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A game theoretical perspective on the quantum probabilities associated with a GHZ state

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Abstract

In the standard approach to quantum games, players' strategic moves are local unitary transformations on an entangled state that is subsequently measured. Players' payoffs are then obtained as expected values of the entries in the payoff matrix of the classical game on a set of quantum probabilities obtained from the quantum measurement. In this paper, we approach quantum games from a diametrically opposite perspective. We consider a classical three-player symmetric game along with a known expression for a set of quantum probabilities relevant to a tripartite Einstein–Podolsky–Rosen (EPR) experiment that depends on three players' directional choices in the experiment. We define the players' strategic moves as their directional choices in an EPR setting and then express their payoff relations in the resulting quantum game in terms of their directional choices, the entries of the payoff matrix, and the quantum probability distribution relevant to the tripartite EPR experiment.

Keywords Quantum games · Tripartite EPR experiment · GHZ state · Quantum probabilities

1 Introduction

In the standard scheme [1,2] of a quantized version of a non-cooperative game [3], the players share an entangled state, their strategic moves are local unitary transformations on the state, and the quantum measurement [4] generates the players' payoffs. The resulting players' payoffs in the quantum game can be understood as the expected values of the entries in the payoff matrix of the (classical) game [5–7] arising from a set of quantum probabilities [4]. The key concerns in determining the players' payoffs relations in the quantum game are (a) What are the players' moves in the quantum

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game? (b) Which set of quantum probabilities is obtained by quantum measurement? and (c) How the players' strategic moves are related to the set of quantum probabilities?

This brings us to question whether the unitary transformations are really necessary in the set-up of a quantum game. A proposed scheme [8–11] for playing a quantum game in which players' strategic moves are not unitary transformations uses the setting of an Einstein–Podolsky–Rosen (EPR) experiment [4,12–16]. Two players are located in space-like separated regions and share a singlet state. In a run of the experiment, each player decides one out of the two available directions and a quantum measurement is performed. This leads to obtaining a (normalized) set of quantum probabilities along with a listing of the directional choices the players make in each run of the experiment. As the players' directional choices determine the quantum probability distribution, the setting can be used to develop a quantum version of a two-player game. A multipartite EPR experiment would then be required for a multiplayer quantum game.

In this paper, we consider a classical three-player symmetric game, along with a reported expression for a quantum probability distribution, which is relevant to the three-partite EPR experiment. We then define players' directional choices in the experiment as their strategic moves and express players' payoff relations in the quantum game in terms of the three directional choices and the entries of the payoff matrix.

This paper thus provides a game theoretic perspective on the peculiarity of quantum probabilities. The first perspective along game theoretical lines on quantum probabilities that are associated with a GHZ state [4] was provided by Vaidman [17]. Vaidman proposed a set of rules defining a game that can only be won by a team of three players when they share a GHZ state. The present paper extends Vaidman's perspective by considering Nash equilibria in the set of symmetric games played by a team of three players in a non-cooperative game setting. Vaidman presented his game without invoking Hilbert space as is the case in the present paper.

2 Three-player games with mixed strategies

Consider a three-player (non-cooperative) game in which the players Alice (A), Bob (B), and Chris (C) make their strategic moves simultaneously. The players are assumed located at distance and are unable to communicate to one another. Each player has to decide between two choices, called the *pure strategies*, and in repeated version of the game they can also play the *mixed strategies*. The payoff relations depend on the game matrix, the players' pure strategies, and the probability distribution on pure strategies.

To be specific, we assume that the player A's pure strategies are S_1, S_2; the player B's pure strategies are S_1', S_2'; and the player C's pure strategies are S_1'', S_2''. Also, the game is defined by the following pure strategy payoff relations [18]

$$\begin{array}{ll} \Pi_{A,B,C}\left(S_1, S_1', S_1''\right) = \alpha_1, \beta_1, \gamma_1; & \Pi_{A,B,C}\left(S_1, S_2', S_2''\right) = \alpha_5, \beta_5, \gamma_5; \\ \Pi_{A,B,C}\left(S_2, S_1', S_1''\right) = \alpha_2, \beta_2, \gamma_2; & \Pi_{A,B,C}\left(S_2, S_1', S_2''\right) = \alpha_6, \beta_6, \gamma_6; \\ \Pi_{A,B,C}\left(S_1, S_2', S_1''\right) = \alpha_3, \beta_3, \gamma_3; & \Pi_{A,B,C}\left(S_2, S_2', S_1''\right) = \alpha_7, \beta_7, \gamma_7; \\ \Pi_{A,B,C}\left(S_1, S_1', S_2''\right) = \alpha_4, \beta_4, \gamma_4; & \Pi_{A,B,C}\left(S_2, S_2', S_2''\right) = \alpha_8, \beta_8, \gamma_8. \end{array} \quad (1)$$

For example, $\Pi_{A,B,C}\left(S_1, S_2', S_1''\right) = \alpha_3, \beta_3, \gamma_3$ states that the players A, B, and C obtain the payoffs α_3, β_3, and γ_3, respectively, when they play the pure strategies S_1, S_2', and S_1'', respectively.

In a repeated version of this game, a player can choose between his/her two pure strategies with some probability, which defines his/her mixed strategy. We specify a mixed strategy by $x, y, z \in [0, 1]$ for players A, B, and C, respectively. These are the probabilities with which the players A, B, and C play the pure strategies S_1, S_1', and S_1'', respectively. They, then, play the pure strategies S_2, S_2', and S_2'' with probabilities $(1-x)$, $(1-y)$, and $(1-z)$, respectively, and the mixed strategy payoff relations, therefore, read

$$\begin{aligned} \Pi_{A,B,C}(x,y,z) = {} & xyz\Pi_{A,B,C}\left(S_1, S_1', S_1''\right) + x(1-y)z\Pi_{A,B,C}\left(S_1, S_2', S_1''\right) \\ & + xy(1-z)\Pi_{A,B,C}\left(S_1, S_1', S_2''\right) + x(1-y)(1-z)\Pi_{A,B,C}\left(S_1, S_2', S_2''\right) \\ & + (1-x)yz\Pi_{A,B,C}\left(S_2, S_1', S_1''\right) + (1-x)(1-y)z\Pi_{A,B,C}\left(S_2, S_2', S_1''\right) \\ & + (1-x)y(1-z)\Pi_{A,B,C}\left(S_2, S_1', S_2''\right) \\ & + (1-x)(1-y)(1-z)\Pi_{A,B,C}\left(S_2, S_2', S_2''\right), \end{aligned} \quad (2)$$

that can also be written as

$$\Pi_{A,B,C}(x,y,z) = \sum_{i,j,k=1,2} \Pr_c\left(S_i, S_j', S_k''\right)\Pi_{A,B,C}\left(S_i, S_j', S_k''\right), \quad (3)$$

where $\Pr_c\left(S_i, S_j', S_k''\right)$ are the factorizable probabilities, and for instance, $\Pr_c\left(S_1, S_1', S_2''\right) = xy(1-z)$ and $\Pr_c\left(S_2, S_2', S_1''\right) = (1-x)(1-y)z$.

2.1 Symmetric three-player games

Symmetric three-player games are defined by the condition that a player's payoff is decided by his/her strategic move and not by his/her identity. Mathematically, this is expressed by the conditions

$$\Pi_A(x,y,z) = \Pi_A(x,z,y) = \Pi_B(y,x,z) = \Pi_B(z,x,y) = \Pi_C(y,z,x) = \Pi_C(z,y,x), \quad (4)$$

i.e. the player A's payoff when s/he plays x remains the same either when player B plays y, whereas player C plays y or when player B plays x, whereas player C play x. The payoff relations (2) satisfy the conditions (4) when [18]

$$\begin{array}{llll} \beta_1=\alpha_1, & \beta_2=\alpha_3, & \beta_3=\alpha_2, & \beta_4=\alpha_3, \\ \beta_5=\alpha_6, & \beta_6=\alpha_5, & \beta_7=\alpha_6, & \beta_8=\alpha_8, \\ \gamma_1=\alpha_1, & \gamma_2=\alpha_3, & \gamma_3=\alpha_3, & \gamma_4=\alpha_2, \\ \gamma_5=\alpha_6, & \gamma_6=\alpha_6, & \gamma_7=\alpha_5, & \gamma_8=\alpha_8, \\ & \alpha_6=\alpha_7, & \alpha_3=\alpha_4. & \end{array} \tag{5}$$

A symmetric three-player game can, therefore, be defined by only six constants α_1, α_2, α_3, α_5, α_6, and α_8. In the rest of this paper, we will define these six constants to be α, β, δ, ϵ, θ, and ω, where $\alpha_1=\alpha$, $\alpha_2=\beta$, $\alpha_3=\delta$, $\alpha_5=\epsilon$, $\alpha_6=\theta$, and $\alpha_8=\omega$. The pure strategy payoff relations (1) in this symmetric game are then re-expressed as

$$\begin{array}{ll} \Pi_{A,B,C}\left(S_1,S_1',S_1''\right)=\alpha,\alpha,\alpha; & \Pi_{A,B,C}\left(S_1,S_2',S_2''\right)=\epsilon,\theta,\theta; \\ \Pi_{A,B,C}\left(S_2,S_1',S_1''\right)=\beta,\delta,\delta; & \Pi_{A,B,C}\left(S_2,S_1',S_2''\right)=\theta,\epsilon,\theta; \\ \Pi_{A,B,C}\left(S_1,S_2',S_1''\right)=\delta,\beta,\delta; & \Pi_{A,B,C}\left(S_2,S_2',S_1''\right)=\theta,\theta,\epsilon; \\ \Pi_{A,B,C}\left(S_1,S_1',S_2''\right)=\delta,\delta,\beta; & \Pi_{A,B,C}\left(S_2,S_2',S_2''\right)=\omega,\omega,\omega. \end{array} \tag{6}$$

The mixed strategy payoff relations in Eq. (2) then take the form

$$\begin{aligned} \Pi_{A,B,C}(x,y,z) = {} & xyz(\alpha,\alpha,\alpha)+x(1-y)z(\delta,\beta,\delta)+xy(1-z)(\delta,\delta,\beta) \\ & +x(1-y)(1-z)(\epsilon,\theta,\theta)+(1-x)yz(\beta,\delta,\delta) \\ & +(1-x)(1-y)z(\theta,\theta,\epsilon)+(1-x)y(1-z)(\theta,\epsilon,\theta) \\ & +(1-x)(1-y)(1-z)(\omega,\omega,\omega). \end{aligned} \tag{7}$$

3 Quantum probability distribution for a GHZ state

Now, consider a GHZ state

$$|\psi\rangle=(|0\rangle_1\,|0\rangle_2\,|0\rangle_3+|1\rangle_1\,|1\rangle_2\,|1\rangle_3)/\sqrt{2}, \tag{8}$$

that is shared among three the three players, where $|i\rangle_j$ is the i-th state of the j-th qubit and the setting of the generalized EPR experiments. Each player measures the dichotomic observable $\vec{n}.\vec{\sigma}$ where $\vec{n}=\vec{a},\vec{b},\vec{c}$ and $\vec{\sigma}$ is a vector the components of which are standard Pauli matrices. The family of observables $\vec{n}.\vec{\sigma}$ covers all possible dichotomic observables for a qubit system [4].

Kaszlikowski and Żukowski [19] show that the probability of obtaining the result $m=\pm1$ for the player A, when s/he plays the strategy $\vec{a}$, the result $l=\pm1$ for the player B, when s/he plays the strategy $\vec{b}$ and the result $k=\pm1$ for the player C, when s/he plays the strategy $\vec{c}$ is given by

$$\Pr_{QM}(m,l,k;\vec{a},\vec{b},\vec{c})=\frac{1}{8}\left[1+mla_3b_3+mka_3c_3+lkb_3c_3+mlk\sum_{r,p,s=1}^{3}M_{rps}a_rb_pc_s\right], \tag{9}$$

where a_r, b_p, c_s are components of vectors $\vec{a},\vec{b},\vec{c}$ and where nonzero elements of the tensor M_{rps} are $M_{111}=1$, $M_{122}=-1$, $M_{212}=-1$, $M_{221}=-1$. In view of this, the only terms in the product $a_r b_p c_s$ that contribute towards the probability $\Pr_{QM}(m, l, k; \vec{a}, \vec{b}, \vec{c})$ are $a_1b_1c_1$, $a_1b_2c_2$, $a_2b_1c_2$, and $a_2b_2c_1$. Equation (9) can therefore be written as

$$\Pr_{QM}(m, l, k; \vec{a}, \vec{b}, \vec{c}) = \frac{1}{8}[1 + mla_3b_3 + mka_3c_3 + lkb_3c_3 + mlk(a_1b_1c_1 - a_1b_2c_2 - a_2b_1c_2 - a_2b_2c_1)]. \quad (10)$$

Note that Eq. (9) gives a quantum probability distribution without reference to the underlying Hilbert space, unitary transformations, or quantum measurement.

We consider playing a three-player quantum game in which the strategic moves of the players *A, B*, and *C* consist of choosing the directions $\vec{a}$, $\vec{b}$, and $\vec{c}$, respectively. The players's payoff relations are then expressed in terms of the quantum probability distribution given in Eq. (9).

3.1 Players sharing a GHZ state and when choosing a direction is a player's strategic move

Let $\vec{a} = \vec{a}(a_1, a_2, a_3)$, $\vec{b} = \vec{b}(b_1, b_2, b_3)$, $\vec{c} = \vec{c}(c_1, c_2, c_3)$ be the players' directional choices that we consider as their strategies. Denoting the quantum probabilities by $\Pr_Q$, the set of quantum probabilities can be obtained from Eq. (10) as follows

$$\begin{aligned}
\Pr_Q(S_1, S_1', S_1'') &= \Pr_Q[(\vec{a}, m = +1), (\vec{b}, l = +1), (\vec{c}, k = +1)] \\
&= \frac{1}{8}[1 + a_3b_3 + a_3c_3 + b_3c_3 + \Delta]; \\
\Pr_Q(S_1, S_2', S_1'') &= \Pr_Q[(\vec{a}, m = +1), (\vec{b}, l = -1), (\vec{c}, k = +1)] \\
&= \frac{1}{8}[1 - a_3b_3 + a_3c_3 - b_3c_3 - \Delta]; \\
\Pr_Q(S_1, S_1', S_2'') &= \Pr_Q[(\vec{a}, m = +1), (\vec{b}, l = +1), (\vec{c}, k = -1)] \\
&= \frac{1}{8}[1 + a_3b_3 - a_3c_3 - b_3c_3 - \Delta]; \\
\Pr_Q(S_1, S_2', S_2'') &= \Pr_Q[(\vec{a}, m = +1), (\vec{b}, l = -1), (\vec{c}, k = -1)] \\
&= \frac{1}{8}[1 - a_3b_3 - a_3c_3 + b_3c_3 + \Delta]; \\
\Pr_Q(S_2, S_1', S_1'') &= \Pr_Q[(\vec{a}, m = -1), (\vec{b}, l = +1), (\vec{c}, k = +1)] \\
&= \frac{1}{8}[1 - a_3b_3 - a_3c_3 + b_3c_3 - \Delta]; \\
\Pr_Q(S_2, S_2', S_1'') &= \Pr_Q[(\vec{a}, m = -1), (\vec{b}, l = -1), (\vec{c}, k = +1)] \\
&= \frac{1}{8}[1 + a_3b_3 - a_3c_3 - b_3c_3 + \Delta];
\end{aligned}$$

$$\Pr_Q\left(S_2, S_1', S_2''\right) = \Pr_Q[(\vec{a}, m=-1), (\vec{b}, l=+1), (\vec{c}, k=-1)]$$
$$= \frac{1}{8}[1 - a_3b_3 + a_3c_3 - b_3c_3 + \Delta];$$
$$\Pr_Q\left(S_2, S_2', S_2''\right) = \Pr_Q[(\vec{a}, m=-1), (\vec{b}, l=-1), (\vec{c}, k=-1)]$$
$$= \frac{1}{8}[1 + a_3b_3 + a_3c_3 + b_3c_3 - \Delta]; \tag{11}$$

where $\Delta = a_1b_1c_1 - a_1b_2c_2 - a_2b_1c_2 - a_2b_2c_1$. We define players A's, B's, C's payoff relations in the quantum game as follows

$$\Pi_{A,B,C}(\vec{a}, \vec{b}, \vec{c}) = \sum_{i,j,k=1}^{2} \Pr_Q\left(S_i, S_j', S_k''\right) \Pi_{A,B,C}\left(S_i, S_j', S_k''\right), \tag{12}$$

i.e. these are obtained as the expectation of payoff entries (6) on the set of quantum probabilities (11). For the symmetric game defined in Eq. (6), the payoffs to the players A, B, and C, given in (12), can then be expanded as follows:

$$\begin{aligned}
&\Pi_{A,B,C}(\vec{a}, \vec{b}, \vec{c}) \\
&= \frac{1}{8}\Big\{[1 + a_3b_3 + a_3c_3 + b_3c_3 + \Delta](\alpha, \alpha, \alpha) + [1 - a_3b_3 + a_3c_3 - b_3c_3 - \Delta](\delta, \beta, \delta) \\
&\quad + [1 + a_3b_3 - a_3c_3 - b_3c_3 - \Delta](\delta, \delta, \beta) + [1 - a_3b_3 - a_3c_3 + b_3c_3 + \Delta](\epsilon, \theta, \theta) \\
&\quad + [1 - a_3b_3 - a_3c_3 + b_3c_3 - \Delta](\beta, \delta, \delta) + [1 + a_3b_3 - a_3c_3 - b_3c_3 + \Delta](\theta, \theta, \epsilon) \\
&\quad + [1 - a_3b_3 + a_3c_3 - b_3c_3 + \Delta](\theta, \epsilon, \theta) + [1 + a_3b_3 + a_3c_3 + b_3c_3 - \Delta](\omega, \omega, \omega)\Big\}.
\end{aligned} \tag{13}$$

Let $a_3 = b_3 = c_3 = 0$, i.e. when the players' unit vectors are confined to the X–Y plane, the payoff relations (13) can be written as

$$\begin{aligned}
&\Pi_{A,B,C}(\vec{a}, \vec{b}, \vec{c}) \\
&= \frac{1}{8}\Big\{(1 + \Delta)(\alpha, \alpha, \alpha) + (1 - \Delta)(\delta, \beta, \delta) + (1 - \Delta)(\delta, \delta, \beta) + (1 + \Delta)(\epsilon, \theta, \theta) \\
&\quad + (1 - \Delta)(\beta, \delta, \delta) + (1 + \Delta)(\theta, \theta, \epsilon) + (1 + \Delta)(\theta, \epsilon, \theta) + (1 - \Delta)(\omega, \omega, \omega)\Big\}.
\end{aligned} \tag{14}$$

It is apparent from above that the resulting payoff relations (14) in the quantum game cannot be put into a form that is same as for the classical mixed strategy game, i.e. Eq. (7). This raises the question whether there exist constraints that can be placed on the players' directional choices, i.e. the unit vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$, such that the payoff relations (13) in the quantum game are reduced to the players' payoffs in the classical game allowing mixed strategies (7). In order to find an answer to this, we set

$$\Pi_{A,B,C}(\vec{a}, \vec{b}, \vec{c}) = \Pi_{A,B,C}(x, y, z), \tag{15}$$

and equate the right sides of Eqs. (13, 7), i.e.

$$\frac{1}{8}\left[1 + a_3 b_3 + a_3 c_3 + b_3 c_3 + \Delta\right] = xyz, \tag{16}$$

$$\frac{1}{8}\left[1 - a_3 b_3 + a_3 c_3 - b_3 c_3 - \Delta\right] = x(1-y)z, \tag{17}$$

$$\frac{1}{8}\left[1 + a_3 b_3 - a_3 c_3 - b_3 c_3 - \Delta\right] = xy(1-z), \tag{18}$$

$$\frac{1}{8}\left[1 - a_3 b_3 - a_3 c_3 + b_3 c_3 + \Delta\right] = x(1-y)(1-z), \tag{19}$$

$$\frac{1}{8}\left[1 - a_3 b_3 - a_3 c_3 + b_3 c_3 - \Delta\right] = (1-x)yz, \tag{20}$$

$$\frac{1}{8}\left[1 + a_3 b_3 - a_3 c_3 - b_3 c_3 + \Delta\right] = (1-x)(1-y)z, \tag{21}$$

$$\frac{1}{8}\left[1 - a_3 b_3 + a_3 c_3 - b_3 c_3 + \Delta\right] = (1-x)y(1-z), \tag{22}$$

$$\frac{1}{8}\left[1 + a_3 b_3 + a_3 c_3 + b_3 c_3 - \Delta\right] = (1-x)(1-y(1-z). \tag{23}$$

Now, by adding Eqs. (16) and (17) we obtain

$$\frac{1}{4}(1 + a_3 c_3) = xz, \tag{24}$$

adding Eqs. (16) and (18) gives

$$\frac{1}{4}(1 + a_3 b_3) = xy, \tag{25}$$

adding Eqs. (16) and (20) gives

$$\frac{1}{4}(1 + b_3 c_3) = yz, \tag{26}$$

adding Eqs. (18) and (19) gives

$$\frac{1}{4}(1 - a_3 c_3) = x(1-z). \tag{27}$$

Now, we add Eqs. (24) and (27) to obtain $x = \frac{1}{2}$. Adding Eqs. (22) and (23) gives

$$\frac{1}{4}(1 + a_3 c_3) = (1-x)(1-z), \tag{28}$$

and substitution from Eq. (24) and $x = \frac{1}{2}$ gives $z = \frac{1}{2}$. Similarly, adding Eqs. (17) and (19) gives

$$\frac{1}{4}(1 - a_3 b_3) = x(1-y), \tag{29}$$

and adding Eqs. (21) and (23) gives

$$\frac{1}{4}(1 + a_3 b_3) = (1 - x)(1 - y). \tag{30}$$

By adding Eqs. (29) and (30), we obtain $y = \frac{1}{2}$, and thus, $(x, y, z) = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ is obtained as the solution of Eqs. (16)–(23).

Therefore, the mixed strategy payoff relations (7) can be recovered from the payoffs relations (13) for the quantum game only for the special case when $(x, y, z) = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$. This is because the quantum probability distribution for the GHZ state, from which the payoff relations (13) are constructed, is inherently non-factorizable. In the research area of quantum games, recovering the mixed strategy classical payoff relations from the payoff relations for a quantum game is quite often considered an essential requirement. When the underlying quantum probabilities in a quantum game are obtained from the GHZ state, this requirement is not satisfied except for a very special case, i.e. $(x, y, z) = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$.

Considering the payoff relations (13) in the quantum game, a Nash equilibrium (NE) is a directional triple $(\vec{a}^*, \vec{b}^*, \vec{c}^*)$ that satisfies the following constraints:

$$\begin{aligned}
&\Pi_A\left(\vec{a}^*, \vec{b}^*, \vec{c}^*\right) - \Pi_A\left(\vec{a}, \vec{b}^*, \vec{c}^*\right) \geq 0, \\
&\Pi_B\left(\vec{a}^*, \vec{b}^*, \vec{c}^*\right) - \Pi_B\left(\vec{a}^*, \vec{b}, \vec{c}^*\right) \geq 0, \\
&\Pi_C\left(\vec{a}^*, \vec{b}^*, \vec{c}^*\right) - \Pi_C\left(\vec{a}^*, \vec{b}^*, \vec{c}\right) \geq 0,
\end{aligned} \tag{31}$$

for all $\vec{a}$, $\vec{b}$, and $\vec{c}$. For the symmetric game, these Nash inequalities take the form

$$\begin{aligned}
\Pi_A(\vec{a}^*, \vec{b}^*, \vec{c}^*) - \Pi_A(\vec{a}, \vec{b}^*, \vec{c}^*) &= \frac{1}{8}\left[(a_3^* - a_3)\Delta_1\gamma_1 + \gamma_2(a_1^* - a_1)\Delta_2 - \gamma_2(a_2^* - a_2)\Delta_3\right] \geq 0, \\
\Pi_A(\vec{a}^*, \vec{b}^*, \vec{c}^*) - \Pi_A(\vec{a}^*, \vec{b}, \vec{c}^*) &= \frac{1}{8}\left[(b_3^* - b_3)\Delta_1'\gamma_1 - \gamma_2(b_2^* - b_2)\Delta_2' + \gamma_2(b_1^* - b_1)\Delta_3'\right] \geq 0, \\
\Pi_A(\vec{a}^*, \vec{b}^*, \vec{c}^*) - \Pi_A(\vec{a}^*, \vec{b}^*, \vec{c}) &= \frac{1}{8}\left[(c_3^* - c_3)\Delta_1''\gamma_1 - \gamma_2(c_2^* - c_2)\Delta_2'' + \gamma_2(c_1^* - c_1)\Delta_3''\right] \geq 0,
\end{aligned} \tag{32}$$

where

$$\gamma_1 = \alpha - \beta - \epsilon + \omega \quad \text{and} \quad \gamma_2 = \alpha - 2\delta - \beta + \epsilon + 2\theta - \omega, \tag{33}$$

and

$$\begin{aligned}
&\Delta_1 = b_3 + c_3, \quad \Delta_2 = b_1 c_1 - b_2 c_2, \quad \Delta_3 = b_1 c_2 + b_2 c_1, \\
&\Delta_1' = a_3 + c_3, \quad \Delta_2' = a_1 c_2 + a_2 c_1, \quad \Delta_3' = a_1 c_1 - a_2 c_2, \\
&\Delta_1'' = a_3 + b_3, \quad \Delta_2'' = a_1 b_2 + a_2 b_1, \quad \Delta_3'' = a_1 b_1 - a_2 b_2.
\end{aligned} \tag{34}$$

3.2 Three-player Prisoners' Dilemma

Prisoner's Dilemma (PD) is a non-cooperative game [5–7] that is widely known in the areas of economics, social, and political sciences. In recent years, quantum physics has been added to this list. It was investigated early in the history of quantum games and provided significant motivation for further work in this area [1,2].

Two-player PD is about two suspects, considered here as the players in a game, who have been arrested on the allegations of having committed a crime, but there is no enough available evidence to convict them. The investigators come up with an ingenious plan to make the suspects confess their crime.

They are taken to separate cells and are not allowed to communicate. They are contacted individually and, along with being dictated a set of rules, are asked to choose between two choices (strategies): *to Confess* ($\mathfrak{D}$) and *Not to Confess* ($\mathfrak{C}$), where $\mathfrak{C}$ and $\mathfrak{D}$ stand for cooperation and defection. These are the well-known wordings for the available choices for them and refer to the choice they make to the fellow prisoner, and not to the authorities.

The rules state that if neither prisoner confesses, i.e. ($\mathfrak{C}$, $\mathfrak{C}$), both are given freedom; when one prisoner confesses ($\mathfrak{D}$) and the other does not ($\mathfrak{C}$), i.e. ($\mathfrak{C}$, $\mathfrak{D}$) or ($\mathfrak{D}$, $\mathfrak{C}$), the prisoner who confesses ($\mathfrak{D}$) gets freedom as well as a financial reward, while the prisoner who did not confess ends up in prison for a longer term. If both prisoners confess, i.e. ($\mathfrak{D}$, $\mathfrak{D}$), both are given a reduced term.

In the two-player case, involving the players A and B the strategy pair ($\mathfrak{D}$, $\mathfrak{D}$) comes out as the unique NE (and the rational outcome) of the game, leading to the situation of both ending up in jail with reduced term. The game offers a dilemma as the rational outcome ($\mathfrak{D}$, $\mathfrak{D}$) differs from the outcome ($\mathfrak{C}$, $\mathfrak{C}$), which is an available choice, and for which both prisoners obtain freedom.

With the above notation, the three-player PD can be defined by making the following associations

$$S_1 \sim \mathfrak{C}, \quad S_2 \sim \mathfrak{D}, \quad S_1' \sim \mathfrak{C}, \quad S_2' \sim \mathfrak{D}, \quad S_1'' \sim \mathfrak{C}, \quad S_2'' \sim \mathfrak{D}, \tag{35}$$

and afterwards imposing the following conditions [20]:

(a) The strategy S_2 is a dominant choice [6] for each player. For Alice, this requires

$$\begin{aligned} \Pi_A\left(S_2, S_1', S_1''\right) &> \Pi_A\left(S_1, S_1', S_1''\right), \\ \Pi_A\left(S_2, S_2', S_2''\right) &> \Pi_A\left(S_1, S_2', S_2''\right), \\ \Pi_A\left(S_2, S_1', S_2''\right) &> \Pi_A\left(S_1, S_1', S_2''\right), \end{aligned} \tag{36}$$

and similar inequalities hold for players Bob and Chris.

(b) A player is better off if more of his/her opponents choose to cooperate. For Alice, this requires

$$\begin{aligned} \Pi_A\left(S_2, S_1', S_1''\right) &> \Pi_A\left(S_2, S_1', S_2''\right) > \Pi_A\left(S_2, S_2', S_2''\right), \\ \Pi_A\left(S_1, S_1', S_1''\right) &> \Pi_A\left(S_1, S_1', S_2''\right) > \Pi_A\left(S_1, S_2', S_2''\right), \end{aligned} \tag{37}$$

and similar relations hold for Bob and Chris.

(c) If one player's choice is fixed, the other two players are left in the situation of a two-player PD. For Alice, this requires

$$\begin{aligned}
&\Pi_A\left(S_1, S_1', S_2''\right) > \Pi_A\left(S_2, S_2', S_2''\right),\\
&\Pi_A\left(S_1, S_1', S_1''\right) > \Pi_A\left(S_2, S_1', S_2''\right),\\
&\Pi_A\left(S_1, S_1', S_2''\right) > (1/2)\left\{\Pi_A\left(S_1, S_2', S_2''\right) + \Pi_A\left(S_2, S_1', S_2''\right)\right\},\\
&\Pi_A\left(S_1, S_1', S_1''\right) > (1/2)\left\{\Pi_A\left(S_1, S_1', S_2''\right) + \Pi_A\left(S_2, S_1', S_1''\right)\right\},
\end{aligned} \tag{38}$$

and similar relations hold for Bob and Chris.

Translating the above conditions while using the notation introduced in (6) requires

$$\begin{aligned}
&\text{a) } \beta > \alpha, \quad \omega > \epsilon, \quad \theta > \delta,\\
&\text{b) } \beta > \theta > \omega, \quad \alpha > \delta > \epsilon,\\
&\text{c) } \delta > \omega, \quad \alpha > \theta, \quad \delta > (1/2)(\epsilon + \theta), \quad \alpha > (1/2)(\delta + \beta),
\end{aligned} \tag{39}$$

which defines the generalized three-player PD. For example, [20], by letting

$$\alpha = 7, \quad \beta = 9, \quad \delta = 3, \quad \epsilon = 0, \quad \omega = 1, \quad \theta = 5, \tag{40}$$

all of these conditions hold.

4 Three-player quantum Prisoners' Dilemma with GHZ state

The values in (40) give $\gamma_1 = -1$ and $\gamma_2 = 1$. With the deltas given in (34), the Nash inequalities (32) take the form

$$\begin{aligned}
&\Pi_A(\vec{a}^*, \vec{b}^*, \vec{c}^*) - \Pi_A(\vec{a}, \vec{b}^*, \vec{c}^*)\\
&\quad = \frac{1}{8}\left[-(a_3^* - a_3)(b_3 + c_3) + (a_1^* - a_1)(b_1c_1 - b_2c_2) - (a_2^* - a_2)(b_1c_2 + b_2c_1)\right] \geq 0,\\
&\Pi_A(\vec{a}^*, \vec{b}^*, \vec{c}^*) - \Pi_A(\vec{a}^*, \vec{b}, \vec{c}^*)\\
&\quad = \frac{1}{8}\left[-(b_3^* - b_3)(a_3 + c_3) - (b_2^* - b_2)(a_1c_2 + a_2c_1) + (b_1^* - b_1)(a_1c_1 - a_2c_2)\right] \geq 0,\\
&\Pi_A(\vec{a}^*, \vec{b}^*, \vec{c}^*) - \Pi_A(\vec{a}^*, \vec{b}^*, \vec{c})\\
&\quad = \frac{1}{8}\left[-(c_3^* - c_3)(a_3 + b_3) - (c_2^* - c_2)(a_1b_2 + a_2b_1) + (c_1^* - c_1)(a_1b_1 - a_2b_2)\right] \geq 0.
\end{aligned} \tag{41}$$

These inequalities show that for the PD game defined in (40), no directional triplet can exist as a NE when the three players have the choice to direct their respective unit vector along any direction, i.e. there are no restrictions placed on the players' directional choices.

The inequalities (32) suggest the following cases:

4.1 Case (a)

Consider $a_3 = b_3 = c_3 = 0$. Nash inequalities (32) then take the form

$$\Pi_A(\vec{a}^*, \vec{b}^*, \vec{c}^*) - \Pi_A(\vec{a}, \vec{b}^*, \vec{c}^*) = \frac{1}{8}\gamma_2 \left[(a_1^* - a_1)\Delta_2 - (a_2^* - a_2)\Delta_3\right] \geq 0,$$
$$\Pi_A(\vec{a}^*, \vec{b}^*, \vec{c}^*) - \Pi_A(\vec{a}^*, \vec{b}, \vec{c}^*) = \frac{1}{8}\gamma_2 \left[-(b_2^* - b_2)\Delta_2' + (b_1^* - b_1)\Delta_3'\right] \geq 0,$$
$$\Pi_A(\vec{a}^*, \vec{b}^*, \vec{c}^*) - \Pi_A(\vec{a}^*, \vec{b}^*, \vec{c}) = \frac{1}{8}\gamma_2 \left[-(c_2^* - c_2)\Delta_2'' + (c_1^* - c_1)\Delta_3''\right] \geq 0, \tag{42}$$

that can also be expressed as

$$\Pi_A(\vec{a}^*, \vec{b}^*, \vec{c}^*) - \Pi_A(\vec{a}, \vec{b}^*, \vec{c}^*) = \frac{1}{8}\gamma_2 \left[a_1^*\Delta_2 - a_2^*\Delta_3 + \varsigma\right] \geq 0,$$
$$\Pi_A(\vec{a}^*, \vec{b}^*, \vec{c}^*) - \Pi_A(\vec{a}^*, \vec{b}, \vec{c}^*) = \frac{1}{8}\gamma_2 \left[-b_2^*\Delta_2' + b_1^*\Delta_3' + \varsigma\right] \geq 0,$$
$$\Pi_A(\vec{a}^*, \vec{b}^*, \vec{c}^*) - \Pi_A(\vec{a}^*, \vec{b}^*, \vec{c}) = \frac{1}{8}\gamma_2 \left[-c_2^*\Delta_2'' + c_1^*\Delta_3'' + \varsigma\right] \geq 0, \tag{43}$$

where

$$\varsigma = a_1 b_2 c_2 + a_2 b_1 c_2 - a_1 b_1 c_1 + a_2 b_2 c_1. \tag{44}$$

Consider the case when $\gamma_2 > 0$, then for given a_1^*, b_1^*, and c_1^*, the restrictions on the directions that the unit vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ can take can be determined. For instance, for $a_2^*, = b_1^*, = c_1^* = 1$, i.e. then $a_2^* = b_2^* = c_2^* = 0$, these constraints become

$$\Delta_2 + \varsigma \geq 0, \quad \Delta_3' + \varsigma \geq 0, \quad \Delta_3'' + \varsigma \geq 0. \tag{45}$$

4.2 Case (b)

Consider $\gamma_2 = 0$ and $a_3 = b_3 = c_3 = 0$. With these constraints, the allowed directions are confined to the X–Y plane and any directional triplet then exists as a NE. In this case, from Eq. (34) we then have $\Delta_1 = \Delta_1' = \Delta_1'' = 0$. As $\vec{a}$, $\vec{b}$, and $\vec{c}$ are unit vectors, we also have $a_2 = \pm\sqrt{1 - a_1^2}$, $b_2 = \pm\sqrt{1 - b_1^2}$, and $c_2 = \pm\sqrt{1 - c_1^2}$.

5 Discussion

We present an analysis of the three-partite EPR experiments that use a GHZ state and its setting is considered in order to play a three-player non-cooperative quantum game. The players' strategic choices are the three directions $\vec{a}$, $\vec{b}$, and $\vec{c}$ along which the dichotomic observables $\vec{n}.\vec{\sigma}$ are measured, where $\vec{n} = \vec{a}, \vec{b}, \vec{c}$ and $\vec{\sigma}$ is a vector whose components are the standard Pauli matrices. Using Kaszlikowski and Żukowski's [19] results for the quantum probabilities involved in such experiments, we develop a three-player quantum game, with the underlying setting of the three-partite EPR experiment This extends an approach to quantum games by Vaidman [17] that does not involve Hilbert space, and/or quantum measurement, and shows how three-player quantum games with EPR experiments can be developed. Players' strategies are their directions in terms of which their payoffs are expressed using Eq. (10). Nash inequalities are used to obtain Nash equilibria as direction triples and the players' payoffs are then compared to their payoffs for the Nash equilibria in the classical game.

For a three-player Prisoners' Dilemma game, defined in (40), we conclude that no directional triplet can exist as a NE when no restrictions are placed on the players' directional choices. A directional triplet, however, can exist as a NE under constraints placed on the directions allowed to the players. This is in accordance with Eisert et al.'s result in Ref. [1], showing that a pair of unitary transformations $(\hat{Q}, \hat{Q})$, where $\hat{Q} \sim \hat{U}(0, \pi/2)$, exists as a NE in PD when the players' allowed actions are restricted to certain subsets of the set SU(2) consisting of all unitary transformations.

As is known [21,22] that the particular subset of unitary transformations that Eisert et al. used in order to obtain the NE of $(\hat{Q}, \hat{Q})$ in two-player quantum Prisoners' Dilemma is not even closed under composition. In particular, in Eisert et al.'s [1] protocol for 2×2 quantum games, the new Nash equilibria, and the classical–quantum transitions that occur are the outcomes of the particular strategy space chosen that is a two-parameter subset of single qubit unitary operators. By choosing a different, but equally plausible, two-parameter strategy a different Nash equilibria with different classical–quantum transitions can arise.

Using an EPR setting, and a shared GHZ state, for a three-player quantum Prisoners' Dilemma game, we present an approach that is driven along purely probabilistic lines with only an implicit reference to the mathematical formalism of quantum theory and showing the constraints on the players' directional choices under which a particular triplet can exist as a NE in the game.

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