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Two-player quantum games: When player strategies are via directional choices

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Abstract
We propose a scheme for a quantum game based on performing an EPR-type experiment and in which each player's spatial directional choices are considered as their strategies. A classical mixed-strategy game is recovered by restricting the players' choices to specific spatial trajectories. We show that for players' directional choices for which the Bell-CHSH inequality is violated, the players' payoffs in the quantum game have no mapping within the classical mixed-strategy game. The scheme provides a more direct link between classical and quantum games.

Keywords Quantum games · EPR experiments · Nash equilibria · Quantum probability

1 Introduction

Broadly speaking, a quantum game [1–4] can be considered as a game [5–7] in which a player's payoff relations involve a set of quantum probabilities [8] that are obtained from each player's strategic actions or strategies. For instance, in the quantum version of a 2×2 game proposed in the Eisert Wilkens Lewenstein (EWL) scheme [2, 3], each player's strategies are local unitary transformations performed on a maximally entangled state. The state evolves unitarily, and the set of quantum probabilities is obtained by projecting the final quantum state of the game to a basis in $2 \otimes 2$ Hilbert space, in terms of which the payoff relations for each player are then expressed. Quantum games are surveyed in Refs. [9, 10] and recent works in this area are in Refs. [12–20]. An extensive list of articles in this area are in Ref. [11].

A strategy profile is a Nash equilibrium (NE) [5–7]—with one strategy associated with each player—such that there remains no motivation for any player for unilater-

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ally deviating from that profile. In the EWL scheme, a NE is a set of local unitary transformations that satisfies the Nash conditions.

A quantization scheme can be viewed as a mechanism that establishes a convincing link between each player's strategies—quantum or classical—and a set of quantum probabilities, obtained from the players' strategies, and in terms of which each player's payoffs are then expressed. As players have access to much larger strategy sets in EWL scheme—relative to the strategy sets available to them in the classical game—Enk and Pike [21] argued that a quantum game in that scheme can be considered as an extended classical game. They argued that the quantized version of a game, in EWL scheme, solves a new classical game—with players' strategy sets extended—without solving the dilemma within the original game. This led to suggestions for using EPR-type experiment [8, 22–26] in constructing quantum games [27–33] and in which each player's strategy set remain classical while resulting in a set of quantum probabilities—thus circumventing Enk and Pike's argument.

It appears to us that historically there have been two distinct approaches in the literature in the area of quantum games. The first approach considers specially designed classical games, for instance, the game proposed by Vaidman in Ref. [4] that involves a winning condition, in which a quantum advantage can be demonstrated directly. The second approach, however, develops quantization procedures for a whole class of classical games, as reported in Refs. [2, 3]. The second approach is distinct from the first; in that a game is not designed in order to demonstrate an advantage in its quantum mechanical implementation—usually tied to crafting a winning condition—but the objective, instead, is to determine how an implementation that allows access to the resources of quantum superposition and entanglement, resulting in a different outcome of the game. The present paper is along the lines of the second approach.

Non-cooperative games using a tripartite EPR experiment with GHZ states are discussed in Refs. [28, 30], and in references therein. A tripartite EPR setting using GHZ states is considered in Ref. [33] that presents a quantum version of a three player non-cooperative game. Each player's strategic choices are three directions $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$, and $\hat{\mathbf{c}}$ along which the dichotomic observables $\mathbf{n} \cdot \boldsymbol{\sigma}$ are measured, where $\mathbf{n} = \hat{\mathbf{a}}, \hat{\mathbf{b}}, \hat{\mathbf{c}}$ and $\boldsymbol{\sigma}$ is a vector whose components are the standard Pauli matrices σ_x, σ_y, and σ_z.

In this paper, we present a scheme for playing a two-player quantum game in which each player's (classical) strategy sets—consist of orientating his/her unit vector along any direction in three dimensions—and dichotomic measurement outcomes of ± 1 along those directions. This scheme therefore uses each player's classical strategies to obtain a set of quantum probabilities in terms of which each player's payoff relations are then expressed. As the players' strategies are directional choices, Nash equilibria of the game emerge as directional pairs. For the players' directional choices for which the Bell-CHSH inequality is violated, the payoffs in the quantum game cannot be mapped to a classical mixed-strategy game. As the players in our scheme have access to classical strategy sets, it provides a more direct link between classical and quantum games.

The mixed-strategy version of a classical game is to be faithfully imbedded within the corresponding quantum game. When each player's strategies are spatial directions, we find that requiring a classical mixed strategy game to be imbedded in the corresponding quantum game results in placing constraints on each player's available

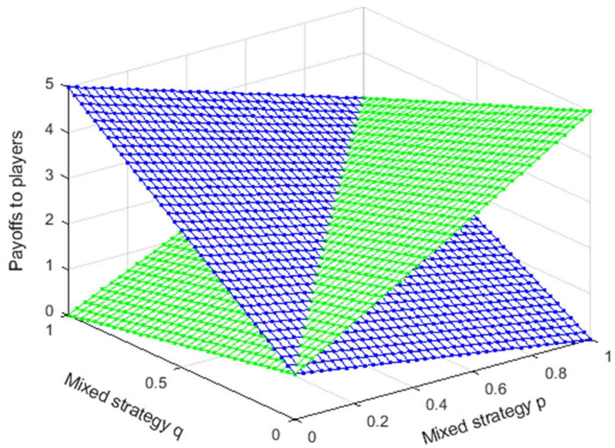

Fig. 1 Plots of the mixed-strategy payoff relations of Eq. (2) with $\alpha = 3$, $\beta = 0$, $\gamma = 5$, and $\delta = 1$ for the Prisoners' Dilemma game. Here, p and q are independent variables in the horizontal plane and the blue plane represents Alice's payoff, whereas the green plane represents Bob's payoff

directional choices. That is, we place restrictions on allowed trajectories on the surface of a unit sphere of the heads of the unit vectors representing each player's strategies.

2 Quantized Prisoners' Dilemma game

Consider the symmetric bimatrix game

$$\begin{array}{cc} & \text{Bob} \\ & \begin{array}{cc} S_1' & S_2' \end{array} \\ \text{Alice} \begin{array}{c} S_1 \\ S_2 \end{array} & \begin{array}{cc} (\alpha, \alpha) & (\beta, \gamma) \\ (\gamma, \beta) & (\delta, \delta) \end{array} \end{array} \tag{1}$$

in which S_1 and S_2 are Alice's moves and S_1' and S_2' are Bob's pure strategies and the entries in the brackets are the players' payoffs. For instance, when Alice plays S_1, whereas Bob plays S_2', Alice's payoff is β and Bob's payoff is γ. Let the players have access to mixed strategies and p is Alice's probability of playing S_1, and thus $(1 - p)$ is the probability of she playing S_2. Likewise, q is Bob's probability of playing S_1', and thus $(1 - q)$ is the probability of he playing S_2'. For the game matrix (1), each players' payoffs in the mixed-strategy game are then obtained as

$$\begin{aligned} \Pi_{\mathrm{A}}(p, q) &= \alpha pq + \beta p(1-q) + \gamma(1-p)q + \delta(1-p)(1-q), \\ \Pi_{\mathrm{B}}(p, q) &= \alpha pq + \gamma p(1-q) + \beta(1-p)q + \delta(1-p)(1-q), \end{aligned} \tag{2}$$

where subscripts A and B are for Alice and Bob, respectively.

For the strategy pair (p^*, q^*) to be a NE—corresponding to the two players—neither player is left with any motivation to unilaterally deviate from it, and this is defined by Nash inequalities

$$\Pi_A(p^*, q^*) - \Pi_A(p, q^*) \geq 0, \quad \Pi_B(p^*, q^*) - \Pi_B(p^*, q) \geq 0. \tag{3}$$

For the game of Prisoners' Dilemma considered in Ref. [2], we have

$$\alpha = 3, \quad \beta = 0, \quad \gamma = 5, \quad \delta = 1, \tag{4}$$

and the inequalities (3) result in obtaining $p^* = 0 = q^*$ and (S_2, S_2') emerging as the unique NE of the game at which $\Pi_{A,B}(0, 0) = 1$.

2.1 EWL scheme

In the quantized version of the game (1) developed in Ref. [2]—henceforth referred to as the EWL scheme—each player's strategies consist of local unitary transformations performed on a maximally entangled state. The state evolves and after passing through an unentangling gate, it is measured in a suitable basis. The game (1) is played with two qubits whose quantum state is described in a $2 \otimes 2$ dimensional Hilbert space.

For this game, a measurement basis for the quantum state of two qubits is chosen as $\left|S_1 S_1'\right\rangle$, $\left|S_1 S_2'\right\rangle$, $\left|S_2 S_1'\right\rangle$, $\left|S_2 S_2'\right\rangle$. An entangled initial quantum state $|\psi_i\rangle$ is obtained by using a two-qubit entangling gate $\hat{J}$, i.e., $|\psi_i\rangle = \hat{J}\left|S_1 S_1'\right\rangle$ where $\hat{J} = \exp\left\{i\gamma S_2 \otimes S_2'/2\right\}$ and $\gamma \in [0, \pi/2]$ is a measure of the game's entanglement. A separable or a product game has $\gamma = 0$, whereas a maximally entangled game has $\gamma = \pi/2$. The players perform their local unitary transformations $\hat{U}_A$ and $\hat{U}_B$ on an initial maximally entangled state $|\psi_i\rangle$. The transformations $\hat{U}_A$ and $\hat{U}_B$ were from the set

$$U(\theta, \phi) = \begin{pmatrix} e^{i\phi}\cos(\theta/2) & \sin(\theta/2) \\ -\sin(\theta/2) & e^{-i\phi}\cos(\theta/2) \end{pmatrix}, \tag{5}$$

where $\theta \in [0, \pi]$, $\phi \in [0, \pi/2]$. Note that EWL defined the unitary operator $\hat{J} = \exp\{i\gamma S_2 \otimes S_2/2\}$ with $\gamma \in [0, \pi/2]$ representing a measure of the game's entanglement. Each player's actions change $|\psi_i\rangle$ to $(\hat{U}_A \otimes \hat{U}_B)\hat{J}\left|S_1 S_1'\right\rangle$ and the state then passes through an untangling gate $\hat{J}^\dagger$ and the state changes to the final state, i.e., $\left|\psi_f\right\rangle = \hat{J}^\dagger(\hat{U}_A \otimes \hat{U}_B)\hat{J}\left|S_1 S_1'\right\rangle$. The state $\left|\psi_f\right\rangle$ is now measured in the basis $\left|S_1 S_1'\right\rangle$, $\left|S_1 S_2'\right\rangle$, $\left|S_2 S_1'\right\rangle$, $\left|S_2 S_2'\right\rangle$. With the quantum probability rule, the players' payoffs are then obtained as

$$\begin{aligned} \Pi_A(\hat{U}_A, \hat{U}_B) &= \alpha \left|\left\langle S_1 S_1' \mid \psi_f\right\rangle\right|^2 + \beta \left|\left\langle S_1 S_2' \mid \psi_f\right\rangle\right|^2 + \gamma \left|\left\langle S_2 S_1' \mid \psi_f\right\rangle\right|^2 + \delta \left|\left\langle S_2 S_2' \mid \psi_f\right\rangle\right|^2, \\ \Pi_B(\hat{U}_A, \hat{U}_B) &= \alpha \left|\left\langle S_1 S_1' \mid \psi_f\right\rangle\right|^2 + \gamma \left|\left\langle S_1 S_2' \mid \psi_f\right\rangle\right|^2 + \beta \left|\left\langle S_2 S_1' \mid \psi_f\right\rangle\right|^2 + \delta \left|\left\langle S_2 S_2' \mid \psi_f\right\rangle\right|^2. \end{aligned} \tag{6}$$

As discussed above, Eq. (6) show the link that this quantization scheme establishes between each player's strategies—consisting of unitary transformations—and the set of four quantum probabilities, i.e., $\left|\left\langle S_1 S_1' \mid \psi_f\right\rangle\right|^2$, $\left|\left\langle S_1 S_2' \mid \psi_f\right\rangle\right|^2$, $\left|\left\langle S_2 S_1' \mid \psi_f\right\rangle\right|^2$, and $\left|\langle S_2 S_2' \mid \psi_f \rangle\right|^2$. The NE for the quantum game consists of a pair $(\hat{U}_A^*, \hat{U}_B^*)$—corresponding to the two players—of local unitary transformations that satisfy the inequalities

$$\Pi_A(\hat{U}_A^*, \hat{U}_B^*) - \Pi_A(\hat{U}_A, \hat{U}_B^*) \geq 0, \quad \Pi_B(\hat{U}_A^*, \hat{U}_B^*) - \Pi_B(\hat{U}_A^*, \hat{U}_B) \geq 0. \tag{7}$$

That is, it is a pair $(\hat{U}_A^*, \hat{U}_B^*)$ from which any unilateral deviation no longer improves player payoff. For (4), a unique quantum NE $(\hat{Q}, \hat{Q})$ was realized where $\hat{Q} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \hat{U}(0, \pi/2)$. Benjamin and Hayden [34] noted that when their two-parameter set is extended to include all local unitary operations, i.e., all of $SU(2)$ [8], the strategy $\hat{Q}$ does not remain an equilibrium, and in the full space of deterministic quantum strategies, there exists no equilibrium for the quantum Prisoners' Dilemma. This was also discussed further in Ref. [35].

3 Quantum probabilities from players' directional choices

In EWL scheme, the players' unitary transformations $\hat{U}_A$ and $\hat{U}_B$ along with the subsequent quantum measurements result in the quantum probability set:

$$\left|\langle S_1 S_1' \mid \psi_f \rangle\right|^2, \quad \left|\langle S_1 S_2' \mid \psi_f \rangle\right|^2, \quad \left|\langle S_2 S_1' \mid \psi_f \rangle\right|^2, \quad \text{and} \quad \left|\langle S_2 S_2' \mid \psi_f \rangle\right|^2. \tag{8}$$

The players' payoff relations (6) are then expressed as expectation values of entries in the game matrix (1) over the quantum probability set (8).

For a three-player symmetric game, a more direct approach in obtaining a set of quantum probabilities is proposed in Ref. [33]. More specifically, this approach cosiders tripartite EPR experiment performed on a GHZ state as a three-player non-cooperative quantum game. Each player's strategies are the three directions $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$, and $\hat{\mathbf{c}}$ along which the dichotomic observables $\hat{\mathbf{n}} \cdot \boldsymbol{\sigma}$ are measured, with the eigenvalues $+1$ or -1 where $\hat{\mathbf{n}} = \hat{\mathbf{a}}, \hat{\mathbf{b}}$, or $\hat{\mathbf{c}}$ and $\boldsymbol{\sigma}$ is a vector whose components are the standard Pauli matrices σ_x, σ_y, and σ_z. A three-player quantum game is developed whose underlying setup is the tripartite EPR experiment.

In the present paper—instead of each player's strategies consisting of local unitary transformations $\hat{U}_A$ and $\hat{U}_B$—we consider player A and B strategies as their directional choices $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$. In an EPR setting, the measurement outcomes along $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ are denoted by $m = \pm 1$ and $n = \pm 1$, respectively. That is, the considered setting requires that a pair of unit vectors $(\hat{\mathbf{a}}, \hat{\mathbf{b}})$ results in a set of quantum probabilities:

$$\Pr_Q(S_1, S_1'), \quad \Pr_Q(S_1, S_2'), \quad \Pr_Q(S_2, S_1'), \quad \Pr_Q(S_2, S_2'), \tag{9}$$

where $\sum \Pr_Q(S_1, S_1') + \Pr_Q(S_1, S_2') + \Pr_Q(S_2, S_1') + \Pr_Q(S_2, S_2') = 1$. Now, acknowledging that there is no unique way in obtaining the set (9) from each players' strategies $(\hat{\mathbf{a}}, \hat{\mathbf{b}})$, we propose to obtain this set as follows

$$\Pr_Q(S_1, S_1') = \Pr_Q[(\hat{\mathbf{a}}, m = +1), (\hat{\mathbf{b}}, l = +1)], \quad \Pr_Q(S_1, S_2') = \Pr_Q[(\hat{\mathbf{a}}, m = +1), (\hat{\mathbf{b}}, l = -1)],$$
$$\Pr_Q(S_2, S_1') = \Pr_Q[(\hat{\mathbf{a}}, m = -1), (\hat{\mathbf{b}}, l = +1)], \quad \Pr_Q(S_2, S_2') = \Pr_Q[(\hat{\mathbf{a}}, m = -1), (\hat{\mathbf{b}}, l = -1)]. \tag{10}$$

For instance, $\Pr_Q(S_1, S_2')$ is the quantum probability that the polarization measurement along $\hat{\mathbf{a}}$ gives the outcome $m = +1$ and polarization measurement along $\hat{\mathbf{b}}$ gives the outcome $n = -1$.

The probabilities (10) are obtained as

$$\Pr_Q(S_1, S_1') = \left|\langle\psi_{ini} \mid (\mid \psi_{+1}\rangle_{\hat{\mathbf{a}}} \otimes |\psi_{+1}\rangle_{\hat{\mathbf{b}}})\right|^2 = \left|\left\langle\psi_{+1}^{\hat{\mathbf{a}}}\psi_{+1}^{\hat{\mathbf{b}}} \mid \psi_{\text{ini}}\right\rangle\right|^2,$$
$$\Pr_Q(S_1, S_2') = \left|\langle\psi_{ini} \mid (\mid \psi_{+1}\rangle_{\hat{\mathbf{a}}} \otimes |\psi_{-1}\rangle_{\hat{\mathbf{b}}})\right|^2 = \left|\left\langle\psi_{+1}^{\hat{\mathbf{a}}}\psi_{-1}^{\hat{\mathbf{b}}} \mid \psi_{\text{ini}}\right\rangle\right|^2,$$
$$\Pr_Q(S_2, S_1') = \left|\langle\psi_{ini} \mid (\mid \psi_{-1}\rangle_{\hat{\mathbf{a}}} \otimes |\psi_{+1}\rangle_{\hat{\mathbf{b}}})\right|^2 = \left|\left\langle\psi_{-1}^{\hat{\mathbf{a}}}\psi_{+1}^{\hat{\mathbf{b}}} \mid \psi_{\text{ini}}\right\rangle\right|^2,$$
$$\Pr_Q(S_2, S_2') = \left|\langle\psi_{ini} \mid (\mid \psi_{-1}\rangle_{\hat{\mathbf{a}}} \otimes |\psi_{-1}\rangle_{\hat{\mathbf{b}}})\right|^2 = \left|\left\langle\psi_{-1}^{\hat{\mathbf{a}}}\psi_{-1}^{\hat{\mathbf{b}}} \mid \psi_{\text{ini}}\right\rangle\right|^2, \tag{11}$$

and each players' payoff relations are then

$$\Pi_{\text{A}}(\hat{\mathbf{a}}, \hat{\mathbf{b}}) = \alpha \Pr_Q(S_1, S_1') + \beta \Pr_Q(S_1, S_2') + \gamma \Pr_Q(S_2, S_1') + \delta \Pr_Q(S_2, S_2'), \tag{12}$$
$$\Pi_{\text{B}}(\hat{\mathbf{a}}, \hat{\mathbf{b}}) = \alpha \Pr_Q(S_1, S_1') + \gamma \Pr_Q(S_1, S_2') + \beta \Pr_Q(S_2, S_1') + \delta \Pr_Q(S_2, S_2'). \tag{13}$$

A directional pair $(\hat{\mathbf{a}}^*, \hat{\mathbf{b}}^*)$ is a NE when the inequalities

$$\Pi_{\text{A}}(\hat{\mathbf{a}}^*, \hat{\mathbf{b}}^*) - \Pi_{\text{A}}(\hat{\mathbf{a}}, \hat{\mathbf{b}}^*) \geq 0, \quad \Pi_{\text{B}}(\hat{\mathbf{a}}^*, \hat{\mathbf{b}}^*) - \Pi_{\text{B}}(\hat{\mathbf{a}}^*, \hat{\mathbf{b}}) \geq 0, \tag{14}$$

are true for any directional choices $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ by players A and B, respectively.

Given each player's strategies consisting of directional choices in three dimensions, the classical mixed-strategy game is recovered from the quantum game if each player's directional choices consist of orientating their respective unit vectors $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ along specific trajectories on the surface of a unit sphere. When the players allow their respective unit vectors $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ to be orientated along directions beyond these trajectories, it results in obtaining the quantum game.

3.1 Orientating a unit vector considered as each player's strategy

In an EPR setting, we note that with player A's strategy $\hat{\mathbf{a}}$, the polarization (or spin) measurement results in the outcome $m = \pm 1$, and with the player B's strategy $\hat{\mathbf{b}}$, the polarization measurement results in the outcome $n = \pm 1$. We consider Pauli's matrices $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ in the eigenbasis $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$,

$|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$:

$$\sigma_x = |0\rangle\langle 1| + |1\rangle\langle 0|, \sigma_y = i(|1\rangle\langle 0| - |0\rangle\langle 1|), \sigma_z = |0\rangle\langle 0| - |1\rangle\langle 1|, \tag{15}$$

with $\boldsymbol{\sigma} = \sigma_x \hat{\imath} + \sigma_y \hat{\jmath} + \sigma_z \hat{k}$ and $\hat{\mathbf{a}} = a_x \hat{\imath} + a_y \hat{\jmath} + a_z \hat{k}$, we have $\boldsymbol{\sigma} \cdot \hat{\mathbf{a}} = a_x\sigma_x + a_y\sigma_y + a_z\sigma_z$, $\boldsymbol{\sigma} \cdot \hat{\mathbf{b}} = b_x\sigma_x + b_y\sigma_y + b_z\sigma_z$ that can be expressed in the diagonal form as $\boldsymbol{\sigma} \cdot \hat{\mathbf{a}} = (a_x - ia_y)|0\rangle\langle 1| + (a_x + ia_y)|1\rangle\langle 0| + a_z(|0\rangle\langle 0| - |1\rangle\langle 1|)$. Let $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ with $|\alpha|^2 + |\beta|^2 = 1$ be the eigenstate of $\boldsymbol{\sigma} \cdot \hat{\mathbf{a}}$ with the eigenvalue $k = \pm 1$, i.e., $(\boldsymbol{\sigma} \cdot \hat{\mathbf{a}})|\psi\rangle = k|\psi\rangle$ or $(\boldsymbol{\sigma} \cdot \hat{\mathbf{a}})(\alpha|0\rangle + \beta|1\rangle) = k(\alpha|0\rangle + \beta|1\rangle)$, or $(\boldsymbol{\sigma} \cdot \hat{\mathbf{a}})(\alpha|0\rangle + \beta|1\rangle) = [\alpha a_z + \beta(a_x - ia_y)]|0\rangle + [\alpha(a_x + ia_y) - \beta a_z]|1\rangle = k(\alpha|0\rangle + \beta|1\rangle)$ which gives $\alpha a_z + \beta(a_x - ia_y) = k\alpha$, $\alpha(a_x + ia_y) - \beta a_z = k\beta$, and the normalized eigenstates for A with eigenvalues $+1$ and -1, respectively, are

$$\begin{aligned} \left|\psi_{+1}^{\hat{\mathbf{a}}}\right\rangle &= \frac{1}{\sqrt{2}}[\sqrt{1+a_z}\,|0\rangle + \frac{a_x + ia_y}{\sqrt{1+a_z}}\,|1\rangle\,], \\ \left|\psi_{-1}^{\hat{\mathbf{a}}}\right\rangle &= \frac{1}{\sqrt{2}}[\sqrt{1-a_z}\,|0\rangle - \frac{a_x + ia_y}{\sqrt{1-a_z}}\,|1\rangle\,]. \end{aligned} \tag{16}$$

Likewise, the eigenstates for B with the eigenvalues $+1$ and -1, respectively, are

$$\begin{aligned} \left|\psi_{+1}^{\hat{\mathbf{b}}}\right\rangle &= \frac{1}{\sqrt{2}}[\sqrt{1+b_z}\,|0\rangle + \frac{b_x + ib_y}{\sqrt{1+b_z}}\,|1\rangle\,], \\ \left|\psi_{-1}^{\hat{\mathbf{b}}}\right\rangle &= \frac{1}{\sqrt{2}}[\sqrt{1-b_z}\,|0\rangle - \frac{b_x + ib_y}{\sqrt{1-b_z}}\,|1\rangle\,]. \end{aligned} \tag{17}$$

From these, we then obtain the eigenstates:

$$\begin{aligned} \left|\psi_{+1}^{\hat{\mathbf{a}}}\psi_{+1}^{\hat{\mathbf{b}}}\right\rangle = \frac{1}{2}[&\sqrt{(1+a_z)(1+b_z)}\,|00\rangle + \sqrt{\frac{1+a_z}{1+b_z}}(b_x + ib_y)\,|01\rangle \\ &+ \sqrt{\frac{1+b_z}{1+a_z}}(a_x + ia_y)\,|10\rangle + \frac{(a_x + ia_y)(b_x + ib_y)}{\sqrt{(1+a_z)(1+b_z)}}\,|11\rangle\,], \end{aligned} \tag{18}$$

$$\begin{aligned} \left|\psi_{+1}^{\hat{\mathbf{a}}}\psi_{-1}^{\hat{\mathbf{b}}}\right\rangle = \frac{1}{2}[&\sqrt{(1+a_z)(1-b_z)}\,|00\rangle - \sqrt{\frac{1+a_z}{1-b_z}}(b_x + ib_y)\,|01\rangle \\ &+ \sqrt{\frac{1-b_z}{1+a_z}}(a_x + ia_y)\,|10\rangle - \frac{(a_x + ia_y)(b_x + ib_y)}{\sqrt{(1+a_z)(1-b_z)}}\,|11\rangle\,], \end{aligned} \tag{19}$$

$$\left|\psi_{-1}^{\hat{\mathbf{a}}}\psi_{+1}^{\hat{\mathbf{b}}}\right\rangle = \frac{1}{2}[\sqrt{(1-a_z)(1+b_z)}\left|00\right\rangle + \sqrt{\frac{1-a_z}{1+b_z}}(b_x+ib_y)\left|01\right\rangle$$
$$-\sqrt{\frac{1+b_z}{1-a_z}}(a_x+ia_y)\left|10\right\rangle - \frac{(a_x+ia_y)(b_x+ib_y)}{\sqrt{(1-a_z)(1+b_z)}}\left|11\right\rangle], \quad (20)$$

$$\left|\psi_{-1}^{\hat{\mathbf{a}}}\psi_{-1}^{\hat{\mathbf{b}}}\right\rangle = \frac{1}{2}[\sqrt{(1-a_z)(1-b_z)}\left|00\right\rangle - \sqrt{\frac{1-a_z}{1-b_z}}(b_x+ib_y)\left|01\right\rangle$$
$$-\sqrt{\frac{1-b_z}{1-a_z}}(a_x+ia_y)\left|10\right\rangle + \frac{(a_x+ia_y)(b_x+ib_y)}{\sqrt{(1-a_z)(1-b_z)}}\left|11\right\rangle]. \quad (21)$$

For instance, the eigenstate (20) corresponds when player A's strategy consists of orientating her unit vector $\hat{\mathbf{a}}$ in one specific spatial direction, whereas player B's strategy consist of orientating his unit vector $\hat{\mathbf{b}}$ in other specific spatial direction and the measurement in an EPR setting generates -1 on A's side and $+1$ on B's side. Quantum probabilities $\Pr_Q(S_1, S_1')$, $\Pr_Q(S_1, S_2')$, $\Pr_Q(S_2, S_1')$, and $\Pr_Q(S_2, S_2')$ are determined from these eigenstates using Eq. (11). That is, with the players' directional choices $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$, the new basis consisting of the kets $\left|\psi_{+1}^{\hat{\mathbf{a}}}\psi_{+1}^{\hat{\mathbf{b}}}\right\rangle$, $\left|\psi_{+1}^{\hat{\mathbf{a}}}\psi_{-1}^{\hat{\mathbf{b}}}\right\rangle$, $\left|\psi_{-1}^{\hat{\mathbf{a}}}\psi_{+1}^{\hat{\mathbf{b}}}\right\rangle$, $\left|\psi_{-1}^{\hat{\mathbf{a}}}\psi_{-1}^{\hat{\mathbf{b}}}\right\rangle$ is prepared onto which the initial state is then projected, during the quantum measurement, to obtain the set of quantum probabilities.

Although the players' strategy sets consist of classical actions of rotating their respective unit vectors in three dimensions, the considered game is genuinely quantum mechanical because the player's payoff relations have an underlying set of quantum mechanical probabilities. In particular, the players have access to directional choices along which Bell's inequalities can be violated. This indicates genuinely quantum mechanical character of this scheme.

In the following, we present the resulting quantum games when the initial quantum states $\left|\psi_{\text{ini}}\right\rangle$ are the product state $\frac{1}{2}(\left|00\right\rangle + \left|01\right\rangle + \left|10\right\rangle + \left|11\right\rangle)$, the maximally entangled state $\frac{1}{\sqrt{2}}(\left|00\right\rangle + i\left|11\right\rangle)$, and the entangled state $\frac{1}{2}(\left|00\right\rangle + \left|01\right\rangle - \left|10\right\rangle + \left|11\right\rangle)$.

4 Game with the quantum state $\left|\psi_{\text{ini}}\right\rangle = \frac{1}{2}(\left|00\right\rangle + \left|01\right\rangle + \left|10\right\rangle + \left|11\right\rangle)$

For this state, we can write

$$\left|\psi_{\text{ini}}\right\rangle = \frac{1}{2}(\left|00\right\rangle + \left|01\right\rangle + \left|10\right\rangle + \left|11\right\rangle) = \frac{(\left|0\right\rangle + \left|1\right\rangle)_{\text{A}}}{\sqrt{2}} \otimes \frac{(\left|0\right\rangle + \left|1\right\rangle)_{\text{B}}}{\sqrt{2}}, \quad (22)$$

i.e., the state is a product state. For this state, we find

$$\Pr(\hat{\mathbf{a}}_{+1}, \hat{\mathbf{b}}_{+1}) = \left|\left\langle \psi_{+1}^{\hat{\mathbf{a}}}\psi_{+1}^{\hat{\mathbf{b}}} \mid \psi_{\text{ini}}\right\rangle\right|^2$$

$$
\begin{aligned}
&= \frac{1}{16(1+a_z)(1+b_z)}\{\left[(1+a_z)(1+b_z)+(1+a_z)b_x\right.\\
&\quad\left.+(1+b_z)a_x+(a_xb_x-a_yb_y)\right]^2\\
&\quad+\left[(1+a_z)b_y+(1+b_z)a_y+(a_xb_y+a_yb_x)\right]^2\},
\end{aligned} \tag{23}
$$

$$
\begin{aligned}
\Pr(\hat{\mathbf{a}}_{+1},\hat{\mathbf{b}}_{-1}) &= \left|\left\langle \psi^{\hat{\mathbf{a}}}_{+1}\psi^{\hat{\mathbf{b}}}_{-1} \mid \psi_{\text{ini}}\right\rangle\right|^2\\
&= \frac{1}{16(1+a_z)(1-b_z)}\{\left[(1+a_z)(1-b_z)-(1+a_z)b_x\right.\\
&\quad\left.+(1-b_z)a_x-(a_xb_x-a_yb_y)\right]^2\\
&\quad+\left[(1+a_z)b_y-(1-b_z)a_y+(a_xb_y+a_yb_x)\right]^2\},
\end{aligned} \tag{24}
$$

$$
\begin{aligned}
\Pr(\hat{\mathbf{a}}_{-1},\hat{\mathbf{b}}_{+1}) &= \left|\left\langle \psi^{\hat{\mathbf{a}}}_{-1}\psi^{\hat{\mathbf{b}}}_{+1} \mid \psi_{\text{ini}}\right\rangle\right|^2\\
&= \frac{1}{16(1-a_z)(1+b_z)}\{\left[(1-a_z)(1+b_z)+(1-a_z)b_x\right.\\
&\quad\left.-(1+b_z)a_x-(a_xb_x-a_yb_y)\right]^2\\
&\quad+\left[(1-a_z)b_y-(1+b_z)a_y-(a_xb_y+a_yb_x)\right]^2\},
\end{aligned} \tag{25}
$$

$$
\begin{aligned}
\Pr(\hat{\mathbf{a}}_{-1},\hat{\mathbf{b}}_{-1}) &= \left|\left\langle \psi^{\hat{\mathbf{a}}}_{-1}\psi^{\hat{\mathbf{b}}}_{-1} \mid \psi_{\text{ini}}\right\rangle\right|^2\\
&= \frac{1}{16(1-a_z)(1-b_z)}\{\left[(1-a_z)(1-b_z)-(1-a_z)b_x\right.\\
&\quad\left.-(1-b_z)a_x+(a_xb_x-a_yb_y)\right]^2\\
&\quad+\left[(1-a_z)b_y+(1-b_z)a_y-(a_xb_y+a_yb_x)\right]^2\}.
\end{aligned} \tag{26}
$$

The payoff to the players (12) can then be expressed as

$$
\begin{aligned}
\Pi_{\text{A,B}}(\hat{\mathbf{a}},\hat{\mathbf{b}}) = \frac{1}{16(1+a_z)}&\left[\frac{(\alpha,\alpha)}{(1+b_z)}\{\left[(1+a_z)(1+b_z)+(1+a_z)b_x\right.\right.\\
&\left.+(1+b_z)a_x+(a_xb_x-a_yb_y)\right]^2\\
&+\left[(1+a_z)b_y+(1+b_z)a_y+(a_xb_y+a_yb_x)\right]^2\}\\
&+\frac{(\beta,\gamma)}{(1-b_z)}\{\left[(1+a_z)(1-b_z)-(1+a_z)b_x+(1-b_z)a_x-(a_xb_x-a_yb_y)\right]^2\\
&\left.+\left[(1+a_z)b_y-(1-b_z)a_y+(a_xb_y+a_yb_x)\right]^2\}\right]\\
&+\frac{1}{16(1-a_z)}\left[\frac{(\gamma,\beta)}{(1+b_z)}\{\left[(1-a_z)(1+b_z)+(1-a_z)b_x\right.\right.\\
&\left.-(1+b_z)a_x-(a_xb_x-a_yb_y)\right]^2\\
&+\left[(1-a_z)b_y-(1+b_z)a_y-(a_xb_y+a_yb_x)\right]^2\}\\
&+\frac{(\delta,\delta)}{(1-b_z)}\{\left[(1-a_z)(1-b_z)-(1-a_z)b_x-(1-b_z)a_x+(a_xb_x-a_yb_y)\right]^2
\end{aligned}
$$

$$+\left[(1-a_z)b_y + (1-b_z)a_y - (a_x b_y + a_y b_x)\right]^2 \}\Big], \tag{27}$$

To convert to polar coordinates, we let $\hat{\mathbf{a}} = (\theta_\mathrm{A}, \phi_\mathrm{A})$ and $\hat{\mathbf{b}} = (\theta_\mathrm{B}, \phi_\mathrm{B})$ with $\theta_\mathrm{A}, \theta_\mathrm{B} \in [0, \pi]$ and $\phi_\mathrm{A}, \phi_\mathrm{B} \in [0, 2\pi)$ and have

$$\begin{aligned} a_x &= \sin\theta_\mathrm{A}\cos\phi_\mathrm{A}, \quad b_x = \sin\theta_\mathrm{B}\cos\phi_\mathrm{B} \\ a_y &= \sin\theta_\mathrm{A}\sin\phi_\mathrm{A}, \quad b_y = \sin\theta_\mathrm{B}\sin\phi_\mathrm{B} \\ a_z &= \cos\theta_\mathrm{A}, \quad b_z = \cos\theta_\mathrm{B}. \end{aligned} \tag{28}$$

This transformation reduces the independent variables $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ to $\theta_\mathrm{A}, \theta_\mathrm{B}, \phi_\mathrm{A}$, and ϕ_B and players payoffs are then expressed as

$$\begin{aligned} \Pi_{\mathrm{A,B}}(\theta_\mathrm{A}, \phi_\mathrm{A}; \theta_\mathrm{B}, \phi_\mathrm{B}) = \frac{1}{4}[&(\alpha, \alpha)(1+\sin\theta_\mathrm{A}\cos\phi_\mathrm{A})(1+\sin\theta_\mathrm{B}\cos\phi_\mathrm{B}) \\ &+(\beta, \gamma)(1+\sin\theta_\mathrm{A}\cos\phi_\mathrm{A})(1-\sin\theta_\mathrm{B}\cos\phi_\mathrm{B}) \\ &+(\gamma, \beta)(1-\sin\theta_\mathrm{A}\cos\phi_\mathrm{A})(1+\sin\theta_\mathrm{B}\cos\phi_\mathrm{B}) \\ &+(\delta, \delta)(1-\sin\theta_\mathrm{A}\cos\phi_\mathrm{A})(1-\sin\theta_\mathrm{B}\cos\phi_\mathrm{B})]. \end{aligned} \tag{29}$$

Note that EWL used the notation $\phi_{\mathrm{A,B}}$ to describe one of the two parameters in terms of which their (restricted) local unitary operators are defined. In this paper, we have used notation $\phi_{\mathrm{A,B}}$ when we change from Cartesian to spherical coordinates in accordance with Eq. (28), i.e., our context is different. Comparing Eq. (29) with Eq. (2), it is noticed that when we take

$$p = (1+\sin\theta_\mathrm{A}\cos\phi_\mathrm{A})/2, \quad q = (1+\sin\theta_\mathrm{B}\cos\phi_\mathrm{B})/2, \tag{30}$$

and thus

$$(1-p) = (1-\sin\theta_\mathrm{A}\cos\phi_\mathrm{A})/2, \quad (1-q) = (1-\sin\theta_\mathrm{B}\cos\phi_\mathrm{B})/2, \tag{31}$$

the quantum payoffs (29) are then reduced to players' classical mixed strategy payoffs (2).

This can be interpreted by stating that the quantum game considered here results in the classical mixed strategy game (in which Alice plays the strategy p, whereas Bob plays the strategy q) is obtained when the tips of Alice's and Bob's unit vectors (representing their strategic choices) are constrained to trajectories on a unit sphere that are defined by

$$\sin\theta_\mathrm{A}\cos\phi_\mathrm{A} = 2p-1, \qquad \sin\theta_\mathrm{B}\cos\phi_\mathrm{B} = 2q-1, \tag{32}$$

and the classical mixed strategy game is recovered by interpreting $\frac{(1+\cos\phi_\mathrm{A})}{2}$ and $\frac{(1+\cos\phi_\mathrm{B})}{2}$ in these equations as the probabilities p and q in the mixed strategy payoff relations (2). Here, $\frac{(1-\cos\phi_\mathrm{A})}{2}$ and $\frac{(1-\cos\phi_\mathrm{B})}{2}$ are then interpreted as $(1-p)$ and $(1-q)$ in (2).

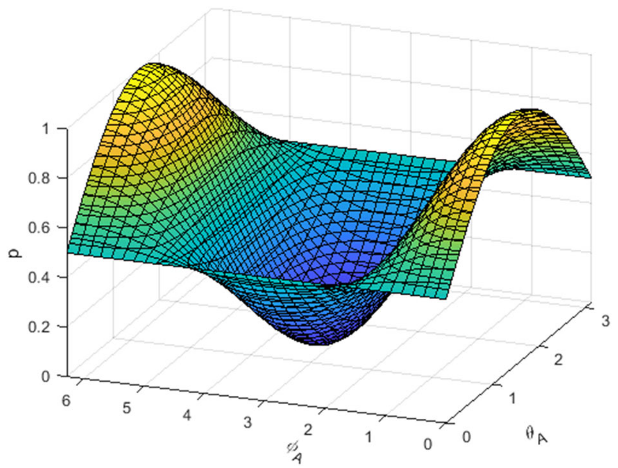

Fig. 2 Alice's mixed strategy p is plotted using Eq. (30) against θ_A and ϕ_A for the product state $|\psi_{\text{ini}}\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$

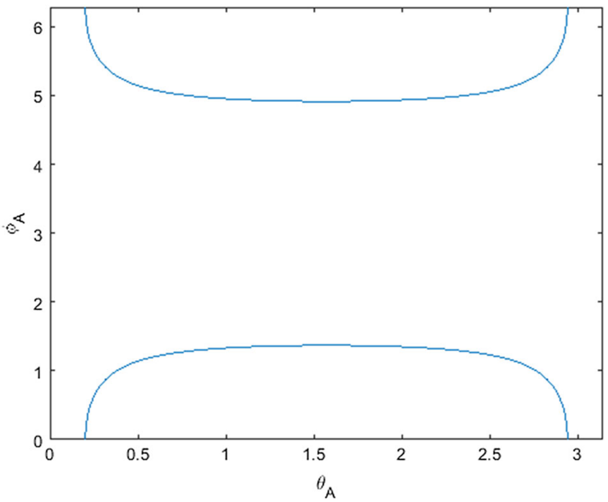

Fig. 3 Plot of θ_A against ϕ_A for $p = 0.6$ obtained from the first equation in (32)

Consider the Prisoners' Dilemma game, as defined by $\alpha = 3, \beta = 0, \gamma = 5, \delta = 1$ in the game matrix (1), a quantized version of which was considered in Ref. [2]. The strategy pair (p, q) is a NE in the classical game, and therefore $(\theta_{\text{A}}^*, \phi_{\text{A}}^*; \theta_{\text{B}}^*, \phi_{\text{B}}^*)$ is a NE for which

$$(1 + \sin\theta_{\text{A}}^* \cos\phi_{\text{A}}^*) = 0 = (1 + \sin\theta_{\text{B}}^* \cos\phi_{\text{B}}^*), \tag{33}$$

and we obtain the NE of the game as

$$(\theta_{\text{A}}^*, \phi_{\text{A}}^*; \theta_{\text{B}}^*, \phi_{\text{B}}^*) = (\pi/2, \pi; \pi/2, \pi), \tag{34}$$

at which the players' payoffs are $\Pi_{A,B}(\pi/2, \pi; \pi/2, \pi) = 1$. That is, playing the game with the state $|\psi_{\text{ini}}\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$ results in the classical mixed-strategy game.

5 Game with the quantum state $|\psi_{\text{ini}}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + i\,|11\rangle)$

For the maximally entangled state $|\psi_{\text{ini}}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + i\,|11\rangle)$ considered in Refs. [2, 3], following set of quantum probabilities are obtained

$$\begin{aligned}
\Pr(\hat{\mathbf{a}}_{+1}, \hat{\mathbf{b}}_{+1}) &= \left|\left\langle \psi^{\hat{\mathbf{a}}}_{+1}\psi^{\hat{\mathbf{b}}}_{+1} \mid \psi_{\text{ini}}\right\rangle\right|^2 = \frac{1}{8}\left|\sqrt{(1+a_z)(1+b_z)} + \frac{(a_x - ia_y)(b_x - ib_y)i}{\sqrt{(1+a_z)(1+b_z)}}\right|^2, \\
&= \frac{1}{4}(1 + a_x b_y + a_y b_x + a_z b_z), \\
\Pr(\hat{\mathbf{a}}_{+1}, \hat{\mathbf{b}}_{-1}) &= \left|\left\langle \psi^{\hat{\mathbf{a}}}_{+1}\psi^{\hat{\mathbf{b}}}_{-1} \mid \psi_{\text{ini}}\right\rangle\right|^2 = \frac{1}{8}\left|\sqrt{(1+a_z)(1-b_z)} - \frac{(a_x - ia_y)(b_x - ib_y)i}{\sqrt{(1+a_z)(1-b_z)}}\right|^2, \\
&= \frac{1}{4}(1 - a_x b_y - a_y b_x - a_z b_z), \\
\Pr(\hat{\mathbf{a}}_{-1}, \hat{\mathbf{b}}_{+1}) &= \left|\left\langle \psi^{\hat{\mathbf{a}}}_{-1}\psi^{\hat{\mathbf{b}}}_{+1} \mid \psi_{\text{ini}}\right\rangle\right|^2 = \frac{1}{8}\left|\sqrt{(1-a_z)(1+b_z)} - \frac{(a_x - ia_y)(b_x - ib_y)i}{\sqrt{(1-a_z)(1+b_z)}}\right|^2, \\
&= \frac{1}{4}(1 - a_x b_y - a_y b_x - a_z b_z), \\
\Pr(\hat{\mathbf{a}}_{-1}, \hat{\mathbf{b}}_{-1}) &= \left|\left\langle \psi^{\hat{\mathbf{a}}}_{-1}\psi^{\hat{\mathbf{b}}}_{-1} \mid \psi_{\text{ini}}\right\rangle\right|^2 = \frac{1}{8}\left|\sqrt{(1-a_z)(1-b_z)} + \frac{(a_x - ia_y)(b_x - ib_y)i}{\sqrt{(1-a_z)(1-b_z)}}\right|^2, \\
&= \frac{1}{4}(1 + a_x b_y + a_y b_x + a_z b_z).
\end{aligned} \tag{35}$$

To express these in polar coordinates, we use Eq. (28), and the quantum probabilities (35) are

$$\begin{aligned}
\Pr(\hat{\mathbf{a}}_{+1}, \hat{\mathbf{b}}_{+1}) &= \frac{1}{4}\{1 + \sin\theta_A \sin\theta_B \sin(\phi_A + \phi_B) + \cos\theta_A \cos\theta_B\}, \\
\Pr(\hat{\mathbf{a}}_{+1}, \hat{\mathbf{b}}_{-1}) &= \frac{1}{4}\{1 - \sin\theta_A \sin\theta_B \sin(\phi_A + \phi_B) - \cos\theta_A \cos\theta_B\}, \\
\Pr(\hat{\mathbf{a}}_{-1}, \hat{\mathbf{b}}_{+1}) &= \frac{1}{4}\{1 - \sin\theta_A \sin\theta_B \sin(\phi_A + \phi_B) - \cos\theta_A \cos\theta_B\}, \\
\Pr(\hat{\mathbf{a}}_{-1}, \hat{\mathbf{b}}_{-1}) &= \frac{1}{4}\{1 + \sin\theta_A \sin\theta_B \sin(\phi_A + \phi_B) + \cos\theta_A \cos\theta_B\}.
\end{aligned} \tag{36}$$

Players' payoffs are obtained as

$$\begin{aligned}
\Pi_{A,B}(\theta_A, \phi_A; \theta_B, \phi_B) &= \Pi(\theta_A, \phi_A; \theta_B, \phi_B) \\
&= (\alpha, \alpha)\Pr(\hat{\mathbf{a}}_{+1}, \hat{\mathbf{b}}_{+1}) + (\beta, \gamma)\Pr(\hat{\mathbf{a}}_{+1}, \hat{\mathbf{b}}_{-1})
\end{aligned}$$

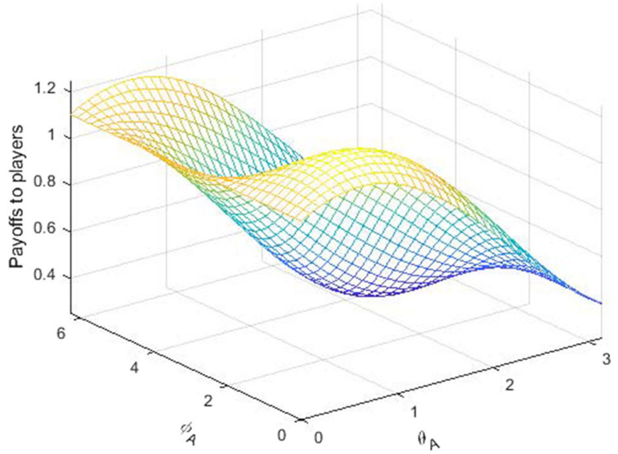

Fig. 4 At $\theta_B = \frac{\pi}{4} = \phi_B$ the payoff relation (37) is plotted against $\theta_A \in [0, \pi]$ and $\phi_B \in [0, 2\pi)$ for $\Delta_2 = 3$ and $\Delta_1 = 2$

$$+(\gamma, \beta)\Pr(\hat{\mathbf{a}}_{-1}, \hat{\mathbf{b}}_{+1}) + (\delta, \delta)\Pr(\hat{\mathbf{a}}_{-1}, \hat{\mathbf{b}}_{-1})$$
$$= \frac{1}{4}\{\Delta_2 + \Delta_1[\sin\theta_A \sin\theta_B \sin(\phi_A + \phi_B) + \cos\theta_A \cos\theta_B]\}, \tag{37}$$

where

$$\Delta_1 = \alpha - \beta - \gamma + \delta \quad \text{and} \quad \Delta_2 = \alpha + \beta + \gamma + \delta. \tag{38}$$

We note that these payoffs cannot be reduced to the mixed strategy payoffs of Eq. (2). Stated alternatively, there do not exist such trajectories for the tips of the players' unit vectors which if followed would result in the mixed-strategy version of the classical game. To determine the NE $(\theta_A^*, \phi_A^*; \theta_B^*, \phi_B^*)$, we require

$$\Pi(\theta_A^*, \phi_A^*; \theta_B^*, \phi_B^*) - \Pi(\theta_A, \phi_A^*; \theta_B^*, \phi_B^*) = (\theta_A^* - \theta_A)\frac{\partial \Pi}{\partial \theta_A} |_*$$
$$= \frac{1}{4}\Delta_1[\cos\theta_A^* \sin\theta_B^* \sin(\phi_A^* + \phi_B^*) - \sin\theta_A^* \cos\theta_B^*](\theta_A^* - \theta_A) \geq 0,$$
$$\Pi(\theta_A^*, \phi_A^*; \theta_B^*, \phi_B^*) - \Pi(\theta_A^*, \phi_A^*; \theta_B, \phi_B^*) = (\theta_B^* - \theta_B)\frac{\partial \Pi}{\partial \theta_B} |_*$$
$$= \frac{1}{4}\Delta_1[\sin\theta_A^* \cos\theta_B^* \sin(\phi_A^* + \phi_B^*) - \cos\theta_A^* \sin\theta_B^*](\theta_B^* - \theta_B) \geq 0,$$
$$\Pi(\theta_A^*, \phi_A^*; \theta_B^*, \phi_B^*) - \Pi(\theta_A^*, \phi_A; \theta_B^*, \phi_B^*) = (\phi_A^* - \phi_A)\frac{\partial \Pi}{\partial \phi_A} |_*$$
$$= \frac{1}{4}\Delta_1[\sin\theta_A^* \sin\theta_B^* \cos(\phi_A^* + \phi_B^*)](\phi_A^* - \phi_A) \geq 0,$$

$$\Pi(\theta_A^*, \phi_A^*; \theta_B^*, \phi_B^*) - \Pi(\theta_A^*, \phi_A^*; \theta_B^*, \phi_B) = (\phi_B^* - \phi_B)\frac{\partial \Pi}{\partial \phi_B} \mid_*$$
$$= \frac{1}{4}\Delta_1[\sin\theta_A^* \sin\theta_B^* \cos(\phi_A^* + \phi_B^*)](\phi_B^* - \phi_B) \geq 0. \tag{39}$$

Now, consider the case when only equalities are involved in the above expressions, i.e.,

$$\cos\theta_A^* \sin\theta_B^* \sin(\phi_A^* + \phi_B^*) - \sin\theta_A^* \cos\theta_B^* = 0, \tag{40}$$
$$\sin\theta_A^* \cos\theta_B^* \sin(\phi_A^* + \phi_B^*) - \cos\theta_A^* \sin\theta_B^* = 0, \tag{41}$$
$$\sin\theta_A^* \sin\theta_B^* \cos(\phi_A^* + \phi_B^*) = 0. \tag{42}$$

As $\theta_A, \theta_B \in [0, \pi]$, these equations would hold true when $\theta_A^*, \theta_B^* = 0, \pi$ and for any ϕ_A, ϕ_B. There both players' payoffs are obtained from Eq. (37) as

$$\Pi_{A,B}(\theta_A^*, \phi_A; \theta_B^*, \phi_B) = \frac{1}{4}\{\Delta_2 + \Delta_1 \cos\theta_A^* \cos\theta_B^*\} = \frac{1}{2}(\alpha + \delta), \quad \frac{1}{2}(\beta + \gamma). \tag{43}$$

Alternatively, Eq. (42) holds when $\cos(\phi_A^* + \phi_B^*) = 0$, i.e., $\sin(\phi_A^* + \phi_B^*) = \pm 1$.

We note that for $\sin(\phi_A^* + \phi_B^*) = +1$, Eqs. (40, 41) give $\sin(\theta_A^* - \theta_B^*) = 0$ or $\theta_A^* - \theta_B^* = 0, \pm\pi$. For this NE, both players' payoffs are obtained from Eq. (37) as

$$\Pi_{A,B}(\theta_A^*, \phi_A^*; \theta_B^*, \phi_B^*) = \frac{1}{4}\{\Delta_2 + \Delta_1 \cos(\theta_A^* - \theta_B^*)\} = \frac{1}{2}(\alpha + \delta), \quad \frac{1}{2}(\beta + \gamma). \tag{44}$$

However, for $\sin(\phi_A^* + \phi_B^*) = -1$, Eqs. (40, 41) give $\sin(\theta_A^* + \theta_B^*) = 0$ or $\theta_A^* + \theta_B^* = 0, \pi, 2\pi$. For this NE, both players' payoffs are then obtained from Eq. (37) as

$$\Pi_{A,B}(\theta_A^*, \phi_A^*; \theta_B^*, \phi_B^*) = \frac{1}{4}\{\Delta_2 + \Delta_1 \cos(\theta_A^* + \theta_B^*)\} = \frac{1}{2}(\alpha + \delta), \quad \frac{1}{2}(\beta + \gamma). \tag{45}$$

Therefore, for all these equilibria, both players' payoffs are same, i.e., either $\frac{1}{2}(\alpha + \delta)$ or $\frac{1}{2}(\beta + \gamma)$. We also note that for the edges located at

$$\begin{gathered}(0, 0; 0, 0), (0, 0; 0, 2\pi), (0, 2\pi; 0, 0), (0, 2\pi; 0, 2\pi);\\ (0, 0; \pi, 0), (0, 0; \pi, 2\pi), (0, 2\pi; \pi, 0), (0, 2\pi; \pi, 2\pi);\\ (\pi, 0; 0, 0), (\pi, 0; 0, 2\pi), (\pi, 2\pi; 0, 0), (\pi, 2\pi; 0, 2\pi);\\ (\pi, 0; \pi, 0), (\pi, 0; \pi, 2\pi), (\pi, 2\pi; \pi, 0), (\pi, 2\pi; \pi, 2\pi),\end{gathered} \tag{46}$$

and we have $\theta_A^*, \theta_B^* = 0$ or π, and therefore $\sin\theta_A^* = 0 = \sin\theta_B^*$. That is, Eqs. (40, 41, 42) are true for all these edges, and both players' payoffs at these are the same as given by Eq. (45).

6 Game with the quantum state $|\psi_{\text{ini}}\rangle = \frac{1}{2}(|00\rangle + |01\rangle - |10\rangle + |11\rangle)$

This is an entangled state for which we find

$$
\begin{aligned}
\Pr(\hat{\mathbf{a}}_{+1}, \hat{\mathbf{b}}_{+1}) &= \left|\left\langle \psi_{+1}^{\hat{\mathbf{a}}}\psi_{+1}^{\hat{\mathbf{b}}} \mid \psi_{\text{ini}}\right\rangle\right|^2 \\
&= \frac{1}{16(1+a_z)(1+b_z)}\{\left[(1+a_z)(1+b_z) + (1+a_z)b_x\right.\\
&\quad \left. -(1+b_z)a_x + (a_x b_x - a_y b_y)\right]^2 + \\
&\quad \left[(1+a_z)b_y - (1+b_z)a_y + (a_x b_y + a_y b_x)\right]^2\}, \\
\Pr(\hat{\mathbf{a}}_{+1}, \hat{\mathbf{b}}_{-1}) &= \left|\left\langle \psi_{+1}^{\hat{\mathbf{a}}}\psi_{-1}^{\hat{\mathbf{b}}} \mid \psi_{\text{ini}}\right\rangle\right|^2 \\
&= \frac{1}{16(1+a_z)(1-b_z)}\{\left[(1+a_z)(1-b_z) - (1+a_z)b_x\right.\\
&\quad \left. -(1-b_z)a_x - (a_x b_x - a_y b_y)\right]^2 \\
&\quad + \left[(1+a_z)b_y + (1-b_z)a_y + (a_x b_y + a_y b_x)\right]^2\}, \\
\Pr(\hat{\mathbf{a}}_{-1}, \hat{\mathbf{b}}_{+1}) &= \left|\left\langle \psi_{-1}^{\hat{\mathbf{a}}}\psi_{+1}^{\hat{\mathbf{b}}} \mid \psi_{\text{ini}}\right\rangle\right|^2 \\
&= \frac{1}{16(1-a_z)(1+b_z)}\{\left[(1-a_z)(1+b_z) + (1-a_z)b_x\right.\\
&\quad \left. +(1+b_z)a_x - (a_x b_x - a_y b_y)\right]^2 \\
&\quad + \left[(1-a_z)b_y + (1+b_z)a_y - (a_x b_y + a_y b_x)\right]^2\}, \\
\Pr(\hat{\mathbf{a}}_{-1}, \hat{\mathbf{b}}_{-1}) &= \left|\left\langle \psi_{-1}^{\hat{\mathbf{a}}}\psi_{-1}^{\hat{\mathbf{b}}} \mid \psi_{\text{ini}}\right\rangle\right|^2 \\
&= \frac{1}{16(1-a_z)(1-b_z)}\{\left[(1-a_z)(1-b_z) - (1-a_z)b_x\right.\\
&\quad \left. +(1-b_z)a_x + (a_x b_x - a_y b_y)\right]^2 \\
&\quad + \left[(1-a_z)b_y - (1-b_z)a_y - (a_x b_y + a_y b_x)\right]^2\}. \qquad (47)
\end{aligned}
$$

The transformations (28) reduce the independent variables $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ to $\theta_{\text{A}}, \theta_{\text{B}}, \phi_{\text{A}}$, and ϕ_{B} and players payoffs can be expressed as

$$
\begin{aligned}
&\Pi_{\text{A,B}}(\theta_{\text{A}}, \phi_{\text{A}}; \theta_{\text{B}}, \phi_{\text{B}}) \\
&= \frac{1}{4}\{\Delta_2 - \Delta_1[\sin\theta_{\text{A}}\sin\theta_{\text{B}}\sin\phi_{\text{A}}\sin\phi_{\text{B}} \\
&\quad + \sin\theta_{\text{A}}\cos\theta_{\text{B}}\cos\phi_{\text{A}} - \cos\theta_{\text{A}}\sin\theta_{\text{B}}\cos\phi_{\text{B}}]\}. \qquad (48)
\end{aligned}
$$

We note that as was the case for the state $\frac{1}{\sqrt{2}}(|00\rangle + i\,|11\rangle)$, these payoffs cannot be reduced to the classical mixed strategy payoffs in the game. In other words, there do not exist such trajectories for the tips of each players' unit vectors which if followed can result in the classical mixed-strategy game. To determine the NE $(\theta_A^*, \phi_A^*; \theta_B^*, \phi_B^*)$, we require

$$
\begin{aligned}
&\Pi(\theta_A^*, \phi_A^*; \theta_B^*, \phi_B^*) - \Pi(\theta_A, \phi_A^*; \theta_B^*, \phi_B^*) \\
&\quad = (\theta_A^* - \theta_A)\frac{\partial \Pi}{\partial \theta_A} \mid_* \\
&\quad = -\frac{1}{4}\Delta_1[\cos\theta_A^* \sin\theta_B^* \sin\phi_A^* \sin\phi_B^* + \cos\theta_A^* \cos\theta_B^* \cos\phi_A^* \\
&\qquad + \sin\theta_A^* \sin\theta_B^* \cos\phi_B^*](\theta_A^* - \theta_A) \geq 0, \\
&\Pi(\theta_A^*, \phi_A^*; \theta_B^*, \phi_B^*) - \Pi(\theta_A^*, \phi_A^*; \theta_B, \phi_B^*) \\
&\quad = (\theta_B^* - \theta_B)\frac{\partial \Pi}{\partial \theta_B} \mid_* \\
&\quad = -\frac{1}{4}\Delta_1[\sin\theta_A^* \cos\theta_B^* \sin\phi_A^* \sin\phi_B^* - \sin\theta_A^* \sin\theta_B^* \cos\phi_A^* \\
&\qquad - \cos\theta_A^* \cos\theta_B^* \cos\phi_B^*](\theta_B^* - \theta_B) \geq 0, \\
&\Pi(\theta_A^*, \phi_A^*; \theta_B^*, \phi_B^*) - \Pi(\theta_A^*, \phi_A; \theta_B^*, \phi_B^*) \\
&\quad = (\phi_A^* - \phi_A)\frac{\partial \Pi}{\partial \phi_A} \mid_* \\
&\quad = -\frac{1}{4}\Delta_1[\sin\theta_A^* \sin\theta_B^* \cos\phi_A^* \sin\phi_B^* \\
&\qquad - \sin\theta_A^* \cos\theta_B^* \sin\phi_A^*](\phi_A^* - \phi_A) \geq 0, \\
&\Pi(\theta_A^*, \phi_A^*; \theta_B^*, \phi_B^*) - \Pi(\theta_A^*, \phi_A^*; \theta_B^*, \phi_B) \\
&\quad = (\phi_B^* - \phi_B)\frac{\partial \Pi}{\partial \phi_B} \mid_* \\
&\quad = -\frac{1}{4}\Delta_1[\sin\theta_A^* \sin\theta_B^* \sin\phi_A^* \cos\phi_B^* \\
&\qquad + \cos\theta_A^* \sin\theta_B^* \sin\phi_B^*](\phi_B^* - \phi_B) \geq 0.
\end{aligned}
\tag{49}
$$

We firstly consider the case when only equalities are involved in the above expressions, i.e.,

$$
\begin{aligned}
&\sin\theta_B^*(\cos\theta_A^* \sin\phi_A^* \sin\phi_B^* + \sin\theta_A^* \cos\phi_B^*) + \cos\theta_A^* \cos\theta_B^* \cos\phi_A^* = 0, \\
&\sin\theta_A^*(\cos\theta_B^* \sin\phi_A^* \sin\phi_B^* - \sin\theta_B^* \cos\phi_A^*) - \cos\theta_A^* \cos\theta_B^* \cos\phi_B^* = 0, \\
&\sin\theta_A^*(\sin\theta_B^* \cos\phi_A^* \sin\phi_B^* - \cos\theta_B^* \sin\phi_A^*) = 0, \\
&\sin\theta_B^*(\sin\theta_A^* \sin\phi_A^* \cos\phi_B^* + \cos\theta_A^* \sin\phi_B^*) = 0,
\end{aligned}
\tag{50}
$$

where $\theta_A, \theta_B \in [0, \pi]$ and $\phi_A, \phi_B \in [0, 2\pi)$. Now, we consider the following cases:

6.1 Case $\sin\theta_A^* = 0 = \sin\theta_B^*$

For $\sin\theta_A^* = 0 = \sin\theta_B^*$, we also have $\cos\theta_A^* = \pm 1$ and $\cos\theta_B^* = \pm 1$ and this results the first two equation in (50) to give $\pm\cos\phi_A^* = 0$ and $\pm\cos\phi_B^* = 0$. This gives

$$\theta_A^* = 0, \pi; \qquad \theta_B^* = 0, \pi; \qquad \phi_A^* = \pi/2, 3\pi/2; \text{ and } \phi_B^* = \pi/2, 3\pi/2, \tag{51}$$

which result in the set of solutions for $(\theta_A^*, \phi_A^*; \theta_B^*, \phi_B^*)$ as

$$\begin{aligned}
&(0, \pi/2; 0, \pi/2), (0, \pi/2; 0, 3\pi/2), (0, 3\pi/2; 0, \pi/2), (0, 3\pi/2; 0, 3\pi/2),\\
&(0, \pi/2; \pi, \pi/2), (0, \pi/2; \pi, 3\pi/2), (0, 3\pi/2; \pi, \pi/2), (0, 3\pi/2; \pi, 3\pi/2),\\
&(\pi, \pi/2; 0, \pi/2), (\pi, \pi/2; 0, 3\pi/2), (\pi, 3\pi/2; 0, \pi/2), (\pi, 3\pi/2; 0, 3\pi/2),\\
&(\pi, \pi/2; \pi, \pi/2), (\pi, \pi/2; \pi, 3\pi/2), (\pi, 3\pi/2; \pi, \pi/2), (\pi, 3\pi/2; \pi, 3\pi/2),
\end{aligned} \tag{52}$$

and the players' payoffs at these Nash equilibria are obtained from Eq. (48) as

$$\Pi_{A,B}(\theta_A^*, \phi_A^*; \theta_B^*, \phi_B^*) = \frac{1}{4}\Delta_2 = \frac{1}{4}(\alpha + \beta + \gamma + \delta). \tag{53}$$

6.2 Case $\sin\theta_A^* \neq 0$ and $\sin\theta_B^* \neq 0$

When $\sin\theta_A^* \neq 0$ and $\sin\theta_B^* \neq 0$, we have from the last two equations in (50)

$$\begin{aligned}
\sin\theta_B^* \cos\phi_A^* \sin\phi_B^* - \cos\theta_B^* \sin\phi_A^* &= 0,\\
\sin\theta_A^* \sin\phi_A^* \cos\phi_B^* + \cos\theta_A^* \sin\phi_B^* &= 0,
\end{aligned} \tag{54}$$

that can be expressed as

$$\cos\phi_A^* \sin\phi_B^* - \cot\theta_B^* \sin\phi_A^* = 0, \tag{55}$$

$$\sin\phi_A^* \cos\phi_B^* + \cot\theta_A^* \sin\phi_B^* = 0. \tag{56}$$

Now, the first two equations in (50) are

$$\sin\theta_B^*(\cos\theta_A^* \sin\phi_B^* \sin\phi_A^* + \sin\theta_A^* \cos\phi_B^*) + \cos\theta_A^* \cos\theta_B^* \cos\phi_A^* = 0, \tag{57}$$

$$\sin\theta_A^*(\cos\theta_B^* \sin\phi_A^* \sin\phi_B^* - \sin\theta_B^* \cos\phi_A^*) - \cos\theta_A^* \cos\theta_B^* \cos\phi_B^* = 0, \tag{58}$$

and given that $\sin\theta_A^* \neq 0$ and $\sin\theta_B^* \neq 0$, we divide Eq. (57) with $\sin\theta_B^*$ and Eq. (58) by $\sin\theta_A^*$ to obtain

$$\cos\theta_A^*(\sin\phi_B^* \sin\phi_A^* + \cot\theta_B^* \cos\phi_A^*) + \sin\theta_A^* \cos\phi_B^* = 0, \tag{59}$$

$$\cos\theta_B^*(\sin\phi_A^* \sin\phi_B^* - \cot\theta_A^* \cos\phi_B^*) - \sin\theta_B^* \cos\phi_A^* = 0. \tag{60}$$

Now, divide Eq. (59) by $\sin\theta_A^*$ and divide Eq. (60) by $\sin\theta_B^*$ to obtain

$$\cot\theta_A^*(\sin\phi_B^* \sin\phi_A^* + \cot\theta_B^* \cos\phi_A^*) + \cos\phi_B^* = 0, \tag{61}$$

$$\cot\theta_B^*(\sin\phi_A^* \sin\phi_B^* - \cot\theta_A^* \cos\phi_B^*) - \cos\phi_A^* = 0. \tag{62}$$

As Eqs. (61, 62) are to be considered along with Eqs. (55, 56), we rewrite (61, 62) as

$$(\cot\theta_A^*\sin\phi_B^*)\sin\phi_A^* + \cot\theta_A^*\cot\theta_B^*\cos\phi_A^* + \cos\phi_B^* = 0, \tag{63}$$
$$(\cot\theta_B^*\sin\phi_A^*)\sin\phi_B^* - \cot\theta_A^*\cot\theta_B^*\cos\phi_B^* - \cos\phi_A^* = 0, \tag{64}$$

and substitute from (55, 56) to (63, 64) to obtain

$$(\cos\phi_A^*\cos\phi_B^* + \cot\theta_A^*\cot\theta_B^*)\cos\phi_A^* = 0, \tag{65}$$
$$(\cos\phi_A^*\cos\phi_B^* + \cot\theta_A^*\cot\theta_B^*)\cos\phi_B^* = 0. \tag{66}$$

The above solution of Eqs. (65, 66, 55, 56) are obtained under the requirement that $\sin\theta_A^* \neq 0$ and $\sin\theta_B^* \neq 0$. This leads us to consider the following cases:

6.2.1 Case $\sin\theta_A^* \neq 0$, $\sin\theta_B^* \neq 0$ and $\cos\phi_A^* = 0 = \cos\phi_B^*$

In this case, we have a solution for which $\sin\phi_A^* = \pm 1$ and $\sin\phi_B^* = \pm 1$, and from Eqs. (55, 56), we then have $\cot\theta_A^* = 0 = \cot\theta_B^*$, i.e., $\cos\theta_A^* = 0 = \cos\theta_B^*$ and therefore $\sin\theta_A^* = \pm 1$ and $\sin\theta_B^* = \pm 1$. Players' payoffs at these Nash equilibria are then obtained from Eq. (48) as

$$\Pi_{A,B}(\theta_A^*, \phi_A^*; \theta_B^*, \phi_B^*) = \frac{1}{4}(\Delta_2 \pm \Delta_1) = \frac{1}{2}(\alpha + \delta), \quad \frac{1}{2}(\beta + \gamma). \tag{67}$$

6.2.2 Case $\sin\theta_A^* \neq 0$, $\sin\theta_B^* \neq 0$ and $\sin\phi_A^* = 0 = \sin\phi_B^*$

In this case, we have a solution for which $\cos\phi_A^* = \pm 1$ and $\cos\phi_B^* = \pm 1$, and from (65, 66), we then have

$$\pm(\pm 1 + \cot\theta_A^*\cot\theta_B^*) = 0, \tag{68}$$

whereas (55, 56) hold true. That is, when $\cot\theta_A^*\cot\theta_B^* = \pm 1$ or when $\cot\theta_A^* = \pm 1$ and $\cot\theta_B^* = \pm 1$, i.e.,

$$\sin\phi_A^* = 0 = \sin\phi_B^*, \quad \cot\theta_A^* = \pm 1 \text{ and } \cot\theta_B^* = \pm 1. \tag{69}$$

As $\theta_A, \theta_B \in [0, \pi]$, we have $\cos\theta_A^* = \pm\frac{1}{\sqrt{2}}$, $\sin\theta_A^* = \frac{1}{\sqrt{2}}$ and $\cos\theta_B^* = \pm\frac{1}{\sqrt{2}}$, $\sin\theta_B^* = \frac{1}{\sqrt{2}}$. Therefore, $\sin\theta_A^*\cos\theta_B^* = \pm\frac{1}{2}$ and $\cos\theta_A^*\sin\theta_B^* = \pm\frac{1}{2}$. Also, then we have $\cos\phi_A^* = \pm 1$ and $\cos\phi_B^* = \pm 1$. This yields

$$\begin{aligned}
&\Pi_{A,B}(\theta_A^*, \phi_A^*; \theta_B^*, \phi_B^*) \\
&= \frac{1}{4}\{\Delta_2 - \Delta_1[\sin\theta_A^*\cos\theta_B^*\cos\phi_A^* - \cos\theta_A^*\sin\theta_B^*\cos\phi_B^*]\}, \\
&= \frac{1}{4}\{\Delta_2 - \Delta_1[\pm(\pm\frac{1}{2}) \pm (\pm\frac{1}{2})]\},
\end{aligned}$$

$$= \frac{1}{4}\{\Delta_2 - \frac{1}{2}\Delta_1[\pm 1 \pm 1]\} = \frac{1}{4}\{\Delta_2 - \frac{1}{2}\Delta_1(2, -2, 0)\},$$
$$= \frac{1}{4}(\Delta_2 \pm \Delta_1), \quad \frac{1}{4}\Delta_2,$$
$$= \frac{1}{2}(\alpha + \delta), \quad \frac{1}{2}(\beta + \gamma), \quad \frac{1}{4}(\alpha + \beta + \gamma + \delta). \tag{70}$$

6.2.3 Case $\sin\theta_A^* \neq 0$, $\sin\theta_B^* \neq 0$ and $\cos\phi_A^* \neq 0$ and $\cos\phi_B^* \neq 0$

Referring to (65, 66), we then have

$$\cos\phi_A^* \cos\phi_B^* + \cot\theta_A^* \cot\theta_B^* = 0, \tag{71}$$

which must hold true along with Eqs. (55, 56). That is, the problem then is to find a solution for $(\theta_A^*, \phi_A^*; \theta_B^*, \phi_B^*)$ from Eqs. (55, 56, 71). Equations (55, 56) can be written as

$$\cos\phi_A^* \sin\phi_B^* = \cot\theta_B^* \sin\phi_A^*, \quad \sin\phi_A^* \cos\phi_B^* = -\cot\theta_A^* \sin\phi_B^*, \tag{72}$$

and on multiplying the sides together, we obtain

$$\cos\phi_A^* \cos\phi_B^* \sin\phi_A^* \sin\phi_B^* = -\sin\phi_A^* \sin\phi_B^* \cot\theta_A^* \cot\theta_B^*,$$

from which Eq. (71) follows as given below:

$$\cos\phi_A^* \cos\phi_B^* + \cot\theta_A^* \cot\theta_B^* = 0. \tag{73}$$

As Eq. (71) follows from (55, 56), it is not required to consider Eq. (71) and can rewrite Eqs. (55, 56) as

$$\cos\phi_A^* \sin\phi_B^* - \cot\theta_B^* \sin\phi_A^* = 0, \quad \sin\phi_A^* \cos\phi_B^* + \cot\theta_A^* \sin\phi_B^* = 0. \tag{74}$$

When $\sin\phi_A^* \neq 0$ and $\sin\phi_B^* \neq 0$, the above equations can be written as

$$\cot\phi_A^* \sin\phi_B^* = \cot\theta_B^*, \qquad -\sin\phi_A^* \cot\phi_B^* = \cot\theta_A^*. \tag{75}$$

Note that substituting from Eq. (75) into Eqs. (61, 62) and Eqs. (55, 56) changes them to identities. From (75), we obtain

$$\theta_A^* = \text{arccot}(-\sin\phi_A^* \cot\phi_B^*), \qquad \theta_B^* = \text{arccot}(\cot\phi_A^* \sin\phi_B^*). \tag{76}$$

As $\sin\theta_A^* \neq 0$, $\sin\theta_B^* \neq 0$ and $\cos\phi_A^* \neq 0$, $\cos\phi_B^* \neq 0$, the players' payoffs are obtained as

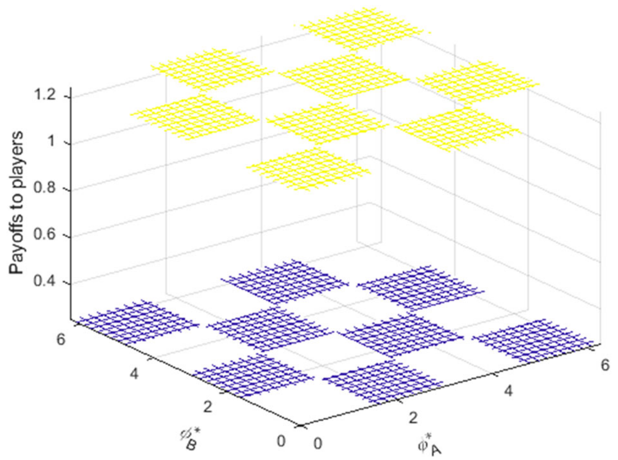

Fig. 5 An infinite number of Nash equilbria exist when the game is played with the state $|\psi_{ini}\rangle = \frac{1}{2}(|00\rangle + |01\rangle - |10\rangle + |11\rangle)$. Players' payoffs at these equilibria $\Pi_{A,B}(\phi_A^*; \phi_B^*)$ and given in Eq. (78) are plotted for $\Delta_2 = 3$ and $\Delta_1 = 2$ againt variables $\phi_A^*, \phi_B^* \in [0, 2\pi)$ considered independent. The ϕ_A^*, ϕ_B^* plane is found to be divided into rectangular patches with respect to the variation of players' payoffs. Angles θ_A^*, θ_B^* that correspond to ϕ_A^*, ϕ_B^* are determined from Eq. (76)

$$\Pi_{\mathrm{A,B}}(\theta_{\mathrm{A}}^*, \phi_{\mathrm{A}}^*; \theta_{\mathrm{B}}^*, \phi_{\mathrm{B}}^*) = \frac{1}{4}\{\Delta_2 - \Delta_1 \sin\theta_{\mathrm{A}}^* \sin\theta_{\mathrm{B}}^*[\sin\phi_{\mathrm{A}}^* \sin\phi_{\mathrm{B}}^* + \cot\theta_{\mathrm{B}}^* \cos\phi_{\mathrm{A}}^* - \cot\theta_{\mathrm{A}}^* \cos\phi_{\mathrm{B}}^*]\}, \tag{77}$$

and by substituting from Eqs. (75) to (77), we obtain

$$\begin{aligned}\Pi_{\mathrm{A,B}}(\theta_{\mathrm{A}}^*, \phi_{\mathrm{A}}^*; \theta_{\mathrm{B}}^*, \phi_{\mathrm{B}}^*) &= \Pi_{\mathrm{A,B}}(\phi_{\mathrm{A}}^*; \phi_{\mathrm{B}}^*) \\ &= \frac{1}{4}\{\Delta_2 - \Delta_1 \sin[\mathrm{arccot}(-\sin\phi_{\mathrm{A}}^* \cot\phi_{\mathrm{B}}^*)] \sin[\mathrm{arccot}(\cot\phi_{\mathrm{A}}^* \sin\phi_{\mathrm{B}}^*)] \\ &\quad \times[\sin\phi_{\mathrm{A}}^* \sin\phi_{\mathrm{B}}^* + \cot\phi_{\mathrm{A}}^* \cos\phi_{\mathrm{A}}^* \sin\phi_{\mathrm{B}}^* + \sin\phi_{\mathrm{A}}^* \cos\phi_{\mathrm{B}}^* \cot\phi_{\mathrm{B}}^*]\}. \end{aligned} \tag{78}$$

An example, consider the case when $\phi_{\mathrm{A}}^* = \pi/4$ and $\phi_{\mathrm{B}}^* = 3\pi/4$ for which $\theta_{\mathrm{A}}^* = 0.95532 = \theta_{\mathrm{B}}^*$. As the pair $(\theta_{\mathrm{A}}^*, \theta_{\mathrm{B}}^*)$ can be determined from a pair $(\phi_{\mathrm{A}}^*, \phi_{\mathrm{B}}^*)$ that is arbitrarily chosen, there exist an infinite set of Nash equilibria. With $(\phi_{\mathrm{A}}^*, \phi_{\mathrm{B}}^*) \in [0, 2\pi)$, the players' payoffs at all these equilibria can be plotted as below with ϕ_{A}^* and ϕ_{B}^* taken as independent coordinates.

The above plot in a different range of values for $\phi_{\mathrm{A}}^*, \phi_{\mathrm{B}}^*$ is given below.

7 Players' directional choices and the violation of Bell-CHSH inequality

The proposed setup for playing a two-player quantum game uses the setting of an EPR type experiment. Consider such an experiment that is designed to test the Bell-

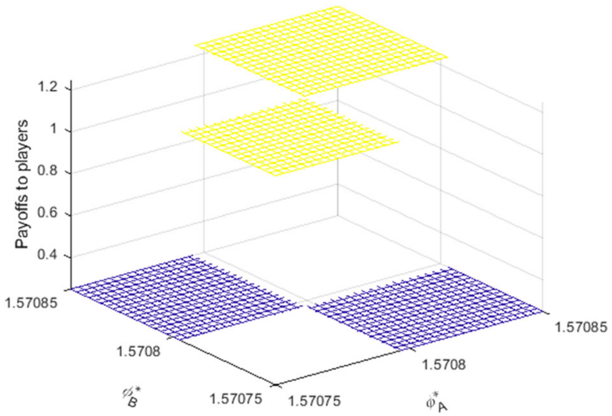

Fig. 6 Players' payoffs $\Pi_{A,B}(\phi^*_A;\phi^*_B)$ for the state $|\psi_{\rm ini}\rangle = \frac{1}{2}(|00\rangle + |01\rangle - |10\rangle + |11\rangle)$ as given in Eq. (78) are plotted for $\Delta_2 = 3$ and $\Delta_1 = 2$ againt the variables ϕ^*_A, ϕ^*_B in a different range

CHSH inequality [8] in which two correlated particles 1 and 2 fly apart in opposite directions from some common source. Subsequently, each of the particles enters its own measuring apparatus which can measure either along $\hat{\mathbf{a}}$ or $\hat{\mathbf{a}}'$ for particle 1 and $\hat{\mathbf{b}}$ or $\hat{\mathbf{b}}'$ for particle 2. The possible values of these variables may be taken to be $+1$ and -1, and the source emits a very large number of particle pairs. We let

$$\hat{\mathbf{a}} = (\theta_{\rm A}, \phi_{\rm A}), \hat{\mathbf{a}}' = (\theta'_{\rm A}, \phi'_{\rm A}), \quad \hat{\mathbf{b}} = (\theta_{\rm B}, \phi_{\rm B}), \quad \hat{\mathbf{b}}' = (\theta'_{\rm B}, \phi'_{\rm B}), \tag{79}$$

where $\theta_{\rm A}, \theta_{\rm B}, \theta'_{\rm A}, \theta'_{\rm B} \in [0, \pi]$ and $\phi_{\rm A}, \phi_{\rm B}, \phi'_{\rm A}, \phi'_{\rm B} \in [0, 2\pi)$. Bell-CHSH inequality can be written as $|\Lambda| \leq 2$ where

$$\begin{aligned}\Lambda = 2[&\Pr(\hat{\mathbf{a}}_{+1}, \hat{\mathbf{b}}_{+1}) + \Pr(\hat{\mathbf{a}}_{-1}, \hat{\mathbf{b}}_{-1}) + \Pr(\hat{\mathbf{a}}_{+1}, \hat{\mathbf{b}}'_{+1}) + \Pr(\hat{\mathbf{a}}_{-1}, \hat{\mathbf{b}}'_{-1})\\ &+ \Pr(\hat{\mathbf{a}}'_{+1}, \hat{\mathbf{b}}_{+1}) + \Pr(\hat{\mathbf{a}}'_{-1}, \hat{\mathbf{b}}_{-1}) + \Pr(\hat{\mathbf{a}}'_{+1}, \hat{\mathbf{b}}'_{-1}) + \Pr(\hat{\mathbf{a}}'_{-1}, \hat{\mathbf{b}}'_{+1}) - 2]\end{aligned} \tag{80}$$

Now, for the state $|\psi_{\rm ini}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + i\,|11\rangle)$, considered above, we have

$$\begin{aligned}\Pr(\hat{\mathbf{a}}_{+1}, \hat{\mathbf{b}}_{+1}) &= \frac{1}{4}\{1 + \sin\theta_{\rm A}\sin\theta_{\rm B}\sin(\phi_{\rm A} + \phi_{\rm B}) + \cos\theta_{\rm A}\cos\theta_{\rm B}\},\\ \Pr(\hat{\mathbf{a}}_{-1}, \hat{\mathbf{b}}_{-1}) &= \frac{1}{4}\{1 + \sin\theta_{\rm A}\sin\theta_{\rm B}\sin(\phi_{\rm A} + \phi_{\rm B}) + \cos\theta_{\rm A}\cos\theta_{\rm B}\},\\ &\dots\\ \Pr(\hat{\mathbf{a}}'_{-1}, \hat{\mathbf{b}}'_{+1}) &= \frac{1}{4}\{1 - \sin\theta'_{\rm A}\sin\theta'_{\rm B}\sin(\phi'_{\rm A} + \phi'_{\rm B}) - \cos\theta'_{\rm A}\cos\theta'_{\rm B}\},\end{aligned} \tag{81}$$

and we obtain

$$\Lambda = \sin\theta_{\rm A}\sin\theta_{\rm B}\sin(\phi_{\rm A} + \phi_{\rm B}) + \sin\theta_{\rm A}\sin\theta'_{\rm B}\sin(\phi_{\rm A} + \phi'_{\rm B}) + \sin\theta'_{\rm A}\sin\theta_{\rm B}\sin(\phi'_{\rm A} + \phi_{\rm B})$$

$$
\begin{aligned}
&- \sin\theta'_{\rm A} \sin\theta'_{\rm B} \sin(\phi'_{\rm A} + \phi'_{\rm B}) + \cos\theta_{\rm A} \cos\theta_{\rm B} \\
&+ \cos\theta_{\rm A} \cos\theta'_{\rm B} + \cos\theta'_{\rm A} \cos\theta_{\rm B} - \cos\theta'_{\rm A} \cos\theta'_{\rm B},
\end{aligned}
$$

that can be expressed as

$$
\begin{aligned}
\Lambda = {} & \sin\theta_{\rm A}[\sin\theta_{\rm B} \sin(\phi_{\rm A} + \phi_{\rm B}) + \sin\theta'_{\rm B} \sin(\phi_{\rm A} + \phi'_{\rm B})] \\
& + \sin\theta'_{\rm A}[\sin\theta_{\rm B} \sin(\phi'_{\rm A} + \phi_{\rm B}) - \sin\theta'_{\rm B} \sin(\phi'_{\rm A} + \phi'_{\rm B})] \\
& + \cos\theta_{\rm A}(\cos\theta_{\rm B} + \cos\theta'_{\rm B}) + \cos\theta'_{\rm A}(\cos\theta_{\rm B} - \cos\theta'_{\rm B}).
\end{aligned} \tag{82}
$$

We take, for instance, $\phi_{\rm A} = \phi_{\rm B} = \phi'_{\rm A} = \phi'_{\rm B} = \pi/4$ and this reduces (82) to

$$
\begin{aligned}
\Lambda = {} & \sin\theta_{\rm A}(\sin\theta_{\rm B} + \sin\theta'_{\rm B}) + \sin\theta'_{\rm A}(\sin\theta_{\rm B} - \sin\theta'_{\rm B}) \\
& + \cos\theta_{\rm A}(\cos\theta_{\rm B} + \cos\theta'_{\rm B}) + \cos\theta'_{\rm A}(\cos\theta_{\rm B} - \cos\theta'_{\rm B}).
\end{aligned} \tag{83}
$$

Now, consider the case when $\theta_{\rm A} = \pi/4$, $\theta'_{\rm A} = 3\pi/4$, $\theta_{\rm B} = \pi/2$, $\theta'_{\rm B} = \pi/4$, and we obtain $\Lambda = 1 + \sqrt{2} \geq 2$, and Bell's inequality is violated. For the state $|\psi_{\rm ini}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + i\,|11\rangle)$, and with $\phi_{\rm A} = \phi_{\rm B} = \phi'_{\rm A} = \phi'_{\rm B} = \pi/4$, the players' payoffs (37) are then obtained as

$$
\begin{aligned}
\Pi_{\rm A,B}\,(\pi/4, \pi/4; \pi/2, \pi/4) = {} & \frac{1}{4}\{\alpha(1 + 1/\sqrt{2}) \\
& + \beta(1 - 1/\sqrt{2}) + \gamma(1 - 1/\sqrt{2}) + \delta(1 + 1/\sqrt{2})\}.
\end{aligned} \tag{84}
$$

To know whether these players' payoffs in the quantum game can be embedded within the classical game, we refer to the players' payoffs (2) in the mixed strategy game. We require $\Pi_{\rm A}(p, q) = \Pi_{\rm B}(p, q)$ in accordance with the players' payoff relations (37) in the quantum game. This results in $\beta = \gamma$, and the players' payoffs in the mixed-strategy classical game (2) become

$$
\Pi_{\rm A,B}(p, q) = \alpha pq + \beta(p + q - 2pq) + \delta(1 - p)(1 - q). \tag{85}
$$

The players' payoffs in the quantum game for the directional choice $(\pi/4, \pi/4; \pi/2, \pi/4)$, and at which Bell's inequalities are violated, are

$$
\Pi_{\rm A,B}\,(\pi/4, \pi/4; \pi/2, \pi/4) = \frac{1}{4}\{\alpha(1 + 1/\sqrt{2}) + \beta(2 - \sqrt{2}) + \delta(1 + 1/\sqrt{2})\}. \tag{86}
$$

Comparing (85) with (86) gives

$$
pq = \frac{1}{4}(1 + 1/\sqrt{2}), \quad p + q - 2pq = \frac{1}{4}(2 - \sqrt{2}), \quad (1 - p)(1 - q) = \frac{1}{4}(1 + 1/\sqrt{2}), \tag{87}
$$

and from which we obtain $p+q=1$ and $q=\frac{1\pm\sqrt{-1/\sqrt{2}}}{2}$, showing that for the directional choice $(\pi/4, \pi/4; \pi/2, \pi/4)$ on behalf of the two players, and at which the players' payoffs are given by (86), the players' payoffs in the quantum game have no mapping within the classical mixed-strategy game.

8 Discussion

This paper presents a quantization scheme for playing two-player games in which each player's strategy consist of orientating a unit vector in three dimensions. In the usual approach, a Nash equilibrium is a pair of unitary operators $(\hat{U}_{\text{A}}^*, \hat{U}_{\text{B}}^*)$ defined by the inequalities (7). For the given initial quantum state $|\psi_{\text{ini}}\rangle$, the proposed quantum game uses an EPR setting in which player A's and player B's strategies consist of orientating the unit vector $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$, respectively. The polarization (or spin) measurements in an EPR setting result in the outcome $m=\pm 1$ along $\hat{\mathbf{a}}$ and $n=\pm 1$ along $\hat{\mathbf{b}}$. The players' payoff relations in the considered scheme involves a set of quantum probabilities that are obtained, according to Eq. (11) from each player's strategies, entries of the matrix of the game, and the initial quantum state $|\psi_{\text{ini}}\rangle$. The payoff relations in the quantum game are defined in terms of this set as described by Eqs. (12, 13). That is, the set of underlying quantum probabilities are generated by each player's strategies—consisting of the players' directional choices—along with the initial quantum state $|\psi_{\text{ini}}\rangle$.

With directional choices as player's strategies, the NE in the quantum game consists of a pair of unit vectors $(\hat{\mathbf{a}}^*, \hat{\mathbf{b}}^*)$ in three dimensional space. Also, the classical mixed strategy game is recovered—for certain initial states $|\psi_{\text{ini}}\rangle$—when each player's directional choices $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ follow the assigned trajectories in space.

The scheme is analyzed for three initial states $|\psi_{\text{ini}}\rangle$. We show that playing the game with the quantum state $|\psi_{\text{ini}}\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$ results in the classical mixed strategy game in which Alice's and Bob's directional choices are given by (30, 31). These express their strategies p and q in the classical mixed strategy game in terms of the angles $\theta_{\text{A}}, \phi_{\text{A}}; \theta_{\text{B}}, \phi_{\text{B}}$—representing player A's and player B's directional choices. For given values of p and q, Eqs. (30, 31) therefore represent the trajectories on the surface of a unit sphere traced by the tips of the unit vectors $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$, respectively. Playing the game with the maximally entangled state $|\psi_{\text{ini}}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + i\,|11\rangle)$ results in obtaining the players' payoff relations (37) that cannot be reduced to the classical mixed-strategy payoff relations. That is interpreted by stating that there do not exist such trajectories on the unit sphere such that when these trajectories are followed by the tips of each player's strategic choices, the quantum game results in the classical mixed-strategy game.

Playing the game with the state $|\psi_{\text{ini}}\rangle = \frac{1}{2}(|00\rangle + |01\rangle - |10\rangle + |11\rangle)$ results in a number of Nash equilibria appearing as the edge cases. For the non-edge cases, we determine that there exist an infinite number of Nash equilibria. At these Nash equilibria we consider $\phi_{\text{A}}^*, \phi_{\text{B}}^* \in [0, 2\pi)$ as independent variables from which the angles $\theta_{\text{A}}^*, \theta_{\text{B}}^* \in [0, \pi]$ can be obtained using Eq. (76). Corresponding to these angles, the players' payoffs at the Nash equilibria, i.e., $\Pi_{\text{A,B}}(\phi_{\text{A}}^*; \phi_{\text{B}}^*)$ are obtained by Eq. (78).

The ϕ_A^*, ϕ_B^* plane is found to be divided into rectangular patches with the corresponding variation of the players' payoffs into two distinct values.

We agree with the perspective that if quantum advantage (or an improved game-theoretical outcome) does not emerge in a quantum game, it does not necessarily change a quantum game to a classical game. The games considered in this paper are truly quantum as they involve quantum superposition and entanglement. In particular, the players' payoff relations are defined from underlying quantum mechanical probability distributions and that the corresponding classical games are recoverable by restricting players' directional choices along specific trajectories in three dimensions.

Considering Bell-CHSH inequality for the directional choice $(\pi/4, \pi/4; \pi/2, \pi/4)$ on behalf of two players, we show that Bell's inequalities are violated. For these directional choices, the players' payoffs in the quantum game are shown to have no mapping within the classical mixed-strategy game. An EPR setting provides the route for the players' access to quantum probability distributions that can violate Bell's inequalities. As the quantum game involves classical strategy sets, Enk and Pike's argument [21] is circumvented.

9 Conclusion

Game theory is widely used in a number of disciplines, and this paper presents a scheme for two-player quantum games that establishes a more direct link between a classical game and its quantum version. Players in the quantum game have access to classical strategy sets as is the case in the corresponding classical game, allowing us to circumvent Enk and Pike's argument. As the contribution of this paper to the theory of quantum games is built on the EPR paradox, a possible future research direction can be to interpret the EPR paradox as a strategic quantum game. Also, the proposed scheme motivates studying refinements of the NE concept using an EPR setting with players' moves consisting of directional choices.

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