. The arrangement we use consists of two players sharing four biased coins to play the game, assuming that the referee has the means to set constraints on their biases.

The referee has four coins and seizes them as \( S_1', S_2', S_1, S_2 \) to be Alice’s coins and \( S_1', S_2' \) to be Bob’s coins. In a run, the referee hands over the \( S_1, S_2 \) coins to Alice and the \( S_1', S_2' \) coins Bob. Alice’s and Bob’s strategies consist of choosing one coin out of the two that each player receives in a run. The pair of chosen coins in a run is one of the \( (S_1, S_1'), (S_1, S_2'), (S_2, S_1'), (S_2, S_2') \). The players return the two chosen coins to the referee who tosses them together and records the outcome.

The referee collects the four coins (two tossed and two untossed) and repeats the same procedure over a large number of runs.

Referee defines and makes public the players’ payoff relations that depend on a) the outcomes of a large number of tosses of four biased coins, while two coins are tossed in each run, b) the players’ strategies, and c) the real numbers defining the matrix of the game.

We now state that the statistical behavior of the four biased coins, expressed over a large number of tosses, is described by:

\[
\begin{pmatrix}
S_1' \\
S_2'
\end{pmatrix}
=\begin{pmatrix}
+1 & -1 & +1 & -1 \\
-1 & +1 & -1 & +1 \\
p_1 & p_2 & p_3 & p_4 \\
p_5 & p_6 & p_7 & p_8 \\
p_9 & p_{10} & p_{11} & p_{12} \\
p_{13} & p_{14} & p_{15} & p_{16}
\end{pmatrix},
\]

where the \( \mathcal{H} \) state of a coin is denoted by +1 and the \( \mathcal{T} \) state by −1. The joint probabilities are factorizable for coins, that is, one can find four numbers \( r, s, r', s' \) and \( s' \in [0,1] \) from which the joint probabilities can be obtained as

\[
p_1 = rr', \quad p_2 = r(1-r'), \quad p_3 = r'(1-r), \quad p_4 = (1-r)(1-r'), \quad p_5 = rs', \quad p_6 = r(1-s'), \quad p_7 = s'(1-r), \quad p_8 = (1-r)(1-s'), \quad p_9 = sr', \quad p_{10} = s(1-r'), \quad p_{11} = r'(1-s), \quad p_{12} = (1-s)(1-r'), \quad p_{13} = ss', \quad p_{14} = s(1-s'), \quad p_{15} = s'(1-s), \quad p_{16} = (1-s)(1-s').
\]

2.2.1 Payoff relations and Nash equilibrium

The referee makes public and uses the following payoff relations:
\[ \Pi_{A,B}(x,y) = \begin{pmatrix} x & 1-x \end{pmatrix}^T \begin{pmatrix} \Pi_{A,B}(S_1,S'_1) & \Pi_{A,B}(S_1,S'_2) \\ \Pi_{A,B}(S_2,S'_1) & \Pi_{A,B}(S_2,S'_2) \end{pmatrix} \begin{pmatrix} y \\ 1-y \end{pmatrix}, \]  

(7)

where \( T \) is for transpose and \( x \) and \( y \) are the probabilities, definable over a large number of runs, with which Alice and Bob choose \( S_1 \) and \( S'_1 \), respectively. Also, that the referee defines

\[ \Pi_{A,B}(S_1,S'_1) = \sum_{i=1}^{4} (a,b)_i p_i, \]

\[ \Pi_{A,B}(S_1,S'_2) = \sum_{i=7}^{12} (a,b)_i p_i, \]

\[ \Pi_{A,B}(S_2,S'_1) = \sum_{i=13}^{18} (a,b)_i p_i, \]

\[ \Pi_{A,B}(S_2,S'_2) = \sum_{i=19}^{24} (a,b)_i p_i, \]

(8)

It can be shown how, and under what circumstances, the payoff relations (7) produce the classical mixed-strategy game and result in the classical NE. For the factorizable joint probabilities (6), obtained by a large number of coin tosses, the NE inequalities (3) read

\[ 4(r-s)(y'(r'-s') + s' - 1/2)(x' - y) \geq 0, \]

\[ -4(r'-s')(x'(r-s) + s - 1/2)(y' - y) \geq 0. \]

(9)

At this stage, the referee sets the coin probabilities \( r, s, r', s' \) to be constrained as

\[ r + s = 1, \quad r' + s' = 1, \]

(10)

which, of course, then results in the strategy pair \( (x',y') = (1/2,1/2) \) to be the NE.

To obtain the players’ payoffs at this NE, from eqs. (7) and (10) we evaluate following quantities

\[ \Pi_A(S_1,S'_1) = (2r - 1)(2r' - 1), \]

\[ \Pi_A(S_1,S'_2) = (2r - 1)(2r' - 1), \]

\[ \Pi_A(S_2,S'_1) = (2s - 1)(2s' - 1), \]

\[ \Pi_A(S_2,S'_2) = (2s - 1)(2s' - 1), \]

(11)

from which the players’ rewards at the NE of \( (x',y') = (1/2,1/2) \) are obtained as

\[ \Pi_A(1/2,1/2) = 0 = \Pi_B(1/2,1/2). \]

(12)

We have thus translated the playing of MP game in an arrangement involving 16 factorizable joint probabilities, obtained from a large number of tosses performed on four biased coins. We have found that, in order to guarantee that factorizable joint probabilities result in the classical game, certain constraints, given in (10), need to be placed on the coin probabilities \( r, s, r', s' \). This translation paves the way for introducing the quantum mechanical joint probabilities in the playing of this game, that may not be factorizable as they are for classical coins.

3. Quantum Games Using the EPR–Bohm Setting

We consider a quantum version of this game that is played using the EPR–Bohm setting. This scheme for playing a quantum version of a two-player two-strategy game was originally developed in ref. 37. The quantum game using the

EPR–Bohm setting involves (refer to Fig. 1):

a) A large number of runs when, in a run, two halves of an EPR pair originate from the same source and move in opposite directions.

b) One half is received by player Alice, while Bob receives the other half. Alice and Bob are located at some distance from each other and are unable to communicate between themselves.

c) The players, however, can communicate about their actions, which they perform on their received halves, to the referee who organizes the game and ensures that the rules of the game are followed.

d) The referee makes available 55 two directions to each player. Call Alice’s two directions \( S_1 \) and \( S'_1 \) and Bob’s two directions \( S_2 \) and \( S'_2 \).

e) In a run, each player has to choose one of two directions at his/her disposal and informs the referee of this choice.

f) After receiving information about the pair of directions, which the players have chosen in a particular run, the referee rotates Stern–Gerlach type detectors along the two chosen directions and performs a quantum measurement.

g) The outcome of the quantum measurement, \( 55 \) on Alice’s side, and on Bob’s side of the Stern–Gerlach detectors, is either +1 or −1.

h) Runs are repeated as the players receive a large number of halves in pairs, when each pair comes from the same source.

i) The referee records the measurement outcomes for all runs, when in each run each player chooses one of the two directions.

j) The referee defines a player’s strategy, over a large number of runs, to be a linear combination (with normalized and real coefficients) of the two directions along which the measurement is performed.

k) The referee has payoff relations that s/he makes public at the start of the game and announces rewards to the players after the completion of runs.

l) The referee constructs these payoff relations in view of a) the matrix (1) of the game being played, b) the list of players’ choices of directions over several runs, and c) the list of measurement outcomes that the referee prepares using his/her Stern–Gerlach apparatus.

The translated MP game, using four biased coins, allows one to express players’ payoff relations in terms of the 16 joint probabilities. The following Section shows that the
physical system in the EPR–Bohm experiments also involve 16 joint probabilities, and thus the above translation provides the natural route for playing a quantum MP game.

3.1 Constraints on quantum mechanical joint probabilities

The payoff relations (7) are defined in view of the fact that the set of 16 joint probabilities satisfy a number of constraints that are imposed by the requirements of a) normalization, b) locality, and c) factorizability.

In order to better appreciate the quantum mechanical probabilities, we consider, for example, the situation when over all runs Alice chooses $S_1$ and Bob chooses $S_2$. Referee rotates Stern–Gerlach detectors along these two directions and then, for example, referring to (5) $p_0$ gives the probability of him/her obtaining $-1$ along Alice's $S_1$ direction and $+1$ along Bob's $S_2$ direction.

3.1.1 Normalization

Normalisation says that when, for example, Alice chooses $S_1$ and Bob chooses $S_2$ for all the runs, the only possible outcomes are $(+1,+1), (+1,-1), (-1,+1), (-1,-1)$. The same is true for other pure strategy pairs $(S_1, S'_1), (S_2, S'_1), (S_2, S)$:

$$\sum_{i=1}^{4} p_i = 1 = \sum_{i=5}^{8} p_i, \quad \sum_{i=9}^{16} p_i = 1 = \sum_{i=13}^{16} p_i.$$  

3.1.2 Locality

The 16 joint probabilities satisfy another set of constraints that are obtained from the requirements stating that in a run:

a) Alice’s outcome of $+1$ or $-1$ (obtained along $S_1$ or $S_2$) is independent of whether Bob chooses $S'_1$ or $S'_2$ in that run.

b) Bob’s outcome of $+1$ or $-1$ (obtained along $S'_1$ or $S'_2$) is independent of whether Alice chooses $S_1$ or $S_2$ in that run.

When translated in terms of joint probabilities, and referring to (5), these requirements state that:

$$p_1 + p_2 = p_5 + p_6, \quad p_1 + p_3 = p_9 + p_{11},$$
$$p_9 + p_{10} = p_{13} + p_{14}, \quad p_5 + p_7 = p_{13} + p_{15},$$
$$p_3 + p_4 = p_7 + p_8, \quad p_{11} + p_{12} = p_{15} + p_{16},$$
$$p_2 + p_4 = p_{10} + p_{12}, \quad p_6 + p_8 = p_{14} + p_{16}.$$  

Quite often one finds in the literature the word “locality” to describe these constraints. As can be seen, the possibility, described in (6), of writing $p_i$ for $1 \leq i \leq 16$ in terms of $r, s, r', s' \in [0, 1]$ also assumes locality. Notice that for a factorizable set of joint probabilities (6) the locality constraints (14) always hold.

3.1.3 Factorizability

Equation (6) state that the joint probabilities can be written in terms of $r, s, r', s' \in [0, 1]$. If this is the case then:

$$r = p_1 + p_2, \quad s = p_9 + p_{10}, \quad r' = p_1 + p_3, \quad s' = p_5 + p_7,$$

and eq. (6) can be reinterpreted as:

$$p_1 = (p_1 + p_2)(p_1 + p_3),$$
$$p_2 = (p_1 + p_2)(1 - p_1 - p_3), \ldots$$

$$p_{16} = (1 - p_9 - p_{10})(1 - p_5 - p_7).$$  

The alert reader may notice that, in the writing of eqs. (6) and (15) and in the possibility of finding $r, s, r', s' \in [0, 1]$ that allows this, it is assumed that joint probabilities satisfy the locality constraints (14).

3.1.4 Cereceda’s analysis

We now refer to a result, reported by Cereceda\(^{47}\) stating that, because of normalization (13), half of the eq. (14) are redundant thus making eight among sixteen probabilities $p_i$ independent. Cereceda has reported that a convenient solution of the system [eqs. (13) and (14)], is the one for which the set of variables:

$$v = \{p_2, p_3, p_6, p_7, p_{10}, p_{11}, p_{13}, p_{16}\},$$

is expressed in terms of the remaining set of variables:

$$\mu = \{p_1, p_4, p_5, p_8, p_9, p_{12}, p_{14}, p_{15}\},$$

is given as:

$$p_2 = (1 - p_1 - p_4 + p_5 - p_8 - p_9 + p_{12} + p_{14} - p_{15})/2,$$
$$p_3 = (1 - p_1 - p_4 + p_5 + p_8 + p_9 - p_{12} - p_{14} + p_{15})/2,$$
$$p_6 = (1 + p_1 - p_4 - p_5 - p_8 - p_9 + p_{12} + p_{14} - p_{15})/2,$$
$$p_7 = (1 + p_1 - p_4 - p_5 + p_8 + p_9 - p_{12} - p_{14} + p_{15})/2,$$
$$p_{10} = (1 - p_1 + p_4 + p_5 - p_8 - p_9 - p_{12} + p_{14} - p_{15})/2,$$
$$p_{11} = (1 + p_1 + p_4 - p_5 + p_8 - p_9 - p_{12} - p_{14} + p_{15})/2,$$
$$p_{13} = (1 - p_1 + p_4 + p_5 - p_8 + p_9 - p_{12} - p_{14} - p_{15})/2,$$
$$p_{16} = (1 + p_1 - p_4 - p_5 + p_8 + p_9 - p_{12} + p_{14} + p_{15})/2.$$  

These relationships arise because the quantum mechanical joint probabilities fulfill both the normalization condition (13) as well as the locality constraints (14).

3.1.5 CHSH inequality

Notice that using (5) the correlation $\langle S_1 S'_1 \rangle$, for example, can be found as:

$$\langle S_1 S'_1 \rangle = \Pr(S_1 = 1, S'_1 = 1) - \Pr(S_1 = 1, S'_1 = -1)$$
$$= \Pr(S_1 = -1, S'_1 = 1) + \Pr(S_1 = -1, S'_1 = -1)$$
$$= p_1 - p_2 + p_4.$$  

The correlations $\langle S_1 S'_1 \rangle, \langle S_1 S'_2 \rangle, \langle S_2 S'_1 \rangle$, and $\langle S_2 S'_2 \rangle$ can similarly be worked out. The CHSH sum of correlations is then defined as:

$$\Delta = |\langle S_1 S'_1 \rangle + \langle S_1 S'_2 \rangle + \langle S_2 S'_1 \rangle - \langle S_2 S'_2 \rangle|,$$

and the CHSH inequality:

$$|\Delta| \leq 2,$$

which holds for any theory of local hidden variables.

Cereceda has reported\(^{52}\) that there exist two sets of joint probabilities that maximally violate the quantum prediction of the CHSH sum of correlations. The first set is given as:

$$p_1 = (2 + \sqrt{2})/8 \quad \text{for all } p_i \in \mu,$$
$$p_6 = (2 - \sqrt{2})/8 \quad \text{for all } p_i \in \nu,$$

whereas the second set is given as:

$$p_7 = (2 + \sqrt{2})/8 \quad \text{for all } p_i \in \mu,$$

$$p_{16} = (1 - p_9 - p_{10})(1 - p_5 - p_7).$$

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\[ p_k = (2 + \sqrt{2})/8 \quad \text{for all } p_k \in \nu, \quad (24) \]

where \( \nu \) and \( \mu \) are defined in (17) and (18). That is, these two sets provide the maximum absolute limit of \( 2\sqrt{2} \) for \( \Delta_{QM} \).

3.1.6 Constraints imposed by Cirel’son limit

Now, alongside the constraints (27) there is another set of constraints on joint probabilities that are imposed by the Cirel’son limit,\( ^{53} \) saying that the quantum prediction of the CHSH sum of correlations \( \Delta \), defined in (21), is bounded in absolute value by \( 2\sqrt{2} \), i.e., \( |\Delta_{QM}| \leq 2\sqrt{2} \). Taking into account\( ^{57} \) the normalization condition (13), the quantity \( \Delta \) is then equivalently expressed as

\[ \Delta = 2(p_1 + p_4 + p_5 + p_8 + p_9 + p_{12} + p_{14} + p_{15} - 2). \quad (25) \]

In the following, the EPR setting, introduced in this section, is used to play the quantum version of the MP game.

4. Quantum Matching Pennies Game

Essentially, our quantum MP game corresponds when the 16 joint probabilities, that appear in the payoff relations (7), are obtained using the EPR–Bohm setting, instead of using a large number of tosses performed on biased coins.

The players’ payoff relations in the quantum MP game, therefore, remain exactly the same as they are defined and made public by the referee in eq. (7) for the translated game that uses factorizable joint probabilities. Players’ strategies also remain exactly the same as they are in the classical game.

The referee is free to prepare any quantum pure or mixed bi-partite state and to forward it to the players. S/he also fixes the four available directions at the start of the game (refer to Fig. 1) that cannot be changed as the game progresses and large number of its runs are carried out. A player’s strategic choices do not go beyond choosing between the two assigned directions.

4.1 Embedding the classical game within the quantum game

Referring to eq. (10) we recall that it expresses the constraints on the coin probabilities. We also notice that the factorizability, expressed by (6), permits one to write the coin probabilities in terms of joint probabilities:

\[
\begin{align*}
    r &= p_1 + p_2, \quad s = p_0 + p_{10}, \\
    r' &= p_1 + p_3, \quad s' = p_5 + p_7,
\end{align*}
\]

which allows us to rewrite the constraints (10) on coin probabilities as

\[ p_1 + p_3 + p_5 + p_7 = 1, \quad (27) \]

This provides the key for embedding the classical game within the quantum game. S/he makes prior (experimental) arrangements in the EPR–Bohm setup ensuring that the constraints (27) on joint probabilities hold during the whole course of playing the game.\( ^{55} \) When this is the case the classical game remains embedded within the corresponding quantum game in that the quantum game attains classical interpretation with the joint probabilities becoming factorizable.

However, the joint probabilities that the EPR–Bohm setting can generate can also be non-factorizable. This permits playing a quantum game in which the constraints (27) hold, while the factorizability condition on joint probabilities is dropped.

We now look at how dropping the factorizability condition for joint probabilities affects the outcome of the game. With the constraints (27) continuing to hold, the referee can then find a pair of NE strategies \((x^*, y^*)\) in the quantum game using the inequalities (3) as usual. Because of non-factorizable joint probabilities the strategy pair \((x^*, y^*)\) may be different from the one which comes out for factorizable joint probabilities.

Notice that the rewards at the NE are identical to the ones given in (4). That is, when the 16 joint probabilities become factorizable, the NE and the players’ payoffs become identical to the ones obtained in the usual mixed strategy solution of the MP game. Also, the 16 joint probabilities, even when they are non-factorizable and, therefore, violate one or more of the set of eq. (16), will always satisfy the normalization constraints (13) as well as the locality constraints (14).

To be consistent with the standard setting\( ^{2} \) for playing a two-player two-strategy game, the referee considers it reasonable to require that in the EPR setting a player plays a pure strategy if s/he chooses the same direction over all the runs and that s/he plays a mixed strategy if s/he has a probability distribution with which s/he chooses between the two directions at her/his disposal. However, identifying pure and mixed strategies in such a way is not of much help as the payoff relations, which referee uses to reward the players, generate the classical mixed strategy game even when the players play “pure strategies”. This, however, remains consistent with the known result in the area of quantum games stating that a pure product initial state leads to the classical mixed strategy game.

4.2 Nash equilibria in the quantum game

We now find the NE that comes out from a set of non-factorizable (and thus quantum mechanical) joint probabilities when the players’ payoff relations in the quantum game are obtained from the eq. (7). For the inequalities defining the NE in the quantum game we obtain

\[
\begin{align*}
    \Pi_A(x^*, y^*) - \Pi_A(x, y^*) &= [\Pi_A(S_1, S'_1) - \Pi_A(S_2, S'_1) - \Pi_A(S_1, S'_2) + \Pi_A(S_2, S'_2)] \\
    &+ [\Pi_A(S_1, S'_2) - \Pi_A(S_2, S'_2)](x^* - x) \\
    &\geq 0, \\
    \Pi_B(x^*, y^*) - \Pi_B(x^*, y) &= [\Pi_B(S_1, S'_1) - \Pi_B(S_1, S'_2) - \Pi_B(S_2, S'_1) + \Pi_B(S_2, S'_2)] \\
    &+ [\Pi_B(S_1, S'_2) - \Pi_B(S_2, S'_2)](y^* - y) \geq 0,
\end{align*}
\]

where eq. (8) and the matrix (1) gives

\[
\begin{align*}
    \Pi_A(S_1, S'_1) &= p_1 - p_2 - p_3 + p_4 = -\Pi_B(S_1, S'_1), \\
    \Pi_A(S_1, S'_2) &= p_5 - p_6 - p_7 + p_8 = -\Pi_B(S_1, S'_2), \\
    \Pi_B(S_2, S'_1) &= p_9 - p_{10} - p_{11} + p_{12} = -\Pi_B(S_2, S'_1), \\
    \Pi_B(S_2, S'_2) &= p_{13} - p_{14} - p_{15} + p_{16} = -\Pi_B(S_2, S'_2),
\end{align*}
\]

where the right sides of these equations express the fact that the quantum game is a zero-sum game as is the classical game.
Using eq. (19) we eliminate the eight probabilities from the inequalities (28) that gives the inequalities for the NE in the quantum game in terms of the probabilities appearing in the set (18):

\[ \Pi_A(x^*, y^*) - \Pi_A(x, y) = 2[x^*[(1 + p_1 + p_2) - (p_5 + p_8 + p_9 + p_{12} + p_{14} + p_{15})] + (p_5 + p_8 + p_{14} + p_{15} - 1)](x^* - x) \geq 0, \]

\[ \Pi_A(x^*, y^*) - \Pi_B(x^*, y) = -2[x^*[(1 + p_1 + p_2) - (p_5 + p_8 + p_9 + p_{12} + p_{14} + p_{15})] + (p_9 + p_{12} + p_{14} + p_{15} - 1)](y^* - y) \geq 0. \]  

As some of the joint probabilities are constrained by (27), using (19) we rewrite these constraints as

\[ p_9 + p_{15} = p_{12} + p_{14}, \quad p_5 + p_{14} = p_8 + p_{15}. \]  

Now, adding the two equations in (31) and subtracting the second from the first gives

\[ p_5 + p_9 = p_8 + p_{12}, \quad p_5 + p_{12} + 2p_{14} = p_8 + p_9 + 2p_{15}, \]  

and we write

\[ p_{12} = p_5 + p_9 - p_8, \quad p_{15} = p_5 + p_{14} - p_8. \]

in order to eliminate arbitrarily the probabilities \( p_{12} \) and \( p_{15} \) from the inequalities (30) to obtain

\[ \Pi_A(x^*, y^*) - \Pi_A(x, y) = 2[x^*[(1 + p_1 + p_2 + p_8) - (3p_5 + 2p_9 + 2p_{14})] + (2p_5 + p_{14} - 1)](x^* - x) \geq 0, \]

\[ \Pi_A(x^*, y^*) - \Pi_B(x^*, y) = -2[x^*[(1 + p_1 + p_2 + p_8) - (3p_5 + 2p_9 + 2p_{14})] + (2p_5 - p_8 + p_9 + p_{14} - 1)](y^* - y) \geq 0. \]

The right sides of these inequalities involve six joint probabilities, which we treat as “independent” and these are \( p_1, p_2, p_5, p_8, p_{14}, p_{15}. \) These inequalities guarantee that for a factorizable set of joint probabilities the classical mixed strategy game of MP emerges.

4.2.1 Nash equilibria for maximally entangled state

Refer to the probability sets (23) and (24) that maximally violate the CHSH inequality. Probabilities in these sets are non-factorizable as for both sets a solution for \( r, s, r', s' \) obtained from the eq. (6) makes one or more of the probabilities \( r, s, r', s' \) to be negative or greater than one. This is also equivalent to stating that for either of the sets (23) and (24) one or more of eq. (6) does not hold, when \( r, s, r', s' \in [0, 1] \) and the constraints (14) imposed by locality hold.

Now a natural question arising here is to ask if these two probability sets can be used for the quantum game of MP. This will indeed be possible if for each of these two sets the constraints given by (27) hold ensuring that the classical MP game is embedded within the quantum. For both the sets (23) and (24) we find that the constraint (27) hold, thus these probability sets, maximally violating the CHSH sum of correlations, can legitimately be used in the quantum MP game.

For the first set (23) the inequalities (34) work out as

\[ \Pi_A(x^*, y^*) - \Pi_A(x, y) = \sqrt{2}(-y^* + 1)(x^* - x) \geq 0, \]

\[ \Pi_B(x^*, y^*) - \Pi_B(x^*, y) = -\sqrt{2}(-x^* + 1)(y^* - y) \geq 0, \]

which give the strategy pairs (1, 0) and (1, 1) as NE. At the strategy pair (1, 0) the players’ payoffs are obtained from eqs. (7) and (29) as \( \Pi_A(1, 0) = 1/\sqrt{2} = -\Pi_B(1, 0) \) whereas at the strategy pair (1, 1) the players’ payoffs are obtained to be the same, i.e., \( \Pi_A(1, 1) = 1/\sqrt{2} = -\Pi_B(1, 1). \)

Similarly, for the second set (24) the NE inequalities (34) are

\[ \Pi_A(x^*, y^*) - \Pi_A(x, y) = \sqrt{2}(y^*-1)(x^*-x) \geq 0, \]

\[ \Pi_B(x^*, y^*) - \Pi_B(x^*, y) = -\sqrt{2}(x^*-1)(y^*-y) \geq 0, \]

giving the strategy pairs (0, 1) and (1, 1) as the NE. At the strategy pair (0, 1) the players’ payoffs work out as \( \Pi_A(0, 1) = -1/\sqrt{2} = -\Pi_B(0, 1) \) whereas at the strategy pair (1, 1) the players’ payoffs are obtained as the same, i.e., \( \Pi_A(1, 1) = -1/\sqrt{2} = -\Pi_B(1, 1). \)

5. Discussion

This paper is motivated by the observation that by having a unique mixed NE the classical MP game offers an opportunity for seeing more clearly how dropping the factorizability condition on joint probabilities may affect this unique NE, which emerges for factorizable joint probabilities in the quantization scheme based on EPR–Bohm experiments.

Notice that in the scheme based on EPR–Bohm experiments the referee’s role is significantly increased as compared to other schemes for playing quantum games. This is because s/he is free to provide any pair of directions to each player and makes quantum measurement(s) on any pure or mixed bi-partite states. The available options for the players are, therefore, reduced in comparison to what is the case in other quantization schemes, and they have exactly the same options as in the classical game.

In a classical two-player two-strategy game each player can play a linear combination (with real and normalized coefficients) of two pure strategies and this remains exactly the same in the our scheme for playing a two-player quantum game.

Joint probabilities in EPR–Bohm experiments, performed on entangled bipartite states, are known to become non-factorizable when players make their strategic choices along certain pairs of directions. This provides the opportunity to look at the possible new outcomes of the game that non-factorizable joint probabilities may generate. In the quantization scheme based on EPR–Bohm experiments the constraints placed on probabilities guarantee that the classical game remains embedded within the quantum game, while probabilities may become non-factorizable.

By constructing quantum games directly from quantum probabilities the suggested approach contributes towards an understanding and potential use of quantum probabilities in the area of game theory. That is, the question addressed in this paper asks whether quantum probabilities have more to offer to game theory. The answer to this we find is “yes”.

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The possibility that CHSH inequality can be rephrased in terms of two-player cooperative games has been reported in literature. In ref. 56 Cleve et al. have reported a game based on CHSH inequality in which the maximum probability of winning the classical game is 3/4 whereas, using a quantum strategy, the players can win this game with probability 0.85, which, as they show using Cirel’son’s limit, is optimal. Also, in ref. 57 Cheon has reported a quantum game in which both players maximize a quantity (utility) defined from spin projections of two particles which they share, whereas the payoff operators are the measurement operator for EPR–Bohm experiment. Cheon then finds the NE of the quantum game that rewards the players far better for particles with maximum entanglement compared to when the particles are uncorrelated.

Both of these studies show that EPR–Bohm experiments can be translated into special games. The contribution of the quantization scheme developed in ref. 37, and that of the present paper, however, is that it shows that, along with the reported possibility of translating EPR–Bohm experiments as special games, one can in fact quantize any two-player game using the framework of EPR–Bohm experiments. Secondly, that we can analyze our quantum game using the non-factorizable property of quantum mechanical joint probabilities.

Nonfactorizability is known48,49,51 to be a necessary but insufficient condition for the violation of Bell’s inequality, the CHSH form of which we consider here. That is, a set of 16 joint probabilities that violates Bell’s inequality will always be non-factorizable, whereas one can find a set of joint probabilities that is non-factorizable and still does not violate the CHSH form of Bell’s inequality. This known result has the following implications when it is considered in our scheme for playing quantum games using EPR–Bohm experiments: As a new solution of the game, which emerges because of dropping the factorizability condition, the relevant joint probabilities may not violate the Bell’s inequality (in its CHSH form) — only those outcomes of the quantum game are to be considered to have a bona fide quantum aspect49 for which the corresponding set of joint probabilities violates the CHSH form of Bell’s inequality. The NE of the quantum game for which the Bell’s inequality is not violated will, therefore, have a pseudo-classical aspect.

Using Bell’s inequality one can identify the pseudoclassical domain from the quantum domain as follows. With the constraints (27) the CHSH inequality (22) using (25) and (33) reduces itself to \(|\Delta_t| \leq 1\), where \(\Delta_t = p_1 + p_2 + 3p_3 - p_5 + 2p_6 + 2p_{14} - 2\). Now, if a set of joint probabilities results in a NE in the quantum game and for this set we have \(|\Delta_t| \leq 1\) then this NE has the pseudoclassical aspect. However, if for this set we have \(|\Delta_t| > 1\) then it has a bona fide quantum aspect. Note that in the quantum MP game the strategy pairs (1, 0) and (1, 1) emerge as NE for the set (23). For these NE we obtain \(\Delta_t = 2/\sqrt{2}\). Similarly, the strategy pairs (0, 1) and (1, 1) emerge as NE for the set (23) and for these NE we obtain \(\Delta_t = -2/\sqrt{2}\). These four NE, therefore, have a bona fide quantum aspect.

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41) J. S. Bell: Physics (N.Y.) I (1964) 195.
52) As the two halves, which the players receive in a run, are quantum mechanical objects, it is assumed that the referee is familiar with quantum mechanics.
53) That is, in each run the referee’s measurement generates one of the four possible pairs \((+1, +1), (+1, -1), (-1, +1),\) and \((-1, -1),\) where the first entry in a bracket is the measurement outcome along Alice’s chosen direction (which is either \(S_1\) or \(S_2\)) and, similarly, the second entry corresponds to Bob’s chosen direction (which is either \(S'_1\) or \(S'_2\)).
55) It is to be noticed that the constraints (27) depend on the game being played and also on the particular outcome that results for factorizable joint probabilities.