Quantum models of Parrondo's games

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ABSTRACT

It is possible to have two games that are losing when played in isolation but that, because of some form of feedback, produce a winning game when played alternately or even in a random mixture. This effect is known as Parrondo's paradox. Quantum mechanics provides novel methods of combining two games through interference and entanglement. Two models of quantum Parrondo's games have been published and these are reviewed here. We speculate on a model of a quantum Parrondo's game using entanglement. Such games could find a use in the development of algorithms for quantum computers.

Keywords: quantum games, Parrondo's games, quantum lattice gas automata

1. INTRODUCTION

The idea of a ratchet and pawl driven in one direction by the Brownian fluctuations of gas molecules, in violation of the second law of thermodynamics, was first proposed at the beginning of the 20th century and later revisited by Feynman.[1] Although it was shown that this could not function as a perpetual motion machine, it has inspired the idea of the Brownian ratchet. Drawing a link between games of chance and Brownian ratchets gave rise to the idea of Parrondo's games or Parrondo's paradox[2,3] where a losing game can be turned into a winning one by combining it with a second losing game that acts as a source of "noise." The noise disrupts the feedback that caused the first game to be losing. A game is losing or winning depending on whether the expectation value of its payoff over a long series of games is negative or positive, respectively. The classical Parrondo's game is cast in the form of a gambling game utilizing a set of biased coins.[2–6] The source of "noise" is the toss of a single biased coin that is marginally losing, while the other game depends on two or more biased coins, the choice of which is determined by the game situation in the form of, for example, dependence on total capital,[2–5] history dependent rules,[6,7] or spatial neighbor dependence.[8] For a recent review of Parrondo's games see Harmer *et al.*[9]

There has been recent interest in reformulating the classical theory of games into the quantum realm,[10–14] in order to better study unsolved problems in quantum information, communication and computation.[15] Quantum strategies can exploit both superposition and entanglement of states. Applications for quantum game theory have been proposed in evolutionary biology[16–18] and economics[19,20] as well as the traditional game theory problems such as the prisoners' dilemma,[11,13,16,21,22] the battle of the sexes,[23,24] and the Monty Hall problem.[25–27]

In quantum games the actions of the players are encoded by a qubit, with the basis states $|0\rangle$ and $|1\rangle$ corresponding to the classical moves. The quantum players have at their disposal instruments for manipulating their qubit without causing decoherence. That is, a move by a quantum player corresponds to acting with a unitary operator on their qubit. By entangling the initial states of the two qubits interesting effects can be obtained.

Meyer and Blumer[28] were the first to combine the ideas of quantum game theory and Parrondo's paradox in an attempt to create a quantum Parrondo's game using a quantum lattice gas automaton. The history dependent Parrondo's game of Parrondo *et al*[6] has been quantized directly by replacing the rotation of a bit, representing a toss of a classical coin, by an SU(2) operation on a qubit.[29,30] Parrondo's games have also been linked

to the creation of quantum algorithms.[31]   In section 2 and 3 we present summaries of the history dependent and capital dependent Parrondo's games and review their quantum analogues. In section 4 we present some additional results with Meyer and Blumer's scheme and speculate on future directions.

2. A HISTORY DEPENDENT QUANTUM PARRONDO'S GAME

In the classical history dependent Parrondo's paradox, game A is the toss of a single biased coin with winning probability $p < 1/2$, while game B is a collection of biased coins, the selection of which is dependent on the results of previous games. In order to achieve ratcheting, the history dependent game needs the feedback from at least the last two results. Thus, game B has four possible coins, B_1, B_2, B_3, and B_4, with winning probabilities p_1, p_2, p_3 and p_4, respectively. The choice of coin is determined by the results of the two previous games as indicated in Fig. 1. An example of probabilities that give rise to a Parrondo's paradox is

$$
\begin{aligned}
p_1 &= 7/10 - \epsilon, \\
p_2 = p_3 &= 1/4 - \epsilon, \\
p_4 &= 9/10 - \epsilon,
\end{aligned}
\tag{1}
$$

for a small positive parameter ϵ. With $p < 1/2$, game A is clearly a losing game, while game B has been shown to losing for any $\epsilon > 0$.[6]   However, combinations of A and B, including selecting the game to be played at each step at random, produce a game with a positive expected payoff. The effect can also be obtained by combining two history dependent games.[7]

The quantum generalization of game A is the the application of an SU(2) operation on a qubit:

$$
\hat{A}(\theta, \gamma, \delta) = \begin{pmatrix} e^{-i(\gamma+\delta)/2} \cos\theta & -e^{-i(\gamma-\delta)/2} \sin\theta \\ e^{i(\gamma-\delta)/2} \sin\theta & e^{i(\gamma+\delta)/2} \cos\theta \end{pmatrix},
\tag{2}
$$

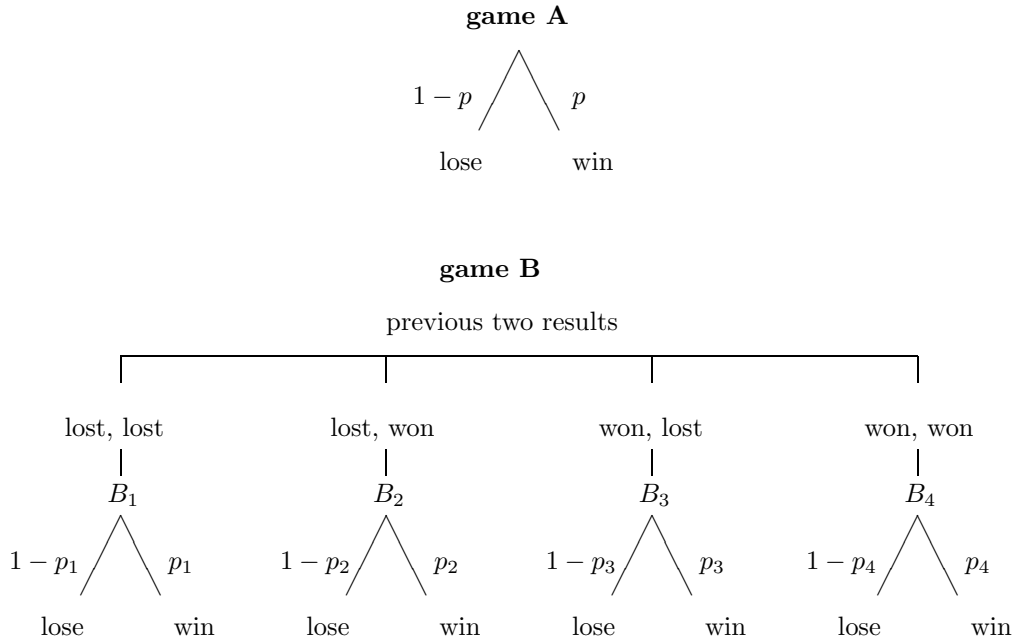**Figure 1.** Winning and losing probabilities for game A and the history dependent game B. Adapted from Flitney *et al.*[30]

where $\theta \in [0, \pi/2]$ and $\gamma, \delta \in [0, 2\pi]$. Game B consists of four SU(2) operations, each of the form of Eq. (2):

$$
\hat{B}(\phi_1, \alpha_1, \beta_1, \phi_2, \alpha_2, \beta_2, \phi_3, \alpha_3, \beta_3, \phi_4, \alpha_4, \beta_4) = \begin{bmatrix} A(\phi_1, \alpha_1, \beta_1) & 0 & 0 & 0 \\ 0 & A(\phi_2, \alpha_2, \beta_2) & 0 & 0 \\ 0 & 0 & A(\phi_3, \alpha_3, \beta_3) & 0 \\ 0 & 0 & 0 & A(\phi_4, \alpha_4, \beta_4) \end{bmatrix}.
$$
(3)

This operator acts upon the state $|\psi(t-2)\rangle \otimes |\psi(t-1)\rangle \otimes |\psi(t)\rangle$, where $|\psi(t-1)\rangle$ and $|\psi(t-2)\rangle$ represent the results of the two previous games and $|\psi(t)\rangle$ is the initial state of the target qubit which is altered by the action of $\hat{B}$. The $\hat{B}$ operator consists of four control-control SU(2) operations as indicated in Fig. 2.
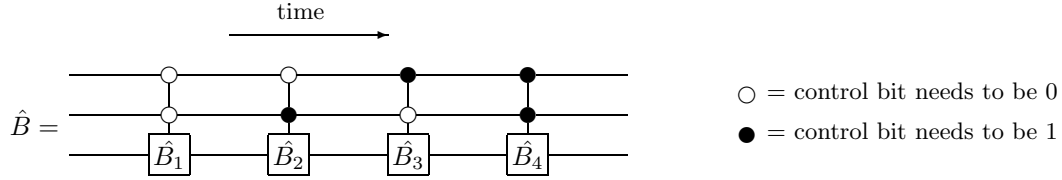

Figure 2. In the history dependent quantum Parrondo game, $\hat{B}$ consists of four control-control rotations depending on the four possible states of the two control qubits.

The information flow in an alternating succession of games A and B is shown in Fig. 3. If the initial state is $|\psi_i\rangle = |00 \ldots 0\rangle$ the quantum game gives no results that cannot be obtained classically. To see quantum effects we require the possibility of interference. By choosing the maximally entangled initial state,

$$
|\psi_i^m\rangle = \frac{1}{\sqrt{2}} (|00 \ldots 0\rangle + |11 \ldots 1\rangle),
$$
(4)

we produce interference effectively between two different games, those with initial states $|00 \ldots 0\rangle$ and $|11 \ldots 1\rangle$. The payoff (i.e., the count of the number of 1's in the final state) is dependent on the phases β_i, $i = 1 \ldots 4$. By judicious selection of the phases the payoff can either be enhanced or diminished. For a sample of results see Flitney et al.[30]
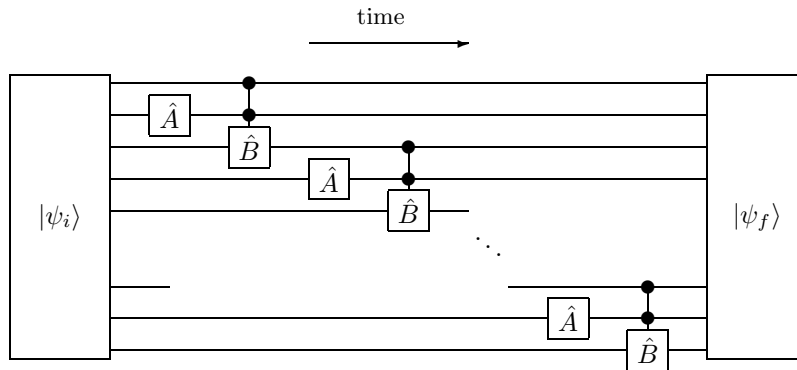

Figure 3. The information flow in qubits (solid lines) in an alternating sequence of A and B. A measurement on $|\psi_f\rangle$ is taken on completion of the sequence of games to determine the payoff. Figure adapted from Flitney et al.[30]

3. QUANTUM PARRONDO AS A LATTICE GAS AUTOMATON

In capital dependent Parrondo's games, game A is again the toss of a single biased coin with winning probability $p = 1/2 - \epsilon$, but game B utilizes two coins whose use depends on the total capital of the player as follows: coin
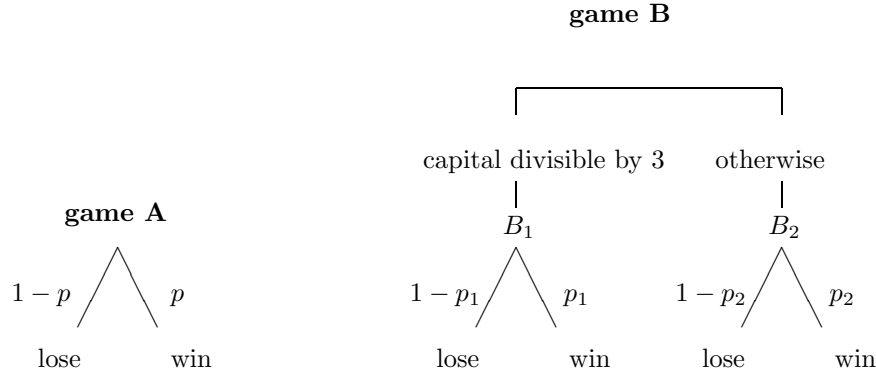
game B

game A

capital divisible by 3   otherwise

B_1 B_2

$1-p$ / \ p $1-p_1$ / \ p_1 $1-p_2$ / \ p_2

lose   win   lose   win   lose   win

Figure 4: Winning and losing probabilities for game A and the capital dependent game B.

B_1 with winning probability p_1 is used if the capital is divisible by three, otherwise B_2 is used with winning probability p_2. This is shown schematically in Fig. 4. By adjusting the probabilities p_1 and p_2, for example

$$p_1 = 1/10 - \epsilon, \qquad p_2 = 3/4 - \epsilon, \qquad \epsilon > 0, \tag{5}$$

we get a net loss over time.[2–4] Although the weighted average of the winning probabilities is positive, the "bad" coin is used more often than the one-third of the time that we might naively expect. By interspersing game A with game B this effect is broken and the combination can now be winning, provided the net positive effect of game B exceeds the negative effect of game A. Some examples of the expected payoff after one hundred games is given in Fig. 5

In Meyer and Blumer's quantum version of this game[28] the capital corresponds to a discretization of the position of a particle undergoing Brownian motion in one dimension under the action of some potential. A potential with a constant gradient sloping towards negative x produces the effect of game A, while game B corresponds to a tilted sawtooth potential. The quantum "coin" is a two state system such as a spin–$\frac{1}{2}$ particle, in a superposition of the $|\uparrow\rangle$ and $|\downarrow\rangle$ states, the eigenstates of σ_z. The quantum analogue of an unbiased coin flip is a unitary transformation represented by the matrix $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$. Let $|x\rangle$ correspond to the gambling capital, and $|\uparrow\rangle$ and $|\downarrow\rangle$ indicate a win or a loss, respectively. That is, one play in the game is effected by the unitary transformation

$$\begin{aligned}
|x, \downarrow\rangle &\rightarrow \frac{1}{\sqrt{2}}\left(|x-1, \downarrow\rangle + i|x+1, \uparrow\rangle\right), \\
|x, \uparrow\rangle &\rightarrow \frac{1}{\sqrt{2}}\left(i|x-1, \downarrow\rangle + |x+1, \uparrow\rangle\right).
\end{aligned} \tag{6}$$

The initial state is chosen to be $\frac{1}{\sqrt{2}}(|0, \uparrow\rangle + |0, \downarrow\rangle)$ so the particle begins with no particular momentum bias and an unbiased game A produces no net drift*. The effect of the potentials are implemented by multiplication by an x-dependent phase factor.[32] The quantum version of the games is the unbiased transition in Eq. (6) multiplied by a phase $e^{-iV(x)}$, where for game A and B, respectively,

$$\begin{aligned}
V_A(x) &= \alpha x, \\
V_B(x) &= \beta(1 - \frac{1}{2}(x \mod 3)) + V_A(x).
\end{aligned} \tag{7}$$

Game A has a negative expected payoff for $\alpha > 0$. The expected payoff is periodic with period $2\pi/\alpha$, returning to zero after this time. However, for an arbitrary time, $\langle x \rangle < 0$, so we conclude that A is a losing game. The

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*In Travaglione and Milburn[34] an unbiased quantum random walk is created by using the Hadamard operator, represented by $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, and starting with the initial state $\frac{1}{\sqrt{2}}(|0, \downarrow\rangle + i|0, \uparrow\rangle)$. The two schemes are equivalent.

situation is similar for game B. Choosing $\alpha = 2\pi/5000$ and $\beta = \pi/3$ gives results for the individual games comparable (within a factor of two) to the classical games with the probabilities of Eq. (5). A and B are losing individually but, for example, the combination $BAAAA$ is winning, demonstrating a quantum version of the Parrondo paradox (see Fig. 6).

4. DISCUSSION AND CONCLUSION

Apart from the cyclical nature of the expectation values for games A and B, this quantum model of Parrondo's paradox differs from its classical counter-part in a number of ways:

- For times small compared with the period of game A, the deviation of the expectation value of the winning combinations from that of the individual games is quadratic compared to linear in the classical case.

- Fewer combinations of A and B give rise to a reversal of fortune (see Fig. 7). In particular, mixing A and B using a fair coin does not yield a winning game.

- The effect holds for a broad range of α and β. (see Fig. 8).

- There is sensitivity to initial conditions. For example, playing B first prior to the sequences $AAAAB$ turns a losing sequence into a winning one, an effect not seen in the classical game where such a change merely results in the same trend with an offset.

The stated model uses interference to combine the two games. By choosing a combination of the games A and B that produced constructive interference in the positive x-direction, a net movement of the particle in that direction is achieved. In quantum mechanics we can also utilize entanglement to produce non-classical outcomes. We can explore the following model. Consider the positions of two spin-$\frac{1}{2}$ particles undergoing Brownian motion under the effect of a one-dimensional potential, as in the previous section. We shall let the positions $|x_1\rangle$ and $|x_2\rangle$ of these particles evolve by Eq. (6) and the position dependent phase factors of Eq. (7). By choosing to initially entangle the spins of the two particles we can get qualitatively new behavior. That is, the initial state is taken to be $\frac{1}{\sqrt{2}}(|00, \uparrow\uparrow\rangle + |00, \downarrow\downarrow\rangle)$. In the absence of a potential to bias the movement of the particles the expectation value of the particle positions remains zero, as we would expect. Instead, if one particle is subject to potential $V_A(x)$ and the other to $V_B(x)$, the situation is similar to playing A and B alternately, except that the coupling between the games in this case is through entanglement of the particle spins. The scheme remains to be explored, but initial results are unspectacular. The nature of the initial state ensures that any net movement in either direction is minimized resulting in $\langle x_1 \rangle \approx \langle x_2 \rangle \approx 0$.

The history dependent quantum Parrondo's game of Flitney et al [30] is able to utilize the additional degrees of freedom, permitted by the introduction of phase factors, to enhance payoffs. Only short sequences of games have been studied with this scheme and for some sequences, for example AAB, the quantum enhancement disappears over longer runs. The scheme could be extended to incorporate the combination of two history dependent games, as is done classically by Kay and Johnson.[7]

Lee and Johnson's approach[31] to quantum algorithms is interesting. In this, A and B represent a partioning of the operations in Grover's search algorithm. Neither step alone is efficient, but the original algorithm is recreated by randomly combining A and B, thus giving a constructive role to randomness in the creation of quantum algorithms.

Classical systems where two losing games can be combined to produce a net winning payoff are known. One of these games can be thought of as noise that disrupts the negative bias of the other, thus giving a constructive role to noise. Quantum mechanical analogues of these systems can also produce interesting effects. Selective constructive interference as well as entanglement can be utilized for combining games. We have indicated some possible models of quantum Parrondo's games.

ACKNOWLEDGMENTS

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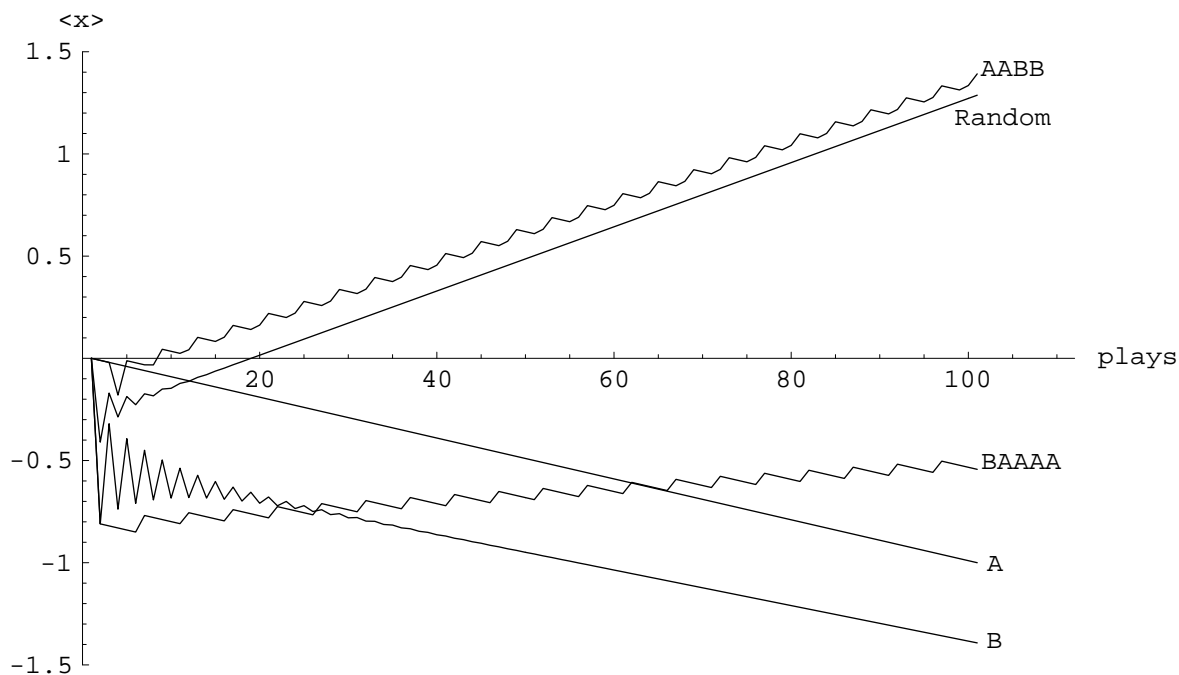
Figure 5. For the classical capital dependent Parrondo's game, the expected gain from playing various sequences of A and B with $p = 1/2 - \epsilon$ for game A, and $p_1 = 1/10 - \epsilon$ and $p_2 = 3/4 - \epsilon$ for game B, where $\epsilon = 0.005$. Random refers to selecting A or B at each play using a fair coin.
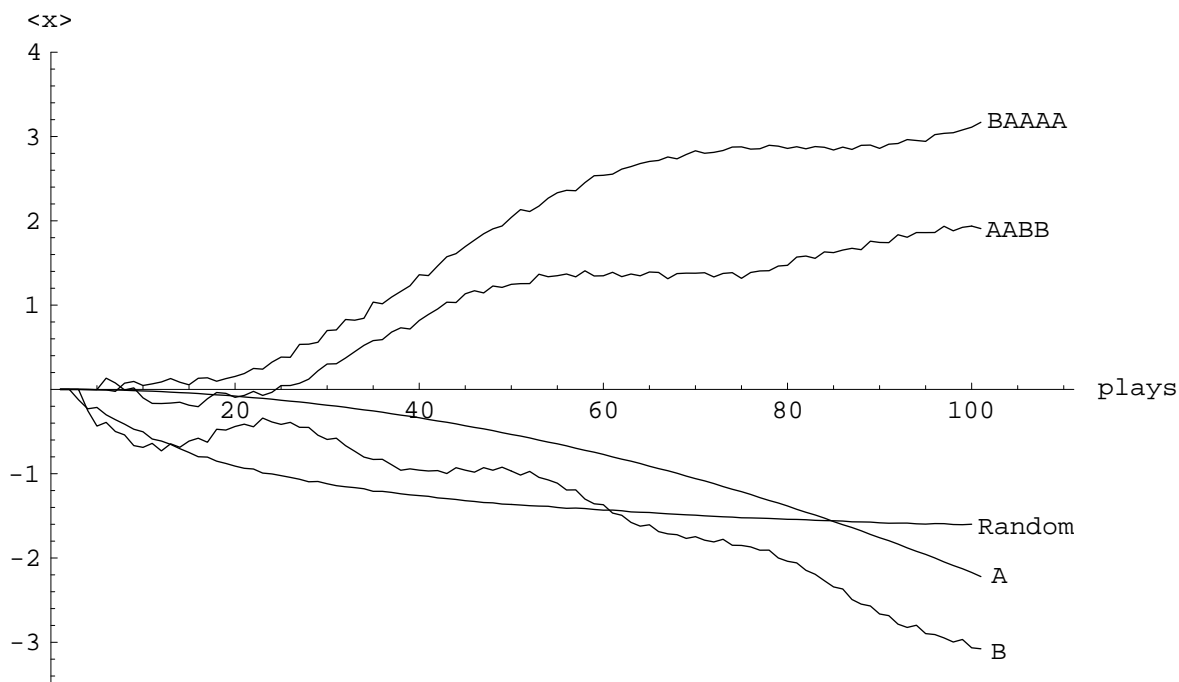

Figure 6. For the quantum capital dependent Parrondo's game, the expectation value of the gain from playing various sequences of A and B with $\alpha = 2\pi/5000$ and $\beta = \pi/3$. Random refers to selecting A or B at each play using a fair coin. The results for the this are the average of the expectation values over 500 runs.
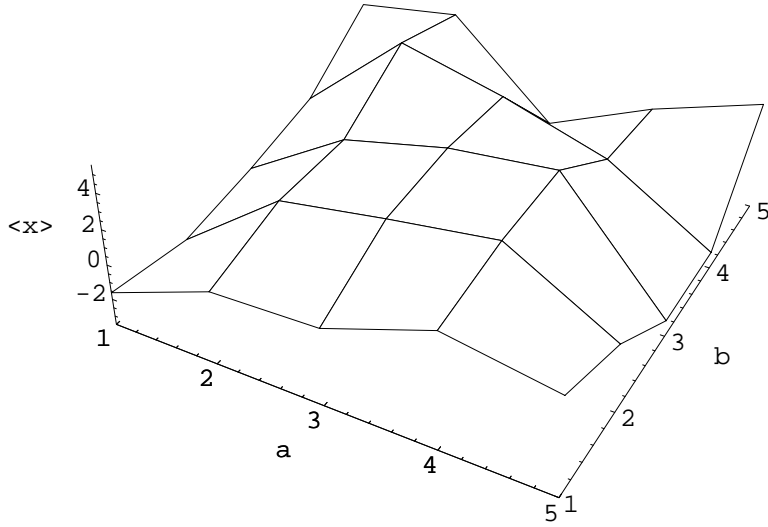
Figure 7. For the quantum capital dependent Parrondo's game, the expectation value of the gain after 100 games by repeating a sequence of a games of A followed by b games of B, with $\alpha = 2\pi/5000$ and $\beta = \pi/3$. For example, $a = 4$, $b = 1$ represents the sequence $AAAAB$ repeated 20 times, giving a total of 100 games.
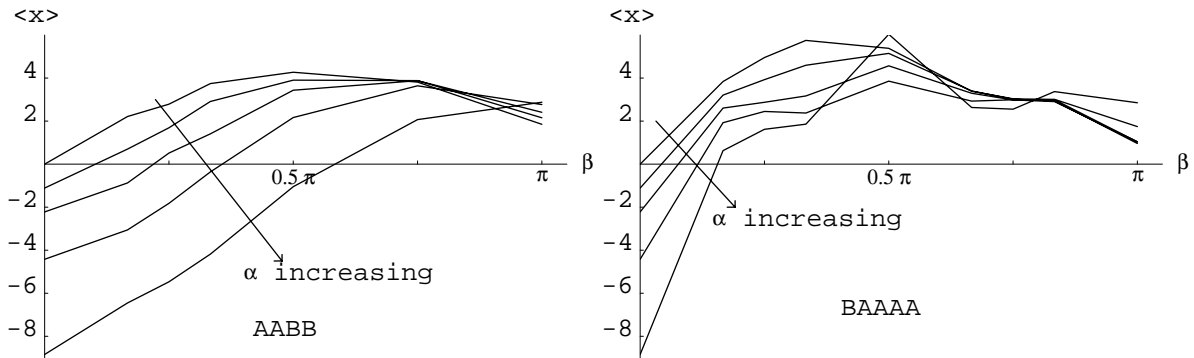

Figure 8. For the quantum capital dependent Parrondo's game, the expectation value of the gain after 100 games using the sequence (a) AABB or (b) BAAAA, for various values of α and β. The curves are for $\alpha = 0$, $\alpha = \pi/5000$, $\alpha = \pi/2500$, $\alpha = \pi/1250$, and $\alpha = \pi/625$.