Quantum Inf Process
DOI 10.1007/s11128-014-0824-9

A probabilistic approach to quantum Bayesian games of incomplete information

Azhar Iqbal · James M. Chappell · Qiang Li · Charles E. M. Pearce · Derek Abbott

Received: 26 January 2014 / Accepted: 3 September 2014

Abstract A Bayesian game is a game of incomplete information in which the rules of the game are not fully known to all players. We consider the Bayesian game of Battle of Sexes that has several Bayesian Nash equilibria and investigate its outcome when the underlying probability set is obtained from generalized Einstein–Podolsky–Rosen experiments. We find that this probability set, which may become non-factorizable, results in a unique Bayesian Nash equilibrium of the game.

Keywords Quantum games · Bayesian Nash equilibria · EPR experiments · Quantum probability

1 Introduction

The standard approach to constructing quantum games [1–57] naturally uses the formalism of quantum mechanics in Hilbert space [58]. In recent years, however, a prob-

A. Iqbal (✉)
Department of Mathematics and Statistics, King Fahd University of Petroleum and Minerals, Dhahran 31261, Kingdom of Saudi Arabia
e-mail: iqbal@eleceng.adelaide.edu.au

A. Iqbal · J. M. Chappell · D. Abbott
School of Electrical and Electronic Engineering, The University of Adelaide, Adelaide, SA 5005, Australia

Q. Li
College of Electrical Engineering, Chongqing University, Chongqing 400030, People's Republic of China

C. E. M. Pearce
School of Mathematical Sciences, The University of Adelaide, Adelaide, SA 5005, Australia

abilistic approach to this research area [28,39,48–50,52–54] has been proposed that uses sets of non-factorizable quantum mechanical probabilities, i.e., without using the quantum mechanical concepts of state vectors, self-adjoint operators, and quantum measurement. This is with a view of making the area of quantum games more accessible to wider mathematical application, as the methods and range of solution concepts of game theory [59–62] are used and exploited, without any real need for invoking the Hilbert space formalism of quantum mechanics.

The probabilistic approach for a two-player two-strategy game directly uses sets of quantum probabilities corresponding to the measurement outcomes on a two-qubit quantum system. As is known, the setting of generalized Einstein–Podolsky–Rosen (EPR) experiments [63–71] performed on this system leads to the consideration of a set of 16 quantum probabilities. Properties of this probability set have been investigated by Cereceda [71], and it has been pointed out that the CHSH form of Bell's inequality [70] can be re-expressed in terms of constraints on the elements from this set. It is observed that only a non-factorizable probability set, as is defined later, can lead to the violation of Bell's inequality and that not every non-factorizable probability set violates Bell's inequality.

A Bayesian game is a game of incomplete information in which the rules of the game are not fully known to all players. In this paper, we study a Bayesian game that is a variant of the well-known Battle of Sexes game, also studied [12,36] in the quantum game literature. In an earlier study [36], using the quantization protocol based on Schmidt decomposition [58], a Bayesian game of incomplete information [62] has been investigated in relation to the violation of Bell's inequality [58,65–68]. The present paper, however, adopts a different approach in that, without referring to the Hilbert space formalism, it finds the outcome of a Bayesian game when the considered probability set can be non-factorizable—this can arise, in an experimental situation, from a setup involving quantum entanglement, such as an EPR-type experiment, where the CHSH form of Bell's inequality is violated.

In its normal form representation, the game matrix of the Bayesian game has the same number of entries as the 16 elements ε_i in the probability set that corresponds to the generalized EPR experiments [71]. We find that the richer structure of the Bayesian game permits a natural embedding of the classical factorizable game within the quantum game. We show that, whereas the classical factorizable Bayesian game of imperfect information has several Nash equilibria, its non-factorizable quantum version obtained from a set of quantum probabilities corresponding to generalized EPR experiments has a unique Nash equilibrium.

The suggested probabilistic approach to obtaining quantum games thus re-expresses players' payoff relations in terms of a set of probabilities that can also arise in a quantum mechanical experiment. As the approach is based on probabilities only, it does not refer to the formalism of quantum mechanics using state vectors, unitary transformations, and quantum measurements. As game theory is a broad area, with applications ranging from trade, politics, sociology, biology, engineering, etc., most researchers in this area are naturally not familiar with the mathematical formalism of quantum mechanics. This paper thus fills in that gap and demonstrates how an unusual game-theoretic outcome for a Bayesian game results when probability sets that are obtained in quantum mechanical experiments are the underlying probabilities of a Bayesian game. In this approach, the quantum game reduces itself to the classical game when the considered probability set becomes factorizable.

The rest of this paper is organized as follows. Sections 2 and 3 present a review of the classical theory of a Bayesian game that is a variant of the game of Battle of Sexes. Section 4 describes quantum probabilities in generalized EPR experiments, their constraints, and how within the quantum game the players' payoff relations are defined in terms of these probabilities. Section 5 analyses the outcome of the Bayesian game of Battle of Sexes with EPR probabilities, and Sect. 6 discusses the results.

2 The Bayesian game of Battle of Sexes and its variant

The game of Battle of Sexes (BoS) describes [62] the following situation. Two people Alice and Bob wish to go out together, and two concerts are available: one with music by Bach, and one with music by Stravinsky. One person prefers Bach, and the other prefers Stravinsky. If they go to different concerts, each of them is equally unhappy listening to the music of either composer. The situation is represented by the following matrix

$$\begin{array}{c} \text{Bob} \\ \begin{array}{cc} \mathcal{B} & \mathcal{S} \end{array} \\ \text{Alice} \begin{array}{c} \mathcal{B} \\ \mathcal{S} \end{array} \begin{pmatrix} (2,1) & (0,0) \\ (0,0) & (1,2) \end{pmatrix} \end{array}, \tag{1}$$

where Bach and Stravinsky are represented by symbols $\mathcal{B}$ and $\mathcal{S}$, respectively. For this game, a Nash equilibrium (NE) is a pair of strategies such that each player's strategy is the best reply to the strategic choice of the other players. In other words, unilateral deviation from a Nash equilibrium by a player in the form of a different choice of strategy will produce a payoff that is less than or equal to what a Nash equilibrium strategy will give to that player. Analysis shows [62] that this game has three mixed-strategy Nash equilibria $(0, 0)$, $\left(\frac{2}{3}, \frac{1}{3}\right)$, $(1, 1)$, where the numbers in parentheses are the Alice's and Bob's probabilities of choosing the strategy $\mathcal{B}$.

An interesting variant of this game [62] is the one in which Alice is unsure whether Bob prefers to join her or prefers to avoid her, whereas Bob knows Alice's preferences. Assume that Alice thinks that with probability $\frac{1}{2}$ Bob wants to go out with her, and with probability $\frac{1}{2}$ Bob wants to avoid her,

$$\begin{array}{c} \text{Alice playing against Bob's two types} \\ \begin{array}{cc} \text{probability } \frac{1}{2} & \text{probability } \frac{1}{2} \\ \begin{array}{cc} \mathcal{B} & \mathcal{S} \end{array} & \begin{array}{cc} \mathcal{B} & \mathcal{S} \end{array} \\ \begin{array}{c} \mathcal{B} \\ \mathcal{S} \end{array} \begin{pmatrix} (2,1) & (0,0) \\ (0,0) & (1,2) \end{pmatrix} & \begin{array}{c} \mathcal{B} \\ \mathcal{S} \end{array} \begin{pmatrix} (2,0) & (0,2) \\ (0,1) & (1,0) \end{pmatrix} \\ \text{Bob's first type} & \text{Bob's second type} \end{array} \end{array}, \tag{2}$$

that is, from Alice's perspective, Bob has two possible types, the first is shown on the left and the second is on the right in (2).

Alice does not know Bob's type and is thus faced with the situation of choosing her rational action that is based on her belief about the action of Bob of each type. Given these beliefs, and her belief about the likelihood of each type, she can calculate her expected payoff in each case. For instance, given that Alice plays $\mathcal{B}$, and Bob of first type (who wishes to meet Alice) plays $\mathcal{B}$, whereas Bob of second type (who wishes to avoid Alice) plays $\mathcal{S}$, then Alice's expected payoff is $\frac{1}{2}(2) + \frac{1}{2}(0) = 1$.

There are 4 possible pairs of actions of Bob's two types given as $(\mathcal{B}, \mathcal{B})$, $(\mathcal{B}, \mathcal{S})$, $(\mathcal{S}, \mathcal{B})$, and $(\mathcal{S}, \mathcal{S})$. Here, for instance, $(\mathcal{S}, \mathcal{B})$ describes that Bob's first type plays $\mathcal{S}$ and Bob's second type plays $\mathcal{B}$. Players' expected payoffs are then obtained as given below.

$$\begin{array}{cccccc} & & \multicolumn{4}{c}{\text{Bob's two types}} \\ & & (\mathcal{B},\mathcal{B}) & (\mathcal{B},\mathcal{S}) & (\mathcal{S},\mathcal{B}) & (\mathcal{S},\mathcal{S}) \\ \text{Alice} & \mathcal{B} & \left(2,\frac{1}{2}\right) & \left(1,\frac{3}{2}\right) & (1,0) & (0,1) \\ & \mathcal{S} & \left(0,\frac{1}{2}\right) & \left(\frac{1}{2},0\right) & \left(\frac{1}{2},\frac{3}{2}\right) & (1,1) \end{array}. \qquad (3)$$

One can then show [62] that the triplet $(\mathcal{B}, (\mathcal{B}, \mathcal{S}))$, where the first entry in the bracket is Alice's action $\mathcal{B}$ and $(\mathcal{B}, \mathcal{S})$ is the pair of actions of the two types of Bob, constitutes a Nash equilibrium. This is a pure strategy Nash equilibrium consisting of three actions, one for Alice and one for each of the two types of Bob, with the property that (a) Alice's action is optimal given the actions of the two types of Bob, (b) the action of each type of Bob is optimal given the action of Alice.

3 Battle of Sexes with imperfect information

We now consider a situation in which neither player knows whether the other wants to meet or not [62]. As before, Alice assumes that Bob will prefer to join her with probability $\frac{1}{2}$ and will prefer to avoid her with probability $\frac{1}{2}$. Moreover, Bob expects with probability ω, Alice will prefer to join him and with probability $(1 - \omega)$ that she will prefer to avoid him. It is assumed that both Alice and Bob know their own preferences. The game can be represented as shown in Fig. 1 [62].

Consider the payoffs of Alice of type 1. She believes that with probability $\frac{1}{2}$, she faces Bob of type 1 and with probability $\frac{1}{2}$ she faces Bob of type 2. Assume that Bob of type 1 chooses $\mathcal{B}$ and Bob of type 2 chooses $\mathcal{S}$. Then, if Alice of type 1 chooses $\mathcal{B}$, her expected payoff is $\frac{1}{2}(2) + \frac{1}{2}(0) = 1$, and if she chooses $\mathcal{S}$, her expected payoff is $\frac{1}{2}(0) + \frac{1}{2}(1) = \frac{1}{2}$.

We represent the action of the two types of Bob by the pairs $(\mathcal{B}, \mathcal{B})$, $(\mathcal{B}, \mathcal{S})$, $(\mathcal{S}, \mathcal{B})$, $(\mathcal{S}, \mathcal{S})$, where the first entry in the bracket is the action of Bob of type 1 and the second entry is the action of Bob of type 2. The expected payoff to Alice of type 1 when she chooses $\mathcal{B}$ or $\mathcal{S}$, against the 4 pairs of actions of the two types of Bob are

$$\begin{array}{cccccc} & & \multicolumn{4}{c}{\text{Bob's two types}} \\ & & (\mathcal{B},\mathcal{B}) & (\mathcal{B},\mathcal{S}) & (\mathcal{S},\mathcal{B}) & (\mathcal{S},\mathcal{S}) \\ \text{Alice of type 1} & \mathcal{B} & 2 & 1 & 1 & 0 \\ & \mathcal{S} & 0 & \frac{1}{2} & \frac{1}{2} & 1 \end{array}, \qquad (4)$$

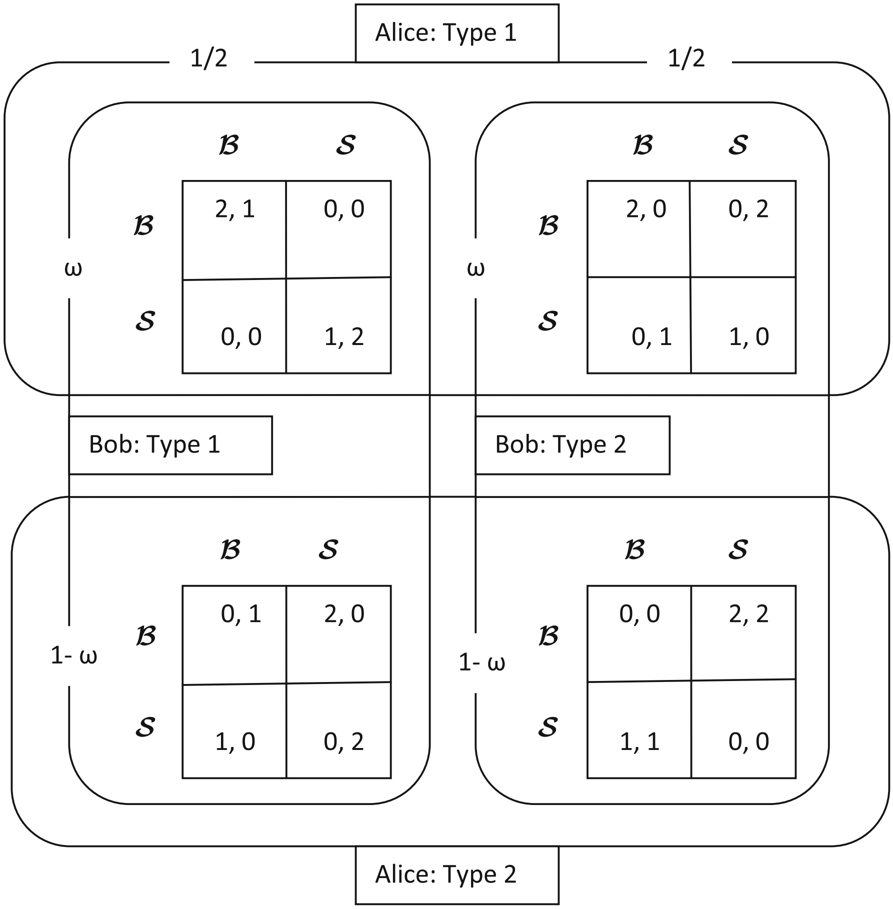

Fig. 1 Game of Battle of Sexes when each player is unsure of other player's preferences [62]

and similarly the payoff to Alice of type 2 is found as

$$\begin{array}{ll} & \qquad\qquad\qquad \text{Bob's two types} \\ & \begin{array}{cccc} (\mathcal{B},\mathcal{B}) & (\mathcal{B},\mathcal{S}) & (\mathcal{S},\mathcal{B}) & (\mathcal{S},\mathcal{S}) \end{array} \\ \text{Alice of type 2}\ \begin{array}{c} \mathcal{B} \\ \mathcal{S} \end{array} & \begin{array}{cccc} \quad 0 \quad & \quad 1 \quad & \quad 1 \quad & \quad 2 \quad \\ \quad 1 \quad & \quad \frac{1}{2} \quad & \quad \frac{1}{2} \quad & \quad 0 \quad \end{array} \end{array}. \tag{5}$$

Consider the case when Alice of type 1 plays $\mathcal{B}$, Alice of type 2 plays $\mathcal{S}$, Bob of type 1 plays $\mathcal{S}$, and Bob of type 2 plays $\mathcal{B}$. We represent this case by the quadruple $(\mathcal{B}, \mathcal{S}), (\mathcal{S}, \mathcal{B})$ where the entries in the first pair are chosen by Alice's two types, respectively, and the entries in the second pair are chosen by Bob's two types, respectively. For this quadruple, the payoffs to Alice's two types can be found from (4, 5) at the entries located at the intersection of the same column with entry $(\mathcal{S}, \mathcal{B})$ and the two row entries at $\mathcal{B}$ and $\mathcal{S}$ corresponding to Alice of type 1 and type 2, respectively:

$$\Pi_{A_1}\{(\mathcal{B},\mathcal{S}),(\mathcal{S},\mathcal{B})\}=1,\ \Pi_{A_2}\{(\mathcal{B},\mathcal{S}),(\mathcal{S},\mathcal{B})\}=\frac{1}{2}, \tag{6}$$

where the subscripts 1 and 2 for A or B give the type of that player.

The following table gives the expected payoffs to Alice's two types against the pairs of actions $(\mathcal{B},\mathcal{B})$, $(\mathcal{B},\mathcal{S})$, $(\mathcal{S},\mathcal{B})$, $(\mathcal{S},\mathcal{S})$ by Bob's two types, respectively,

$$\begin{array}{cccccc} & & & \text{Bob's two types} & & \\ & & (\mathcal{B},\mathcal{B}) & (\mathcal{B},\mathcal{S}) & (\mathcal{S},\mathcal{B}) & (\mathcal{S},\mathcal{S}) \\ & (\mathcal{B},\mathcal{B}) & (2,0) & (1,1) & (1,1) & (0,2) \\ \text{Alice's two types} & (\mathcal{B},\mathcal{S}) & (2,1) & \left(1,\frac{1}{2}\right) & \left(1,\frac{1}{2}\right) & (0,0) \\ & (\mathcal{S},\mathcal{B}) & (0,0) & \left(\frac{1}{2},1\right) & \left(\frac{1}{2},1\right) & (1,2) \\ & (\mathcal{S},\mathcal{S}) & (0,1) & \left(\frac{1}{2},\frac{1}{2}\right) & \left(\frac{1}{2},\frac{1}{2}\right) & (1,0) \end{array}, \tag{7}$$

where in the column on the left, for instance, $(\mathcal{B},\mathcal{S})$ means that Alice of type 1 chooses $\mathcal{B}$ and Alice of type 2 chooses $\mathcal{S}$. The two payoff entries in brackets are the expected payoffs to Alice of type 1 and type 2, respectively. In (7), consider the entry $\left(\frac{1}{2},1\right)$ at the intersection of 3rd row $(\mathcal{S},\mathcal{B})$ and 3rd column $(\mathcal{S},\mathcal{B})$. This means that Alice of the types 1 and 2 chooses $\mathcal{S}$ and $\mathcal{B}$, respectively, and Bob of the types 1 and 2 also chooses $\mathcal{S}$ and $\mathcal{B}$, respectively. The payoff to Alice of type 1 and type 2 are $\frac{1}{2}$ and 1, respectively.

Similarly, the payoffs to Bob's two types can be found as

$$\Pi_{B_1}\{(\mathcal{B},\mathcal{S}),(\mathcal{S},\mathcal{B})\}=\omega(0)+(1-\omega)(2)=2(1-\omega), \tag{8}$$
$$\Pi_{B_2}\{(\mathcal{B},\mathcal{S}),(\mathcal{S},\mathcal{B})\}=\omega(0)+(1-\omega)(1)=1-\omega, \tag{9}$$

and one can write the expected payoffs to Bob's first type as

$$\begin{array}{cccccc} & & & \text{Alice's two types} & & \\ & & (\mathcal{B},\mathcal{B}) & (\mathcal{B},\mathcal{S}) & (\mathcal{S},\mathcal{B}) & (\mathcal{S},\mathcal{S}) \\ \text{Bob's type 1} & \mathcal{B} & 1 & \omega & 1-\omega & 0 \\ & \mathcal{S} & 0 & 2(1-\omega) & 2\omega & 2 \end{array}, \tag{10}$$

and, similarly, the expected payoffs to Bob's second type are obtained as

$$\begin{array}{cccccc} & & & \text{Alice's two types} & & \\ & & (\mathcal{B},\mathcal{B}) & (\mathcal{B},\mathcal{S}) & (\mathcal{S},\mathcal{B}) & (\mathcal{S},\mathcal{S}) \\ \text{Bob's type 2} & \mathcal{B} & 0 & 1-\omega & \omega & 1 \\ & \mathcal{S} & 2 & 2\omega & 2(1-\omega) & 0 \end{array}. \tag{11}$$

As it is the case for payoffs to Alice's two types above, (10) and (11) can be joined together to obtain the payoffs to Bob's two types as

$$
\begin{array}{cc|cccc}
 & & \multicolumn{4}{c}{\text{Bob's two types}} \\
 & & (\mathcal{B},\mathcal{B}) & (\mathcal{B},\mathcal{S}) & (\mathcal{S},\mathcal{B}) & (\mathcal{S},\mathcal{S}) \\
 & (\mathcal{B},\mathcal{B}) & (1,0) & (1,2) & (0,0) & (0,2) \\
\text{Alice's two types} & (\mathcal{B},\mathcal{S}) & (\omega,1-\omega) & (\omega,2\omega) & (2(1-\omega),(1-\omega)) & (2(1-\omega),2\omega) \\
 & (\mathcal{S},\mathcal{B}) & ((1-\omega),\omega) & ((1-\omega),2(1-\omega)) & (2\omega,\omega) & (2\omega,2(1-\omega)) \\
 & (\mathcal{S},\mathcal{S}) & (0,1) & (0,0) & (2,1) & (2,0)
\end{array}, \tag{12}
$$

where the entries in brackets are the expected payoffs to Bob of type 1 and type 2, respectively. Now, (7) and (12) can be joined together to obtain

$$
\begin{array}{cc|cccc}
 & & \multicolumn{4}{c}{\text{Bob}} \\
 & & (\mathcal{B},\mathcal{B}) & (\mathcal{B},\mathcal{S}) & (\mathcal{S},\mathcal{B}) & (\mathcal{S},\mathcal{S}) \\
 & (\mathcal{B},\mathcal{B}) & (2,0),(1,0) & (1,1),(1,2) & (1,1),(0,0) & (0,2),(0,2) \\
\text{Alice} & (\mathcal{B},\mathcal{S}) & (2,1),(\omega,1-\omega) & (1,\frac{1}{2}),(\omega,2\omega) & (1,\frac{1}{2}),(2(1-\omega),(1-\omega)) & (0,0),(2(1-\omega),2\omega) \\
 & (\mathcal{S},\mathcal{B}) & (0,0),((1-\omega),\omega) & (\frac{1}{2},1),((1-\omega),2(1-\omega)) & (\frac{1}{2},1),(2\omega,\omega) & (1,2),(2\omega,2(1-\omega)) \\
 & (\mathcal{S},\mathcal{S}) & (0,1),(0,1) & (\frac{1}{2},\frac{1}{2}),(0,0) & (\frac{1}{2},\frac{1}{2}),(2,1) & (1,0),(2,0)
\end{array} \tag{13}
$$

where, for the two pairs of payoff entries, the first pair is for Alice's two types and the second payoff pair is for Bob's two types. It can be seen that when $\omega = \frac{2}{3}$, for instance, the strategy quadruples $\{(\mathcal{S},\mathcal{B}),(\mathcal{S},\mathcal{S})\}$ and $\{(\mathcal{B},\mathcal{B}),(\mathcal{B},\mathcal{S})\}$ corresponding to the payoff quadruples $(1,2)$, $\left(\frac{4}{3},\frac{2}{3}\right)$, and $(1,1)$, $(1,2)$, respectively, are the pure strategy Bayesian Nash equilibria [61,62].

3.1 Mixed-strategy version

Now, consider the mixed-strategy version of the game in which the players' probabilities of selecting $\mathcal{B}$ from the pure strategies $\mathcal{B}$ and $\mathcal{S}$ are given by the numbers p, q, p', and $q' \in [0,1]$ for Alice of type 1, Alice of type 2, Bob of type 1, and for Bob of type 2, respectively. The mixed-strategy payoffs for Alice's and Bob's two types can then be found from Fig. 1 as

$$
\begin{aligned}
\Pi_{A_1}(p;p',q') &= \frac{1}{2}\begin{pmatrix} p \\ 1-p \end{pmatrix}^T \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} p' \\ 1-p' \end{pmatrix} \\
&\quad + \frac{1}{2}\begin{pmatrix} p \\ 1-p \end{pmatrix}^T \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} q' \\ 1-q' \end{pmatrix}, \\
\Pi_{A_2}(q;p',q') &= \frac{1}{2}\begin{pmatrix} q \\ 1-q \end{pmatrix}^T \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} p' \\ 1-p' \end{pmatrix} \\
&\quad + \frac{1}{2}\begin{pmatrix} q \\ 1-q \end{pmatrix}^T \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} q' \\ 1-q' \end{pmatrix},
\end{aligned}
$$

$$\Pi_{B_1}(p'; p, q) = \omega \begin{pmatrix} p \\ 1-p \end{pmatrix}^T \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} p' \\ 1-p' \end{pmatrix} + (1-\omega) \begin{pmatrix} q \\ 1-q \end{pmatrix}^T \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} p' \\ 1-p' \end{pmatrix},$$

$$\Pi_{B_2}(q'; p, q) = \omega \begin{pmatrix} p \\ 1-p \end{pmatrix}^T \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q' \\ 1-q' \end{pmatrix} + (1-\omega) \begin{pmatrix} q \\ 1-q \end{pmatrix}^T \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q' \\ 1-q' \end{pmatrix}, \tag{14}$$

where T is for transpose, and the subscripts 1 and 2 under A and B refer to the respective player's type. Also, a semicolon is used to separate Alice's and Bob's variables. The following Nash inequalities are then obtained

$$\begin{aligned}
\Pi_{A_1}(p^*; p'^*, q'^*) - \Pi_{A_1}(p, p'^*, q'^*) &= \frac{\partial \Pi_{A_1}}{\partial p} \mid_* (p^* - p) \\
&= \left\{ \frac{3}{2}(p'^* + q'^*) - 1 \right\} (p^* - p) \geq 0, \\
\Pi_{A_2}(q^*; p'^*, q'^*) - \Pi_{A_2}(q, p'^*, q'^*) &= \frac{\partial \Pi_{A_2}}{\partial q} \mid_* (q^* - q) \\
&= 2 \left\{ 1 - (p'^* + q'^*) \right\} (q^* - p) \geq 0, \\
\Pi_{B_1}(p'^*; p^*, q^*) - \Pi_{B_1}(p'; p^*, q^*) &= \frac{\partial \Pi_{B_1}}{\partial p'} \mid_* (p'^* - p') \\
&= \left\{ 3\omega p^* + 3(1-\omega) q^* - 2 \right\} (p'^* - p') \geq 0, \\
\Pi_{B_2}(q'^*; p^*, q^*) - \Pi_{B_2}(q'; p^*, q^*) &= \frac{\partial \Pi_{B_2}}{\partial q'} \mid_* (q'^* - q') \\
&= - \left\{ 3\omega p^* + 3(1-\omega) q^* \right. \\
&\quad \times (q'^* - q') \geq 0,
\end{aligned} \tag{15}$$

where the quadruple $\left\{(p^*, q^*), (p'^*, q'^*)\right\}$ is the Nash equilibrium strategy set, which is indicated by the use of the asterisk label. From the inequalities (15), at $\omega = \frac{2}{3}$, for instance, the pure Bayesian Nash equilibria quadruples can be identified as $\{(0, 1), (0, 0)\}$ and $\{(1, 1), (1, 0)\}$, which correspond to the strategy quadruples $\{(\mathcal{S}, \mathcal{B}), (\mathcal{S}, \mathcal{S})\}$ and $\{(\mathcal{B}, \mathcal{B}), (\mathcal{B}, \mathcal{S})\}$ as can be observed above. Also, it is observed that $\left\{(\frac{1}{2}, 1), (\frac{2}{3}, 0)\right\}$, for instance, is a mixed-strategy Bayesian Nash equilibrium at which the players' payoffs are obtained from Eq. (14) as

$$\Pi_{A_1}\left(\frac{1}{2}; \frac{2}{3}, 0\right) = \frac{2}{3}, \quad \Pi_{A_2}\left(1; \frac{2}{3}, 0\right) = \frac{4}{3},$$
$$\Pi_{B_1}\left(\frac{2}{3}; \frac{1}{2}, 1\right) = \frac{2}{3}, \quad \Pi_{B_2}\left(0; \frac{1}{2}, 1\right) = 2 - \omega. \tag{16}$$

4 Quantum probabilities in generalized Einstein–Podolsky–Rosen experiments

The above analysis of mixed-strategy Bayesian Nash equilibria assumes the underlying probabilities to be factorizable. We now would like to know whether the outcome of the Bayesian game from Fig. 1 is affected when the probabilities become non-factorizable. For this, we would consider the set of non-factorizable quantum probabilities obtained from generalized EPR experiments. The standard setting of such experiments [58,63–71] involves a large number of runs. Two halves of an EPR pair originate from the same source travelling in opposite directions. One-half is received by observer 1, whereas observer 2 receives the other half. The two observers are space like separated and are unable to communicate.

As Fig. 2 shows, the two directions refer to two possible directions along which measurements can be taken, and in a run, the spin or polarization of the received half is measured. We call D_1 and D_2 observer 1's two directions and D_1' and D_2' observer 2's two directions. In a run, each observer selects one direction and thus a directional pair from (D_1, D_2'), (D_1, D_1'), (D_2, D_1'), (D_2, D_2') is selected by the observers in that run. The Stern–Gerlach type detectors are now rotated along these selected directions to perform the quantum measurement. Independent of which directional pair is chosen by the two observers, the outcome of the quantum measurement is either $+1$ or -1 along a measurement direction.

The relevant 16 probabilities are given [71] in the following,

$$\begin{array}{ccc} & & \text{Observer 2} \\ & & \begin{array}{cc} \;\;D_1' & \;\;\;\;D_2' \end{array} \\ & & \begin{array}{cccc} +1 & -1 & +1 & -1 \end{array} \\ \text{Observer 1} & \begin{array}{c} D_1 \begin{array}{c} +1 \\ -1 \end{array} \\ D_2 \begin{array}{c} +1 \\ -1 \end{array} \end{array} & \left(\begin{array}{cc|cc} \varepsilon_1 & \varepsilon_2 & \varepsilon_5 & \varepsilon_6 \\ \varepsilon_3 & \varepsilon_4 & \varepsilon_7 & \varepsilon_8 \\ \hline \varepsilon_9 & \varepsilon_{10} & \varepsilon_{13} & \varepsilon_{14} \\ \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{15} & \varepsilon_{16} \end{array} \right), \end{array} \tag{17}$$

where, for instance, when the observer 1 selects the direction D_2 and observer 2 selects the direction D_1', and the Stern–Gerlach detectors are rotated along these directions, the probability that both experimental outcomes are -1 is ε_{12}, and the probability that the observer 1's experimental outcome is $+1$ and observer 2's experimental outcome is -1 is given by ε_{10}. "Appendix" described how the probabilities ε_j are obtained from a pure state of two qubits.

We note that there are 16 probabilities in the above setting of generalized EPR experiments, and that the Fig. 1 giving a normal form representation of a Bayesian game of incomplete information also has the same number of entries in it. This naturally leads us to consider the situation in which the EPR probabilities are taken as the underlying probabilities of the strategy pairs in the Bayesian game in Fig. 1.

Now, the EPR probabilities can become non-factorizable. As we see in the following, this consideration motivates us to investigate how the outcome of the Bayesian

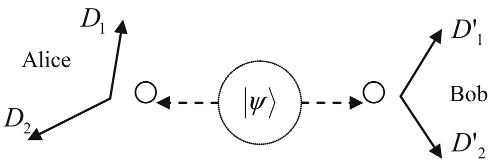

Fig. 2 Setting for generalized Einstein–Podolsky–Rosen experiments. We associate Alice's two directions to the two types of Alice, i.e., D_1 ∼Alice's type 1 and D_2 ∼Alice's type 2. Similarly, D_1' ∼Bob's type 1 and D_2' ∼Bob's type 2

game is affected when the underlying probabilities of this game are obtained from EPR experiments and can thus become non-factorizable.

4.1 Constraints on EPR probabilities and defining the payoff relations

The elements of the probability set ε_j $(1 \leq j \leq 16)$ are known to satisfy certain other constraints that are described as follows. Note that when the directional pair (D_1, D_2') is chosen for all runs of the experiment, the only possible outcomes are $(+1, +1)$, $(+1, -1)$, $(-1, +1)$, $(-1, -1)$. The same is true for other directional pairs (D_1, D_1'), (D_2, D_1'), (D_2, D_2'). This leads to the normalization constraint

$$\sum_{j=1}^{4} \varepsilon_j = 1 = \sum_{j=5}^{8} \varepsilon_j, \quad \sum_{j=9}^{12} \varepsilon_j = 1 = \sum_{j=13}^{16} \varepsilon_j. \tag{18}$$

Also, in a particular run of the EPR experiment, the outcome of $+1$ or -1 (obtained along the direction D_1 or direction D_2) is independent of whether the direction D_1' or the direction D_2' is chosen in that run. Similarly, the outcome of $+1$ or -1 (obtained along D_1' or D_2') is independent of whether the direction D_1 or the direction D_2 is chosen in that run. These requirements, when translated in terms of the probability set ε_j, are expressed as

$$\begin{aligned} &\varepsilon_1+\varepsilon_2 = \varepsilon_5+\varepsilon_6, \quad \varepsilon_1+\varepsilon_3 = \varepsilon_9+\varepsilon_{11}, \quad \varepsilon_9+\varepsilon_{10} = \varepsilon_{13}+\varepsilon_{14}, \ \varepsilon_5+\varepsilon_7 = \varepsilon_{13}+\varepsilon_{15}, \\ &\varepsilon_3+\varepsilon_4 = \varepsilon_7+\varepsilon_8, \ \varepsilon_{11}+\varepsilon_{12} = \varepsilon_{15}+\varepsilon_{16}, \ \varepsilon_2+\varepsilon_4 = \varepsilon_{10}+\varepsilon_{12}, \ \varepsilon_6+\varepsilon_8 = \varepsilon_{14}+\varepsilon_{16}. \end{aligned} \tag{19}$$

A convenient solution of the system (18, 19) is reported by Cereceda [71] to be the one for which the set of probabilities $\upsilon = \{\varepsilon_2, \varepsilon_3, \varepsilon_6, \varepsilon_7, \varepsilon_{10}, \varepsilon_{11}, \varepsilon_{13}, \varepsilon_{16}\}$ is expressed in terms of the remaining set of probabilities $\mu = \{\varepsilon_1, \varepsilon_4, \varepsilon_5, \varepsilon_8, \varepsilon_9, \varepsilon_{12}, \varepsilon_{14}, \varepsilon_{15}\}$ that is given as

$$
\begin{aligned}
\varepsilon_2 &= (1-\varepsilon_1-\varepsilon_4+\varepsilon_5-\varepsilon_8-\varepsilon_9+\varepsilon_{12}+\varepsilon_{14}-\varepsilon_{15})/2,\\
\varepsilon_3 &= (1-\varepsilon_1-\varepsilon_4-\varepsilon_5+\varepsilon_8+\varepsilon_9-\varepsilon_{12}-\varepsilon_{14}+\varepsilon_{15})/2,\\
\varepsilon_6 &= (1+\varepsilon_1-\varepsilon_4-\varepsilon_5-\varepsilon_8-\varepsilon_9+\varepsilon_{12}+\varepsilon_{14}-\varepsilon_{15})/2,\\
\varepsilon_7 &= (1-\varepsilon_1+\varepsilon_4-\varepsilon_5-\varepsilon_8+\varepsilon_9-\varepsilon_{12}-\varepsilon_{14}+\varepsilon_{15})/2,\\
\varepsilon_{10} &= (1-\varepsilon_1+\varepsilon_4+\varepsilon_5-\varepsilon_8-\varepsilon_9-\varepsilon_{12}+\varepsilon_{14}-\varepsilon_{15})/2,\\
\varepsilon_{11} &= (1+\varepsilon_1-\varepsilon_4-\varepsilon_5+\varepsilon_8-\varepsilon_9-\varepsilon_{12}-\varepsilon_{14}+\varepsilon_{15})/2,\\
\varepsilon_{13} &= (1-\varepsilon_1+\varepsilon_4+\varepsilon_5-\varepsilon_8+\varepsilon_9-\varepsilon_{12}-\varepsilon_{14}-\varepsilon_{15})/2,\\
\varepsilon_{16} &= (1+\varepsilon_1-\varepsilon_4-\varepsilon_5+\varepsilon_8-\varepsilon_9+\varepsilon_{12}-\varepsilon_{14}-\varepsilon_{15})/2.
\end{aligned} \tag{20}
$$

This allows us to consider the elements of the set μ as independent variables.

In order to use the EPR setting to play the Bayesian game in Fig. 1, we call the observers 1 and 2 the players Alice and Bob, respectively, of the Bayesian game. We then associate one-half of the EPR pair to the player Alice and the other half to the player Bob. As Alice and Bob have two directions each, we associate Alice's two directions to the two types of Alice, that is, D_1 ∼Alice's type 1 and D_2 ∼Alice's type 2. Similarly, we associate Bob's two directions to the two types of Bob, that is, D_1' ∼Bob's type 1 and D_2' ∼Bob's type 2.

With these associations, and in view of the game in Fig. 1, in a run, each of the two directions D_1 and D_2 is chosen with probability $\frac{1}{2}$. Similarly, the directions D_1' and D_2' are chosen with probabilities ω and $(1-\omega)$, respectively. In view of the payoff relations (14) in the factorizable game, the players' payoff relations in the game with EPR probabilities can now be expressed as

$$
\begin{aligned}
\Pi_{\mathrm{A}_1}(\varepsilon_j) &= \frac{1}{2}\{(2)\varepsilon_1+(0)\varepsilon_2+(0)\varepsilon_3+(1)\varepsilon_4\}\\
&\quad+\frac{1}{2}\{(2)\varepsilon_5+(0)\varepsilon_6+(0)\varepsilon_7+(1)\varepsilon_8\},\\
\Pi_{\mathrm{A}_2}(\varepsilon_j) &= \frac{1}{2}\{(0)\varepsilon_9+(2)\varepsilon_{10}+(1)\varepsilon_{11}+(0)\varepsilon_{12}\}\\
&\quad+\frac{1}{2}\{(0)\varepsilon_{13}+(2)\varepsilon_{14}+(1)\varepsilon_{15}+(0)\varepsilon_{16}\},\\
\Pi_{\mathrm{B}_1}(\varepsilon_j) &= \omega\{(1)\varepsilon_1+(0)\varepsilon_2+(0)\varepsilon_3+(2)\varepsilon_4\}\\
&\quad+(1-\omega)\{(1)\varepsilon_9+(0)\varepsilon_{10}+(0)\varepsilon_{11}+(2)\varepsilon_{12}\},\\
\Pi_{\mathrm{B}_2}(\varepsilon_j) &= \omega\{(0)\varepsilon_5+(2)\varepsilon_6+(1)\varepsilon_7+(0)\varepsilon_8\}\\
&\quad+(1-\omega)\{(0)\varepsilon_{13}+(2)\varepsilon_{14}+(1)\varepsilon_{15}+(0)\varepsilon_{16}\},
\end{aligned} \tag{21}
$$

where $1\le j\le 16$ and $\Pi_{\mathrm{A}_1}(\varepsilon_j)$, $\Pi_{\mathrm{A}_2}(\varepsilon_j)$, $\Pi_{\mathrm{B}_1}(\varepsilon_j)$, and $\Pi_{\mathrm{B}_2}(\varepsilon_j)$ are the payoffs to Alice of type 1, Alice of type 2, Bob of type 1, and Bob of type 2, respectively, expressed in terms of the probabilities ε_j that are defined in (17).

Note that the payoffs (21) are reduced to the payoffs in the mixed-strategy game (14) when the probability set ε_j $(1\le j\le 16)$ is factorizable in terms of the probabilities $p, q, p', q' \in [0, 1]$ as given by Eq. (22).

5 Nash equilibrium inequalities in Bayesian game of Battle of Sexes with EPR probabilities

So as to find the Nash equilibria when the probability set ε_j $(1 \leq j \leq 16)$ are non-factorizable, we notice that when ε_j are factorizable in terms of the probabilities p, q, p', and q', we can write

$$\begin{aligned} \varepsilon_1 &= pp',\ \varepsilon_2 = p(1-p'),\ \varepsilon_3 = (1-p)p',\ \varepsilon_4 = (1-p)(1-p'), \\ \varepsilon_5 &= pq',\ \varepsilon_6 = p(1-q'),\ \varepsilon_7 = (1-p)q',\ \varepsilon_8 = (1-p)(1-q'), \\ \varepsilon_9 &= qp',\ \varepsilon_{10} = q(1-p'),\ \varepsilon_{11} = (1-q)p',\ \varepsilon_{12} = (1-q)(1-p'), \\ \varepsilon_{13} &= qq',\ \varepsilon_{14} = q(1-q'),\ \varepsilon_{15} = (1-q)q',\ \varepsilon_{16} = (1-q)(1-q'). \end{aligned} \tag{22}$$

With this, the payoff relations (21) are reduced to the payoffs given by (14). Also, when ε_j are factorizable, and can be expressed in terms of p, q, p', and q', they satisfy the constraints given by Eqs. (18, 19) and p, q, p', and q' can then be expressed in terms of the probabilities ε_j as follows

$$\begin{aligned} p &= \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_5 + \varepsilon_6),\ q = \frac{1}{2}(\varepsilon_9 + \varepsilon_{10} + \varepsilon_{13} + \varepsilon_{14}), \\ p' &= \frac{1}{2}(\varepsilon_1 + \varepsilon_3 + \varepsilon_9 + \varepsilon_{11}),\ q' = \frac{1}{2}(\varepsilon_5 + \varepsilon_7 + \varepsilon_{13} + \varepsilon_{15}). \end{aligned} \tag{23}$$

From the set of inequalities (15), the expressions describing the Nash equilibria in the factorizable game are

$$\begin{aligned} &\frac{\partial \Pi_{A_1}}{\partial p} \mid_* (p^* - p) \geq 0,\ \frac{\partial \Pi_{A_2}}{\partial q} \mid_* (q^* - q) \geq 0, \\ &\frac{\partial \Pi_{B_1}}{\partial p'} \mid_* (p'^* - p') \geq 0,\ \frac{\partial \Pi_{B_2}}{\partial q'} \mid_* (q'^* - q') \geq 0. \end{aligned} \tag{24}$$

To evaluate these inequalities, we use the relations (20) to express the payoff relations (21) in terms of the elements from the set μ as follows

$$\begin{aligned} \Pi_{A_1}(\varepsilon_j) &= \frac{1}{2}(2\varepsilon_1 + \varepsilon_4 + 2\varepsilon_5 + \varepsilon_8), \\ \Pi_{A_2}(\varepsilon_j) &= \frac{1}{4}(3 - \varepsilon_1 + \varepsilon_4 + \varepsilon_5 - \varepsilon_8 - 3\varepsilon_9 - 3\varepsilon_{12} + 5\varepsilon_{14} + \varepsilon_{15}), \\ \Pi_{B_1}(\varepsilon_j) &= \omega(\varepsilon_1 + 2\varepsilon_4 - \varepsilon_9 - 2\varepsilon_{12}) + (\varepsilon_9 + 2\varepsilon_{12}) \\ \Pi_{B_2}(\varepsilon_j) &= \frac{\omega}{2}(3 + \varepsilon_1 - \varepsilon_4 - 3\varepsilon_5 - 3\varepsilon_8 - \varepsilon_9 + \varepsilon_{12} - 3\varepsilon_{14} - 3\varepsilon_{15}) + (2\varepsilon_{14} + \varepsilon_{15}). \end{aligned} \tag{25}$$

Similarly, using the relations (20), we express p, q, p', q' in (23) in terms of the elements from the set μ as

$$
\begin{aligned}
p &= \frac{1}{2}(1 + \varepsilon_1 - \varepsilon_4 + \varepsilon_5 - \varepsilon_8 - \varepsilon_9 + \varepsilon_{12} + \varepsilon_{14} - \varepsilon_{15}),\\
q &= \frac{1}{2}(1 - \varepsilon_1 + \varepsilon_4 + \varepsilon_5 - \varepsilon_8 + \varepsilon_9 - \varepsilon_{12} + \varepsilon_{14} - \varepsilon_{15}),\\
p' &= \frac{1}{2}(1 + \varepsilon_1 - \varepsilon_4 - \varepsilon_5 + \varepsilon_8 + \varepsilon_9 - \varepsilon_{12} - \varepsilon_{14} + \varepsilon_{15}),\\
q' &= \frac{1}{2}(1 - \varepsilon_1 + \varepsilon_4 + \varepsilon_5 - \varepsilon_8 + \varepsilon_9 - \varepsilon_{12} - \varepsilon_{14} + \varepsilon_{15}).
\end{aligned} \tag{26}
$$

This allows us to use the chain rule to evaluate

$$
\begin{aligned}
\frac{\partial \Pi_{A_1}(\varepsilon_j)}{\partial p} &= \frac{\partial \Pi_{A_1}(\varepsilon_j)}{\partial \varepsilon_1}\frac{\partial \varepsilon_1}{\partial p} + \frac{\partial \Pi_{A_1}(\varepsilon_j)}{\partial \varepsilon_4}\frac{\partial \varepsilon_4}{\partial p} + \frac{\partial \Pi_{A_1}(\varepsilon_j)}{\partial \varepsilon_5}\frac{\partial \varepsilon_5}{\partial p} + \frac{\partial \Pi_{A_1}(\varepsilon_j)}{\partial \varepsilon_8}\frac{\partial \varepsilon_8}{\partial p}\\
&\quad + \frac{\partial \Pi_{A_1}(\varepsilon_j)}{\partial \varepsilon_9}\frac{\partial \varepsilon_9}{\partial p} + \frac{\partial \Pi_{A_1}(\varepsilon_j)}{\partial \varepsilon_{12}}\frac{\partial \varepsilon_{12}}{\partial p} + \frac{\partial \Pi_{A_1}(\varepsilon_j)}{\partial \varepsilon_{14}}\frac{\partial \varepsilon_{14}}{\partial p}\\
&\quad + \frac{\partial \Pi_{A_1}(\varepsilon_j)}{\partial \varepsilon_{15}}\frac{\partial \varepsilon_{15}}{\partial p},
\end{aligned} \tag{27}
$$

that gives $\frac{\partial \Pi_{A_1}(\varepsilon_j)}{\partial p} = 2$. Similarly, we obtain

$$
\frac{\partial \Pi_{A_2}(\varepsilon_j)}{\partial q} = 4, \ \frac{\partial \Pi_{B_1}(\varepsilon_j)}{\partial p'} = -2, \ \frac{\partial \Pi_{B_2}(\varepsilon_j)}{\partial q'} = -2(2\omega + 1). \tag{28}
$$

The inequalities (24) are now written as

$$
\begin{aligned}
&2(p^* - p) \geq 0, \ 4(q^* - q) \geq 0,\\
&-2(p'^* - p') \geq 0, \ -2(2\omega + 1)(q'^* - q') \geq 0,
\end{aligned} \tag{29}
$$

giving

$$
\begin{aligned}
p^* &= \frac{1}{2}(\varepsilon_1^* + \varepsilon_2^* + \varepsilon_5^* + \varepsilon_6^*) = 1, \ q^* = \frac{1}{2}(\varepsilon_9^* + \varepsilon_{10}^* + \varepsilon_{13}^* + \varepsilon_{14}^*) = 1,\\
p'^* &= \frac{1}{2}(\varepsilon_1^* + \varepsilon_3^* + \varepsilon_9^* + \varepsilon_{11}^*) = 0, \ q'^* = \frac{1}{2}(\varepsilon_5^* + \varepsilon_7^* + \varepsilon_{13}^* + \varepsilon_{15}^*) = 0,
\end{aligned} \tag{30}
$$

as a unique Bayesian Nash equilibrium of the game. Using Eqs. (23, 26), we then obtain $\varepsilon_2^* = \varepsilon_6^* = \varepsilon_{10}^* = \varepsilon_{14}^* = 1$ with all the rest of ε_j^* being zeros. The players' payoffs at this equilibrium are obtained as

$$
\Pi_{A_1}(\varepsilon_j^*) = 0, \ \Pi_{A_2}(\varepsilon_j^*) = 2, \ \Pi_{B_1}(\varepsilon_j^*) = 0, \ \Pi_{B_2}(\varepsilon_j^*) = 2, \tag{31}
$$

presenting a dramatic contrast to what is the case when the underlying probabilities are factorizable.

6 Discussion

As is well known, Bell's inequality presents stark difference between the classical and quantum world. The CHSH form of Bell's inequality can be expressed as a constraint on probabilities without referring to the formalism and mathematical machinery of quantum mechanics. Game theory is based on the theory of probability, and this suggests that the quantum contents of quantum games can be given an expression that only employ probabilities without referring to the formalism of quantum mechanics. A probabilistic approach to quantum games is only a matter of perspective as the quantum Bayesian game discussed in this paper can be physically implemented using EPR-type experiments that result in non-factorizable probabilities.

Essentially, this paper discusses a physical implementation of a quantum Bayesian game using EPR experiments. We find that these experiments provide a natural setting for analyzing a quantum Bayesian game. Our analysis uses only probabilities as they facilitate wider access to the area of quantum games. The physical realization of our game is provided by actual EPR experiments where quantum mechanics makes the difference and which can also be expressed in terms of probabilities only.

We study the game of Battle of Sexes with incomplete information that has both pure and mixed Bayesian Nash equilibria. We investigate the situation when the underlying probabilities of this game can become non-factorizable. As is known, the probabilities in generalized EPR experiments can become non-factorizable and in certain cases can maximally violate the corresponding CHSH version of Bell's inequality. When the quantum mechanical probabilities are factorizable, the game attains a classical interpretation. However, when the probabilities are allowed to become non-factorizable, we find that the solution of the game turns out to be entirely different and, in contrast to the classical game that has both pure and mixed Bayesian Nash equilibria, the quantum game has a unique Bayesian Nash equilibrium.

A natural question to ask here is whether any set of probabilities that satisfies the constraints (18,19) is physically realizable? Quantum mechanics is known to impose further constraints, and one such constraint is given by the CHSH version of Bell's inequality [58,70,71]. This constraint states that the quantity Δ defined by

$$\Delta = 2(\varepsilon_1 + \varepsilon_4 + \varepsilon_5 + \varepsilon_8 + \varepsilon_9 + \varepsilon_{12} + \varepsilon_{14} + \varepsilon_{15} - 2) \tag{32}$$

is restricted in the range $\left|\Delta_{QM}\right| \leq 2\sqrt{2}$ by the laws of quantum mechanics, and the CHSH version of Bell's inequality is violated in quantum mechanical experiments when $2 < |\Delta|$. These constraints are imposed by physical realizations, and a maximum value of $\Delta = 4$ emerges when only nonnegative probabilities are considered. It is known that this value is not physically realizable.

Our analysis for a Bayesian game of Battle of Sexes shows that when the underlying probabilities are obtained from generalized EPR experiments, and thus can be non-factorizable, a unique Bayesian Nash equilibrium of the game emerges. This equilibrium corresponds to $\varepsilon_2^* = \varepsilon_6^* = \varepsilon_{10}^* = \varepsilon_{14}^* = 1$ with the remaining ε_j^* being zeros. Substituting these values in Eq. (32) gives $\Delta = -2$, i.e., the unique Bayesian Nash equilibrium is obtained without violating the CHSH version of Bell's inequality.

A particularly interesting situation would be when the violation of CHSH version of Bell's inequality leads to a new outcome of the game. Quantum mechanics is known to put its own constraints on the allowed ranges of the variables $(\varepsilon_1 + \varepsilon_2 + \varepsilon_5 + \varepsilon_6)$, $(\varepsilon_9 + \varepsilon_{10} + \varepsilon_{13} + \varepsilon_{14})$, $(\varepsilon_1 + \varepsilon_3 + \varepsilon_9 + \varepsilon_{11})$, and $(\varepsilon_5 + \varepsilon_7 + \varepsilon_{13} + \varepsilon_{15})$, as described in Eq. (23), and they are different from those imposed by just not permitting probabilities to have negative values. An important question would then be to ask whether this would change or affect the outcome of the game. A consideration of Eqs. (29, 30) shows that the conditions yielding the Bayesian Nash equilibrium are simply too strong to be affected by extra constraints that quantum mechanics can impose on the allowed ranges of the variables $(\varepsilon_1 + \varepsilon_2 + \varepsilon_5 + \varepsilon_6)$, $(\varepsilon_9 + \varepsilon_{10} + \varepsilon_{13} + \varepsilon_{14})$, $(\varepsilon_1 + \varepsilon_3 + \varepsilon_9 + \varepsilon_{11})$, and $(\varepsilon_5 + \varepsilon_7 + \varepsilon_{13} + \varepsilon_{15})$. However, it is possible that for other Bayesian games, the conditions giving the outcome are not so strong and it then would be worthwhile to investigate this question further.

Acknowledgments We acknowledge helpful discussions with Andrew Allison.

7 Appendix

Using the standard notation, the set of 16 probabilities in (17) is obtained from a pure state of two qubits

$$|\psi_0\rangle = \alpha\,|00\rangle + \beta\,|01\rangle + \gamma\,|10\rangle + \delta\,|11\rangle \tag{33}$$

where $\alpha, \beta, \gamma, \delta \in \boldsymbol{C}$ and $|\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2 = 1$. We assume that observer 1's directions D_1 and D_2 are along the unit vectors $\hat{a}$ and $\hat{c}$, respectively. Similarly, observer 2's directions D_1' and D_2' are along the unit vectors $\hat{b}$ and $\hat{d}$, respectively. Without the loss of generality, we also assume that the unit vectors $\hat{a} = [a_x, a_y]$, $\hat{b} = [b_x, b_y]$, $\hat{c} = [c_x, c_y]$, and $\hat{d} = [d_x, d_y]$ are all located in the x–y plane. Observer 1's measurement operators are then $\hat{\sigma} \cdot \hat{a}$ and $\hat{\sigma} \cdot \hat{c}$, respectively. Similarly, observer 2's measurement operators are $\hat{\sigma} \cdot \hat{b}$ and $\hat{\sigma} \cdot \hat{d}$, respectively. Here, $\hat{\sigma} = [\sigma_x, \sigma_y, \sigma_z]$ and $\sigma_x, \sigma_y, \sigma_z$ are Pauli spin matrices.

Consider the probability ε_1 in the (17) that corresponds to observers 1 and 2 measuring along the directions $\hat{a}$ and $\hat{b}$, respectively, and both obtaining the outcome $+1$. For this situation, we require the eigenstates of the operators $(\hat{\sigma} \cdot \hat{a})$ and $(\hat{\sigma} \cdot \hat{b})$, with the eigenvalue of $+1$ for both. These are found to be $\frac{|0\rangle + (a_x + ia_y)|1\rangle}{\sqrt{2}}$ and $\frac{|0\rangle + (b_x + ib_y)|1\rangle}{\sqrt{2}}$, respectively. From these, the eigenstate of the measurement operator $(\hat{\sigma} \cdot \hat{a}) \otimes (\hat{\sigma} \cdot \hat{b})$, with the eigenvalue $+1$, is obtained as

$$|\psi_1\rangle = \frac{1}{2}(|00\rangle + (b_x + ib_y)\,|01\rangle + (a_x + ia_y)\,|10\rangle + (a_x + ia_y)(b_x + ib_y)\,|11\rangle), \tag{34}$$

and the probability ε_1 is then obtained from $|\langle\psi_1|\ \psi_0\rangle|^2$. For the pure state (33), this becomes

$$\varepsilon_1 = \frac{1}{4}\left|\alpha + \beta(b_x - ib_y) + \gamma(a_x - ia_y) + \delta(a_x - ia_y)(b_x - ib_y)\right|^2. \qquad (35)$$

Similarly, for the probability ε_2, along with the eigenstate of $(\hat{\sigma} \cdot \hat{a})$ with eigenvalue $+1$ obtained above, we require the eigenstate of the operator $(\hat{\sigma} \cdot \hat{b})$ with the eigenvalues -1, which is $\frac{|0\rangle - (b_x + ib_y)|1\rangle}{\sqrt{2}}$. From these, the eigenstate of the measurement operator $(\hat{\sigma} \cdot \hat{a}) \otimes (\hat{\sigma} \cdot \hat{b})$, with the eigenvalue -1, is obtained as

$$|\psi_2\rangle = \frac{1}{2}(|00\rangle - (b_x + ib_y)|01\rangle + (a_x + ia_y)|10\rangle - (a_x + ia_y)(b_x + ib_y)|11\rangle), \qquad (36)$$

and, as before, the probability ε_2 is then obtained from $|\langle\psi_2|\ \psi_0\rangle|^2$. For the pure state (33), this becomes

$$\varepsilon_2 = \frac{1}{4}\left|\alpha - \beta(b_x - ib_y) + \gamma(a_x - ia_y) - \delta(a_x - ia_y)(b_x - ib_y)\right|^2, \qquad (37)$$

and the remaining probabilities $\varepsilon_3, \varepsilon_4 \cdots \varepsilon_{16}$ can be obtained similarly.

References

1. Blaquiere, A.: Wave mechanics as a two player game. In: Dynamical Systems and Microphysics, 33. Springer, Berlin (1980). Available at the quantum computation archive maintained by Tom Toffoli: http://pm1.bu.edu/~tt/qcl/pdf/blaquiea198277787a07.pdf
2. Wiesner, S.: Conjugate coding, SIGACT News 15 /1, 78 (1983). Available at the quantum computation archive maintained by Tom Toffoli: http://pm1.bu.edu/~tt/qcl/pdf/wiesners198316024137.pdf
3. Mermin, N.D.: Quantum mysteries revisited. Am. J. Phys. **58**, 731 (1990)
4. Mermin, N.D.: Extreme quantum entanglement in a superposition of macroscopically distinct states. Phys. Rev. Lett. **65**, 1838 (1990)
5. Meyer, D.A.: Quantum strategies. Phys. Rev. Lett. **82**, 1052 (1999)
6. Eisert, J., Wilkens, M., Lewenstein, M.: Quantum games and quantum strategies. Phys. Rev. Lett. **83**, 3077 (1999)
7. Eisert, J., Wilkens, M.: Quantum games. J. Mod. Opt. **47**, 2543 (2000)
8. Vaidman, L.: Variations on the theme of the Greenberger–Horne–Zeilinger Proof. Found. Phys. **29**, 615 (1999)
9. Benjamin, S.C., Hayden, P.M.: Comment on "Quantum games and quantum strategies". Phys. Rev. Lett. **87**, 069801 (2001)
10. van Enk, S.J., Pike, R.: Classical rules in quantum games. Phys. Rev. A **66**, 024306 (2002)
11. Johnson, N.F.: Playing a quantum game with a corrupted source. Phys. Rev. A **63**, 020302(R) (2001)
12. Marinatto, L., Weber, T.: A quantum approach to static games of complete information. Phys. Lett. A **272**, 291 (2000)
13. Iqbal, A., Toor, A.H.: Evolutionarily stable strategies in quantum games. Phys. Lett. A **280**, 249 (2001)
14. Du, J., Li, H., Xu, X., Zhou, X., Han, R.: Entanglement enhanced multiplayer quantum games. Phys. Lett. A **302**, 229 (2002)
15. Du, J., Li, H., Xu, X., Shi, M., Wu, J., Zhou, X., Han, R.: Experimental realization of quantum games on a quantum computer. Phys. Rev. Lett. **88**, 137902 (2002)
16. Piotrowski, E.W., Sladkowski, J.: Quantum market games. Phys. A **312**, 208 (2002)
17. Iqbal, A., Toor, A.H.: Quantum cooperative games. Phys. Lett. A **293**, 103 (2002)
18. Flitney, A.P., Abbott, D.: An introduction to quantum game theory. Fluct. Noise Lett. **2**, R175 (2002)
19. Iqbal, A., Toor, A.H.: Quantum mechanics gives stability to a Nash equilibrium. Phys. Rev. A **65**, 022306 (2002)

20. Piotrowski, E.W., Sladkowski, J.: An invitation to quantum game theory. Int. J. Theor. Phys. **42**, 1089 (2003)
21. Shimamura, J., Özdemir, S.K., Morikoshi, F., Imoto, N.: Entangled states that cannot reproduce original classical games in their quantum version. Phys. Lett. A **328**, 20 (2004)
22. Flitney, A.P., Abbott, D.: Quantum games with decoherence. J. Phys. A **38**, 449 (2005)
23. Du, J., Li, H., Xu, X., Zhou, X., Han, R.-D.: Multi-player and multi-choice quantum game. Chin. Phys. Lett. **19**, 1221 (2002)
24. Han, Y.J., Zhang, Y.S., Guo, G.C.: W state and Greenberger–Horne–Zeilinger state in quantum three-person prisoner's dilemma. Phys. Lett. A **295**, 61 (2002)
25. Iqbal, A., Weigert, S.: Quantum correlation games. J. Phys. A **37**, 5873 (2004)
26. Mendes, R.: The quantum ultimatum game. Quant. Inf. Process. **4**, 1 (2005)
27. Cheon, T., Tsutsui, I.: Classical and quantum contents of solvable game theory on Hilbert space. Phys. Lett. A **348**, 147 (2006)
28. Iqbal, A.: Playing games with EPR-type experiments. J. Phys. A Math. Gen. **38**, 9551 (2005)
29. Nawaz, A., Toor, A.H.: Generalized quantization scheme for two-person non-zero-sum games. J. Phys. A Math. Gen. **37**, 11457 (2004)
30. Özdemir, S.K., Shimamura, J., Imoto, N.: Quantum advantage does not survive in the presence of a corrupt source: optimal strategies in simultaneous move games. Phys. Lett. A **325**, 104 (2004)
31. Cheon, T.: Game theory formulated on Hilbert space. In: AIP Conference Proceedings, Volume 864, pp. 254 260, Quantum Computing: Back Action (2006), Indian Institute of Technology, Kanpur, India. arXiv:quantph/0605134
32. Shimamura, J., Özdemir, S.K., Morikoshi, F., Imoto, N.: Quantum and classical correlations between players in game theory. Int. J. Quant. Inf. **2**, 79 (2004)
33. Chen, Q., Wang, Y., Liu, J.T., Wang, K.L.: N-player quantum minority game. Phys. Lett. A **327**, 98 (2004)
34. Schmid, C., Flitney, A.P., Wieczorek, W., Kiesel, N., Weinfurter, H., Hollenberg, L.C.L.: Experimental implementation of a four-player quantum game. New J. Phys. **12**, 063031 (2010)
35. Ichikawa, T., Tsutsui, I.: Duality, phase structures, and dilemmas in symmetric quantum games. Ann. Phys. **322**, 531 (2007)
36. Cheon, T., Iqbal, A.: Bayesian Nash equilibria and Bell inequalities. J. Phys. Soc. Jpn. **77**, 024801 (2008)
37. Özdemir, S.K., Shimamura, J., Imoto, N.: A necessary and sufficient condition to play games in quantum mechanical settings. New J. Phys. **9**, 43 (2007)
38. Flitney, A.P., Greentree, A.D.: Coalitions in the quantum Minority game: classical cheats and quantum bullies. Phys. Lett. A **362**, 132 (2007)
39. Iqbal, A., Cheon, T.: Constructing quantum games from nonfactorizable joint probabilities. Phys. Rev. E **76**, 061122 (2007)
40. Ichikawa, T., Tsutsui, I., Cheon, T.: Quantum game theory based on the Schmidt decomposition. J. Phys. A Math. Theory **41**, 135303 (2008)
41. Ramzan, M., Khan, M.K.: Noise effects in a three-player Prisoner's Dilemma quantum game. J. Phys. A Math. Theory **41**, 435302 (2008)
42. Flitney, A.P., Hollenberg, L.C.L.: Nash equilibria in quantum games with generalized two-parameter strategies. Phys. Lett. A **363**, 381 (2007)
43. Aharon, N., Vaidman, L.: Quantum advantages in classically defined tasks. Phys. Rev. A **77**, 052310 (2008)
44. Bleiler, S.A.: A formalism for quantum games and an application. arXiv:0808.1389
45. Ahmed, A., Bleiler, S., Khan, F.S.: Octonionization of three player, two strategy maximally entangled quantum games. Int. J. Quant. Inform. **8**(3), 411 (2010)
46. Li, Q., He, Y., Jiang, J.P.: A novel clustering algorithm based on quantum games. J. Phys. A Math. Theor. **42**, 445303 (2009)
47. Li, Q., Iqbal, A., Chen, M., Abbott, D.: Quantum strategies win in a defector-dominated population. Phys. A **391**, 3316 (2012)
48. Chappell, J.M., Iqbal, A., Abbott, D.: Constructing quantum games from symmetric non-factorizable joint probabilities. Phys. Lett. A **374**, 4104 (2010)
49. Chappell, J.M., Iqbal, A., Abbott, D.: Analyzing three-player quantum games in an EPR type setup. PLoS ONE **6**(7), e21623 (2011)
50. Iqbal, A., Abbott, D.: Quantum matching pennies game. J. Phys. Soc. Jpn. **78**, 014803 (2009)

51. Chappell, J.M., Iqbal, A., Lohe, M.A., von Smekal, L.: An analysis of the quantum penny flip game using geometric algebra. J. Phys. Soc. Jpn. **78**, 054801 (2009)
52. Iqbal, A., Cheon, T., Abbott, D.: Probabilistic analysis of three-player symmetric quantum games played using the Einstein–Podolsky–Rosen–Bohm setting. Phys. Lett. A **372**, 6564 (2008)
53. Chappell, J.M., Iqbal, A., Abbott, D.: Analysis of two-player quantum games in an EPR setting using geometric algebra. PLoS ONE **7**(1), e29015 (2012)
54. Chappell, J.M., Iqbal, A., Abbott, D.: N-player quantum games in an EPR setting. PLoS ONE **7**(5), e36404 (2012)
55. Phoenix, S.J.D., Khan, F.S.: The role of correlation in quantum and classical games. Fluct. Noise Lett. **12**, 1350011 (2013)
56. Khan, F.S., Phoenix, S.J.D.: Gaming the quantum. Quant. Inf. Comput. **13**(3–4), 231 (2013)
57. Khan, F.S., Phoenix, S.J.D.: Mini-maximizing two qubit quantum computations. Quant. Inf. Process. **12**, 3807 (2013)
58. Peres, A.: Quantum Theory: Concepts and Methods. Kluwer Academic Publishers, Dordrecht (1995)
59. von Neumann, J., Morgenstern, O.: Theory of Games and Economic Behavior. Princeton University Press, Princeton (1944)
60. Binmore, K.: Game Theory: A Very Short Introduction. Oxford University Press, USA (2007)
61. Rasmusen, E.: Games and Information: An Introduction to Game Theory, 3rd edn. Blackwell Publishers Ltd., Oxford (2001)
62. Osborne, M.J.: An Introduction to Game Theory. Oxford University Press, USA (2003)
63. Einstein, A., Podolsky, B., Rosen, N.: Can quantum-mechanical description of physical reality be considered complete? Phys. Rev. **47**, 777 (1935)
64. Bohm, D.: Quantum Theory. Prentice Hall, Englewood Cliffs (1951)
65. Bell, J.S.: On the Einstein Podolsky Rosen paradox. Physics **1**, 195 (1964)
66. Bell, J.S.: Speakable and Unspeakable in Quantum Mechanics. Cambridge University Press, Cambridge (1987)
67. Bell, J.S.: On the problem of hidden variables in quantum mechanics. Rev. Mod. Phys. **38**, 447 (1966)
68. Aspect, A., Dalibard, J., Roger, G.: Experimental test of Bell's inequalities using time-varying analyzers. Phys. Rev. Lett. **49**, 1804 (1982)
69. Clauser, J.F., Shimony, A.: Bell's theorem. Experimental tests and implications. Rep. Prog. Phys. **41**, 1881 (1978)
70. Clauser, J.F., Horne, M.A., Shimony, A., Holt, R.A.: Proposed experiment to test local hidden-variable theories. Phys. Rev. Lett. **23**, 880 (1969)
71. Cereceda, J.L.: Quantum mechanical probabilities and general probabilistic constraints for Einstein–Podolsky–Rosen–Bohm experiments. Found. Phys. Lett. **13**, 427 (2000)