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MODELLING OF LOW POWER CW LASER BEAM HEATING EFFECTS ON A GaAs SUBSTRATE

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Abstract—A scanning laser beam is a common method used to characterise the optical response of GaAs devices. Laser heating of the substrate, however, can alter the local temperature and hence spuriously shift the values of the electrical parameters of interest. In order to assess the magnitude of this problem, we have solved the steady-state heat equation, with the aid of Kirchhoff's transformation. We show for practical dimensions, that correct temperature prediction does not depend on the lateral boundary conditions. We find that the variable that is most tightly coupled to any temperature increase is the power of the laser beam. Usual approximations for the power dissipation density, in the substrate, were found inadequate. A more complete model that considers power dissipation as an exponential function of substrate depth was found to be necessary. We conclude that for low power applications, i.e. using lasers less than 1 mW, heating effects can be considered negligible. For higher powers our results offer worst-case predictions of the local substrate temperature rise.

1. INTRODUCTION

Gallium arsenide (GaAs) has many important optoelectronic applications. A useful method in characterising its optical response is with the aid of a scanning laser beam, which moves along the surface of a GaAs device or chip. This technique, for example, has been applied to thermal emission measurements[1] evaluation of lattice damage in semiconductors[2], or to study new photogain mechanisms in GaAs MESFETs[3]. Lasers are also commonly used for the optical control of GaAs microwave-integrated circuits[4]. Consequently, it is important to determine if any significant local temperature increase occurs in the GaAs substrate, due to laser-induced heating. Only a few degrees increase in temperature can create a significant undesirable shift in measured electrical parameters.

This paper offers a worst-case analysis, in which the scanning cw laser beam is, in fact, considered stationary. For pulsed laser applications, there are analytical models to predict thermal effects[1,5]. Thus steady-state conditions are assumed throughout our analysis. Our main objective is to determine the extreme value of steady-state local temperature increase of a GaAs chip, when a focused laser beam impinges its surface (see Fig. 1). We assume that pure GaAs is used (effects of electron or hole doping are not considered). For the worst case, we assume perfect transmission of photons through any passivation layers and zero reflection at the surface layers of the chip. The physical parameters[6] are shown in Table 1. The indicated absorption coefficient, α, corresponds to a 670 nm wavelength of the laser beam. The GaAs substrate has a radius, b, of 1 cm and a depth, h, of 150 μm. The power of the laser beam was 1.4 μW with a spot radius of 1 μm, as described in our previous work[3]. The ambient temperature is $T_0 = 300$ K.

First we considered the simplest and worst case, assuming that all the energy is absorbed in a limited region under the laser beam without taking into account the heat propagation in the GaAs substrate. As this produced the unrealistic result that the temperature exceeded the melting point of GaAs, for a laser power of only 1.4 μW, this confirmed the need for a more complex analysis. Thus the differential steady-state heat equation in cylindrical coordinates was formulated and solved with the aid of Kirchhoff's transformation[7] and modified Bessel functions. Different boundary conditions (bc's) were considered and a solution for practical dimensions was obtained.

Using this approach, we consider two cases: (1) the power dissipation density $g(z)$ (W/m^3) as a function of depth z considered constant and (2) a more precise exponential model for $g(z)$. As we shall see, the exponential model is needed to properly explain the dependence of temperature with the various physical parameters.

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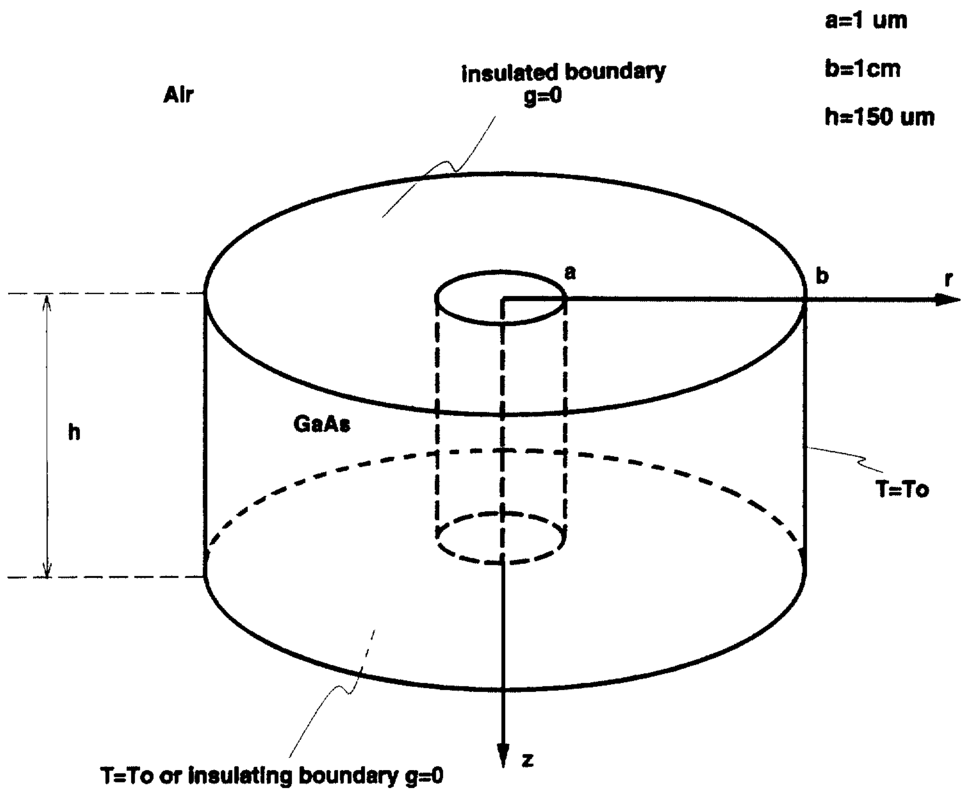

Fig. 1. Diagram defining the dimensions of the laser beam and the wafer

2. THEORY

2.1. *Maximum time for GaAs to melt*

Consider the laser beam radiating power P (W) on the GaAs surface. The worst possible case occurs when the chip has absorbed all the photons in the laser beam, and these are converted to a total of Q (J) of heat confined under the beam (the surrounding GaAs does not act as a heat sink, in this simple model). The GaAs temperature increases and the maximum time, t_{max}, before the GaAs chip starts to melt can be obtained as $t_{max} = Q/P$. Q can be calculated as $mc\Delta T$, where m is the GaAs mass, c is the specific heat capacity and $\Delta T = T_{melt} - T_o$, is the temperature increase needed to melt the GaAs. Considering the mass in the cylinder limited by the laser beam and the substrate, from the parameters in Table 1, a value of $t_{max} = 0.7$ s is obtained. Experiment shows that, in fact, the GaAs chip does not melt[3] under these conditions. Therefore this check confirms that it is necessary to solve the heat equation to explain the correct behaviour.

Table 1. GaAs physical parameters

c (J kg^{-1} K^{-1})	325
ρ (kg m^{-3})	5317.4
T_{melt} (K)	1513
α (m^{-1})	31.1×10^5
k_o (W m^{-1} K^{-1})	57.95

2.2. *The steady-state heat equation and Kirchhoff's transformation*

We assumed that the temperature reaches a steady value. Furthermore, for accurate thermal analysis we must take into account the dependence of the semiconductor thermal conductivity k on the temperature T, which makes the heat equation nonlinear. The heat equation to be solved is then

$$\nabla(k(T)\nabla T) = -g(z).$$

On the top of the chip there is negligible heat loss to the air, so we assume an insulated bc. On the other hand, the bottom of the chip acts as a good enough heat sink, therefore we assume it to be at constant room temperature. For the edge of the chip, we distinguish two different bc's: (1) constant temperature, for a good heat sink, and (2) an insulated bc. Then the bc's of the heat equation are:

$$\text{(a) } T = T_o = 300\,K \quad \text{for } z = h,$$
$$\text{(b) } \partial T/\partial z = 0 \quad \text{for } z = 0,$$
$$\text{(c.1) } T = T_o \quad \text{for } r = b,$$
$$\text{(c.2) } \partial T/\partial r = 0 \quad \text{for } r = b.$$

We will see, for practical dimensions, both solutions for the two (c) bc's are identical.

Kirchhoff's transformation changes the temperature variable T, into a transformed temperature U as

$$U(T) = \int_{T_o}^{T} k(T')/k_o \mathrm{d}T', \qquad (1)$$

where $k_o = k(T_o)$. The Kirchhoff's transformation converts the nonlinear heat equation into a linear one, with linear bc's[8]. Due to the fact that heat transfer is radial and symmetrical ($\partial T/\partial\theta = \partial U/\partial\theta = 0$), the heat equation to be solved is

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{\partial^2 U}{\partial z^2} = -g^* \qquad (2)$$

with bc's

(a) $U = 0$ for $z = h$,

(b) $\partial U/\partial z = 0$ for $z = 0$,

(c.1) $U = 0$ for $r = b$,

(c.2) $\partial U/\partial r = 0$ for $r = b$,

where g^* is the transformed power density defined as $g^* = G\mathrm{e}^{-\xi z}$. We distinguish two cases (g_i^*, $i = 1, 2$):

(1) Constant dissipated power density: we consider the average power in the cylinder limited by the laser beam and the substrate (other approximations were attempted with the depth of the heat cylinder equal to $1/\alpha$, and with the average of g or $g = g(1/\alpha)$ in that cylinder, but less realistic predictions for the temperature were obtained). In this case $\xi = 0$ and $G = P/(\pi a^2 h k_o)$, as shown in Appendix A, then

$$g_1{}^* = \frac{P}{\pi a^2 h k_o}$$

(2) Exponential dissipated power density: that is $\xi = \alpha$, and $G = \alpha P/\pi a^2 k_o$ (see Appendix A). Then

$$g_2{}^* = \frac{\alpha P}{\pi a^2 k_o}\mathrm{e}^{-\alpha z}.$$

After the solution for U has been obtained, the temperature can be determined with the appropriate inverse transformation (see Appendix B).

2.3. *The transformed steady-state heat equation solutions*

2.3.1. *Nonradiated region.* To obtain the temperature in the GaAs chip outside the laser beam region, we must solve the homogeneous heat equation (i.e. $g^* = 0$). Let us name the transformed temperature, U_o. The solution is achieved by using the separation of variables method, i.e. $U_o(r, z) = A_o(z)B_o(r)$. This results in two independent ordinary differential equations

$$\partial^2 A_o/\partial z^2 + p^2 A_o = 0$$

$$r^2(\partial^2 B_o/\partial r^2) + r(\partial B_o/\partial r) - r^2 p^2 B_o = 0,$$

where p^2 is a positive constant (the another possible solution with $-p^2$ was rejected because it does not satisfy the bc's). Taking into account bc's (a) and (b) the solution for A_o is

$$A_{on} = F\cos(p_n z) \qquad (3)$$

$$p_n = (2n-1)\pi/2h; \quad n = 1, 2, 3, \ldots \qquad (4)$$

and F is a constant. On the other hand, the solution for B_o depends on whether bc (c.1) or (c.2) is selected. For bc (c.1) ($B_o = B_{o1}$)

$$B_{o1n} = C_{o1n} f_{o1n}(r)$$
$$f_{o1n}(r) = I_o(p_n r)K_o(p_n b) - I_o(p_n b)K_o(p_n r),$$

where C_{o1n} are n dependent constants, I_o is the modified Bessel function of the first kind and K_o is of the third kind (order $\nu = 0$).

If bc (c.2) is selected, the solution for B_{o2} is

$$B_{o2n} = C_{o2n} f_{o2n}(r)$$
$$f_{o2n}(r) = I_o(p_n r)K_1(p_n b) + I_1(p_n b)K_o(p_n r),$$

where I_1 and K_1 are the same modified Bessel functions mentioned above but with order $\nu = 1$ and C_{o2n} are n dependent constants.

Therefore we have two different global solutions for U_o depending on which of the (c) bc's is selected. For bc's (c.1) and (c.2) we have respectively ($i = 1, 2$)

$$U_{oi} = \sum_{n=1}^{\infty} D_{in} f_{oin}(r)\cos(p_n z),$$

where D_{in} are n dependent constants, to be determined for every (c) bc.

2.3.2. *Radiated region.* The solution U_l, for the region under the laser beam, can be obtained as the sum of the homogeneous equation solution, U_{lh}, and a particular solution U_{lp}. Then $U_l = U_{lh} + U_{lp}$. The homogeneous solution is solved again using the separation of variables method $U_{lh} = A_{lh}(z)B_{lh}(r)$. For A_{lh} we have the same bc as for the nonradiated region. Then $A_{lh}(z, n) = A_{lhn} = A_{on}$ for Equation (3). On the other hand the solution for B_{lh} is

$$B_{lhn} = C_n I_o(p_n r) + F_n K_o(p_n r).$$

Due to $K_o(p_n r)$ diverging when $r \to 0$, then F_n must be equal to 0. Therefore we have for the global homogeneous solution

$$U_{lhi} = \sum_{n=1}^{\infty} C_{in} I_o(p_n r)\cos(p_n z),$$

where C_{in} ($i = 1, 2$) are constants to be determined and dependent on the (c) bc's.

If we express the transformed dissipated power density g^* as

$$g^*(z) = \sum_{n=1}^{\infty} g_n \cos(p_n z) \qquad (5)$$

from Equation (2) we can obtain the particular solution U_{lp} as

$$U_{1p} = \sum_{n=1}^{\infty} \frac{g_n}{p_n^2} \cos(p_n z).$$

We have for the global solution in the region exposed to the laser beam, $U_{1i} = U_{1hi} + U_{1p}$

$$U_{1i} = \sum_{n=1}^{\infty} \left[C_{in} I_o(p_n r) + \frac{g_n}{p_n^2} \right] \cos(p_n z), \quad (6)$$

where $i = 1, 2$ corresponds with the selected (c) bc's.

2.3.3. *C_{in}, D_{in} and g_n constants.* The temperature must be continuous at the sides of the cylinder limited by the laser beam. We assume that the temperature does not change abruptly at this surface. Then T and $\partial T/\partial r$ for $r = a$ are continuous. These conditions for the transformed temperature are

$$U_{oi} = U_{1i} \quad \text{for } r = a,\ i = 1, 2,$$
$$\partial U_{oi}/\partial r = \partial U_{1i}/\partial r \quad \text{for } r = a,\ i = 1, 2.$$

Taking into account the Modified Bessel functions properties[9] we find that for the heat sinking bc (c.1)

$$C_{1n} = -\frac{h_{na}[I_1(p_n a)K_o(p_n b) + I_o(p_n b)K_1(p_n a)]}{I_o(p_n b)},$$

$$D_{1n} = -\frac{h_{na} I_1(p_n a)}{I_o(p_n b)},$$

where

$$h_{na} = g_n / \{ p_n^2 [I_1(p_n a) K_o(p_n a) + I_o(p_n a) K_1(p_n a)] \} \quad (7)$$

and p_n as was defined in Equation (4). For the case of the insulated bc (c.2)

$$C_{2n} = \frac{h_{na}[I_1(p_n a)k_1(p_n b) - I_1(p_n b)K_1(p_n a)]}{I_1(p_n b)},$$

$$D_{2n} = \frac{h_{na} I_1(p_n a)}{I_1(p_n b)}.$$

For solving g_n, we know that the transformed power dissipation density is given by g^* from Equation (5)

$$g^* = G e^{-\xi z} = \sum_{m=1}^{\infty} g_m \cos(p_m z).$$

Multiplying by $\cos(p_n z)$ and integrating with respect to the depth, z, between 0 and h, g_n gives

$$g_n = \frac{2G}{h} \frac{p_n e^{-\xi h}(-1)^{n+1} + \xi}{\xi^2 + p_n^2}.$$

3. RESULTS AND DISCUSSION

3.1. *Solution for practical dimensions*

In practical cases we have that the radius of the GaAs chip, b, can be ten thousand times the radius of the laser beam a ($b \gg a$). A correct solution can be obtained assuming that temperature has asymptotic behaviour ($b \to \infty$). For b tending to infinity[9]

$$K_o(p_n b),\quad K_1(p_n b) \longrightarrow 0;$$
$$I_o(p_n b),\quad I_1(p_n b) \longrightarrow \infty.$$

Then both solutions for the (c) bc's are the same. The solution is independent of the selected bc. Therefore, for practical dimensions we can use the approximation

$$U = \begin{cases} \sum_{n=1}^{\infty} [\frac{g_n}{p_n^2} - h_{na} K_1(p_n a) I_o(p_n r)] \cos(p_n z) & \text{if } r \le a \\ \sum_{n=1}^{\infty} h_{na} K_o(p_n r) I_1(p_{na} a) \cos(p_n z) & \text{if } r > a \end{cases} \quad (8)$$

with p_n and h_{na} defined in Equations (4) and (7).

3.2. *Convergence and accuracy*

The solutions for the transformed temperature in all cases mentioned above are in terms of series that converge (see Appendix C). In order to compute the result, we must chose the necessary number of steps of the series for a correct prediction. In Fig. 2 we can see the three different solutions (for bc's c.1, c.2 and $b \to \infty$), at the top of the substrate and at the middle of the laser beam (0, 0), vs the number of steps selected to compute the series. We can see that for practical dimensions, the three solutions are the same without taking into account the number of steps. Therefore the asymptotic solution in Equation (8) asymptotic is, in fact, a good approximation for practical dimensions.

On the other hand the necessary number of steps for a correct prediction depends on g^*. For a constant transformed power density (g_1^*), 100 terms were enough to obtain the convergence of the series. The transformed exponential power density (g_2^*) needed more computational time; 300 steps were necessary for the convergence.

For the temperature dependence with physical parameters, the necessary number of steps is variable. In the case of g_2^*, for the dependence with the laser beam's radius, 1400 steps were necessary when the radius was very low, although this was only for radii a much smaller than used in practice. The dependence with the depth of the substrate needed 15000 terms for the highest depth. And for highest values of the absorption coefficient, 1000 steps were used. Due to the fact the transformed temperature is directly proportional to the power of the laser beam, there was no necessity to increase the number of terms for the case of temperature vs power. For g_1^*, much fewer terms were necessary in all cases.

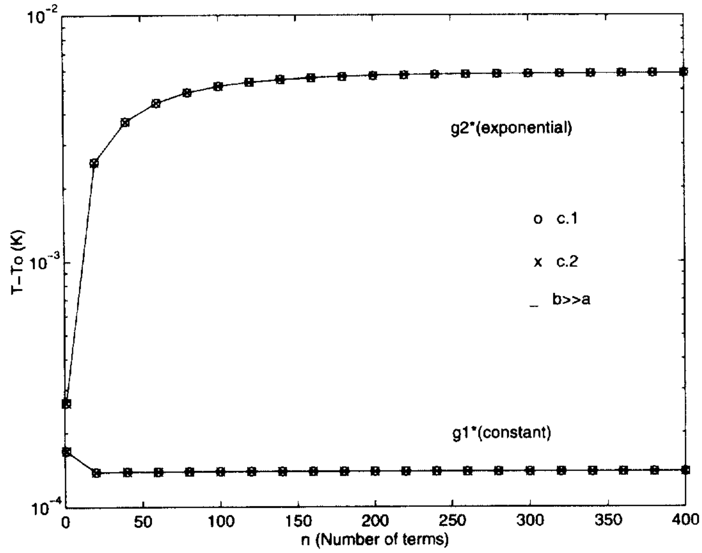

Fig. 2. Temperature rise above room temperature, $T-T_o$, at (0, 0) versus numbers of terms in Equation (6) (P = 1.4 μW, α = 31.1 × 10^5 m^{-1}, a = 1 μm, b = 1 cm, h = 150 μm)

3.3. *Temperature distribution*

The exponential power dissipation density g_2* gave rise to a more realistic temperature dependence with the parameters, as will be explained in the next section. For this case we can see in Fig. 3 the temperature distribution in the substrate under the laser beam, for practical dimensions (see Table 1). The vertical lines indicate the region where the laser beam is applied. Solid lines are constant temperature contours. The maximum temperature is produced at the top of the substrate and the middle of the beam, $T_{max} = T(0, 0)$ = 300.006 K, ie. only a

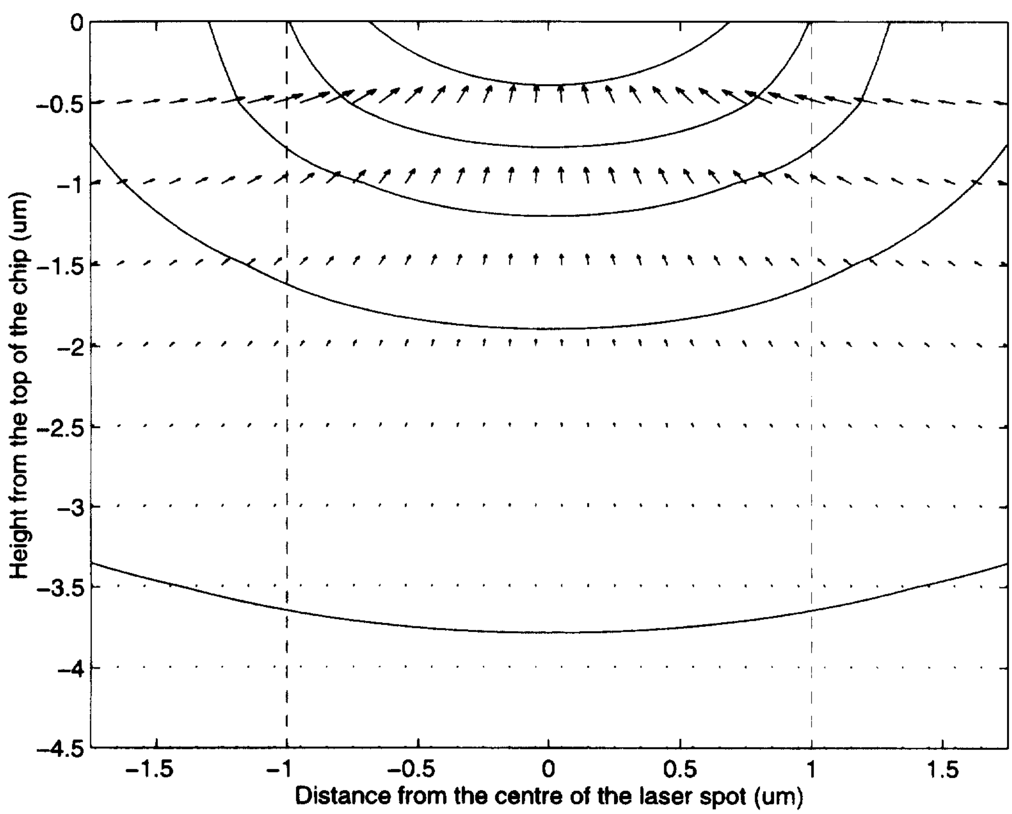

Fig. 3. Constant temperature contours with a quiver plot of the temperature gradient for g_2*. (P = 1.4 μW, α = 31.1 × 10^5 m^{-1}, a = 1 μm, b = 1 cm, h = 150 μm)

6 mK increase over room temperature (in the case of g_1* it was 0.1 mK).

The temperature gradient has been also represented in Fig. 3. This shows that the temperature decreases quickly in both directions, radially and with the depth. In fact, only for a 4 μm depth, or from the middle of the laser at the top of the substrate, the increase of the temperature over T_o is reduced to 1 mK. Then all the heating is practically at the surface and located under the laser beam. This is consistent with[5]. Therefore, the laser beam heating practically does not change any electrical parameter of the semiconductor in this case, for[3].

3.4. *Temperature dependence with the parameters*

In all cases, we refer to maximum temperature at (0, 0). We can see in Fig. 4 the temperature dependence with the laser beam radius. When the radius is reduced, the temperature increases lightly, and is higher for the exponential power density case. This is due to the power dissipation density increase when the radius of the laser is reduced and then higher temperatures are expected. In spite of that, the temperature increase is negligible for the radius range of interest. The worst case was for $a = 0.1$ μm with 300.03 K.

The case of the temperature dependence with the substrate thickness was different for the two power dissipation densities considered. In Fig. 5 we can see the dependence for the exponential density, g_2*. The higher the thickness of the substrate, the higher the temperature. This is due to more absorbed photons, contributing to the temperature increase. When the thickness is high enough, practically all photons have been absorbed and then the temperature tends to be constant. Due to photons being absorbed in a very short depth (63% of them are absorbed within a depth of $1/\alpha = 0.32$ μm), the temperature dependence with the thickness can be neglected. The maximum temperature was only 0.2 mK higher than for practical dimensions ($h = 150$ μm) — hence the effect of thicker h can be ignored at low laser powers.

When the power dissipated was assumed constant, using g_1*, the temperature unrealistically *decreases*, as the thickness is increased — this is because the power is evenly distributed over a greater volume.

With respect to temperature vs absorption coefficient, there was obviously no temperature dependence in the case of g_1*. In the case of g_2*, the temperature increases with absorption coefficient. This is due to the photons being absorbed closer to the top of the substrate. Again, for values of α (10^4–10^8 m^{-1}, in the visible spectrum) there is practically no effect on the temperature. The maximum

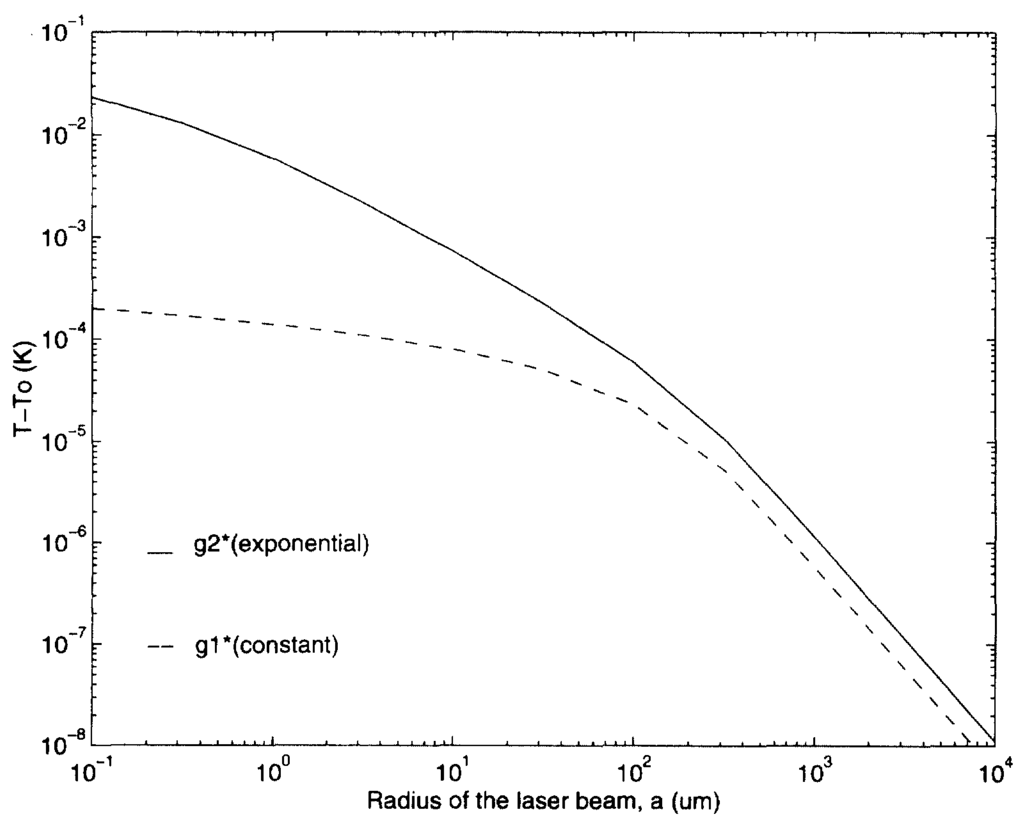

Fig. 4. Temperature dependence with the radius of the laser beam, a. ($P = 1.4$ μW, $\alpha = 31.1 \times 10^5$ m^{-1}, $b = 1$ cm, $h = 150$ μm)

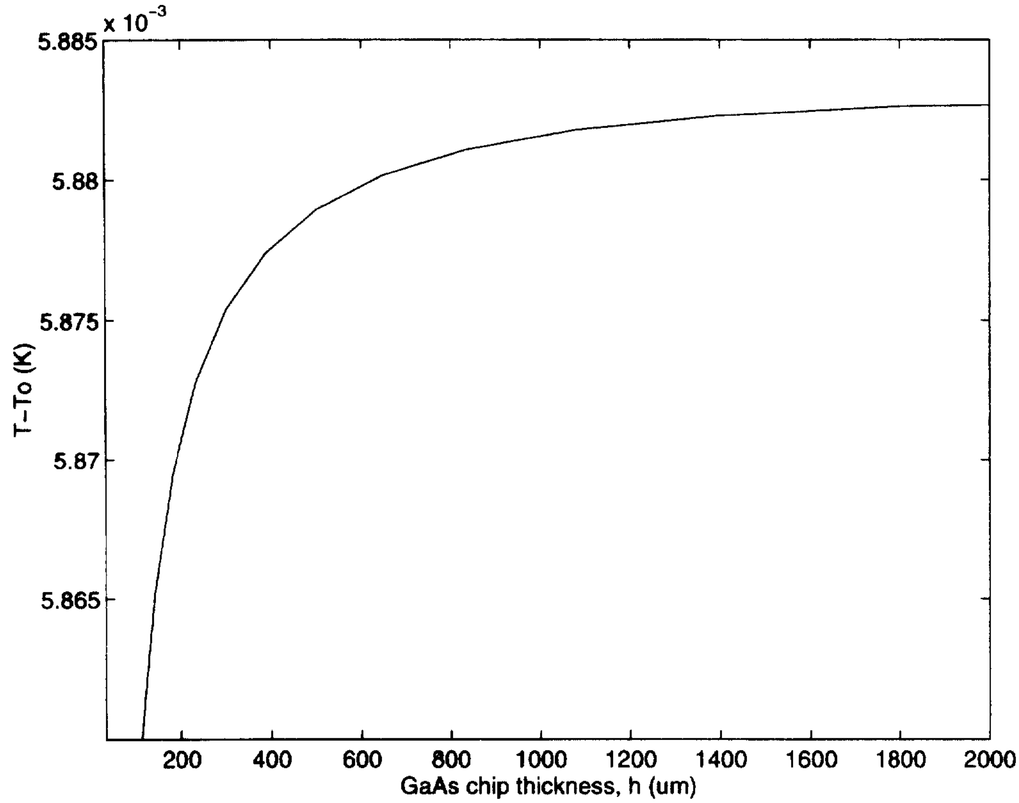

Fig. 5. Temperature dependence with the depth of the substrate h, for g_2*. ($P = 1.4\,\mu$W, $\alpha = 31.1 \times 10^5\,\mathrm{m}^{-1}$, $a = 1\,\mu$m, $b = 1$ cm)

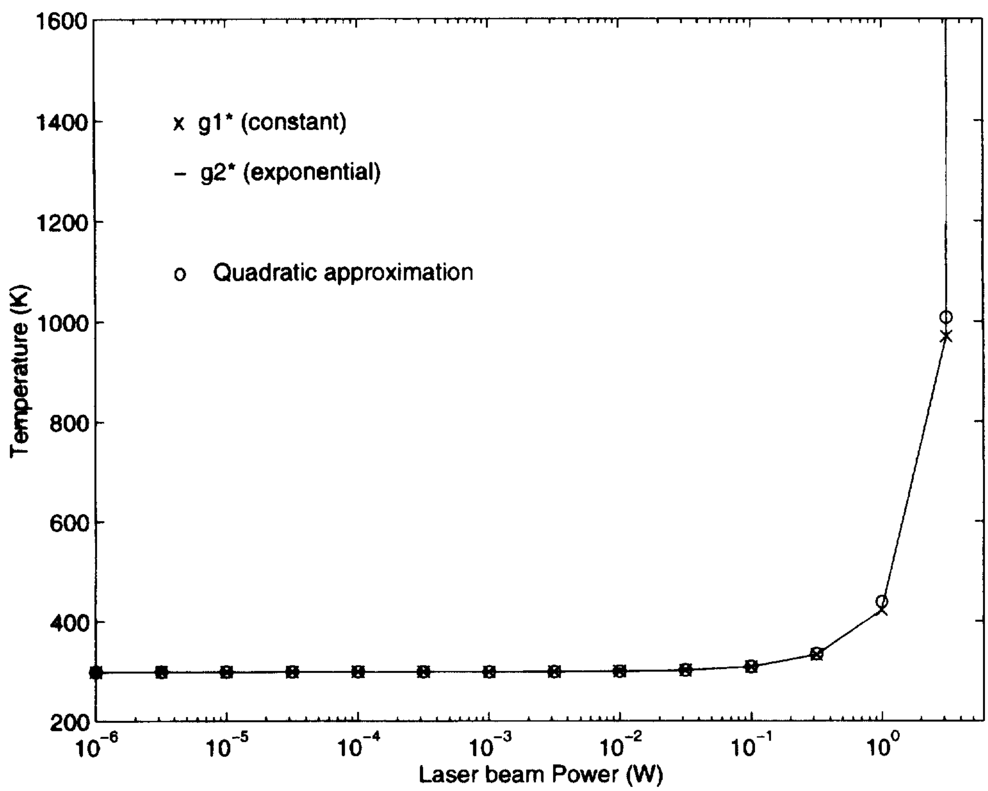

Fig. 6. Temperature dependence with the power of the laser beam. ($\alpha = 31.1 \times 10^5\,\mathrm{m}^{-1}$, $a = 1\,\mu$m, $b = 1$ cm, $h = 150\,\mu$m)

temperature increase achieved was 75 mK. In the case of higher power ($P > 0.1$ W), the temperature increases rapidly with the absorption coefficient. For example, in the wavelength range, 400–750 nm, the corresponding temperature range was 1172–2787 K, at a power of 1 W.

The most important temperature dependence is with the power. As the transformed temperature U is proportional to the power, the temperature dependence is independent of the power density selected. We can see in Fig. 6 that when the power of the laser beam is less than 1 mW there is little increase in the temperature ($T_{max} = 300.1$ K), but for higher power there is a quadratic temperature dependence (the linear approximation can be assumed only for low power). This is consistent with[5]. The model predicts melting of the GaAs substrate for a power of around 4 W.

In conclusion, an asymptotic behaviour for the temperature ($b \rightarrow \infty$), for practical dimensions, can be assumed. Only the power dissipation density with exponential dependence could adequately explain the temperature dependence with all the physical parameters studied. We have shown that the only parameter that affects the temperature, for practical parameters, is the output power of the laser beam. Worst-case cw laser power must be less than 0.1 mW to avoid a shift in the value measured electrical parameters, due to heating.

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APPENDIX A

3.4.1. *Exponential Power Dissipation Density*

In general, the power dissipation varies proportionally with $e^{-\alpha z}$, where z is the depth in the material. Then $g(z) = Ae^{-\alpha z}$, A being a constant to determine. By integrating $g(z)$ over the entire volume of the cylinder which the laser beam is striking, and equating this integral to the total power of the laser beam

$$P = \int_0^h \int_0^{2\pi} \int_0^a g(z) r \, dr \, d\theta \, dz$$

then $P = (A\pi a^2/\alpha)[1 - e^{-\alpha h}]$. Due to $h \gg 1/\alpha \Rightarrow e^{-\alpha h} \ll 1$, then $A \approx \alpha P/\pi a^2$. The power dissipation can be approximated as

$$g(z) \approx \frac{\alpha P}{\pi a^2} e^{-\alpha z} \Longrightarrow g_2{}^* \approx \frac{\alpha P}{\pi a^2 k_o} e^{-\alpha z}.$$

On the other hand

$$g_1{}^* = \langle g_2{}^*(z) \rangle = \frac{1}{h} \int_0^h g_2{}^*(z) dz = \frac{P}{\pi a^2 h k_o}$$

APPENDIX B

3.4.2. *Conversion Of Kirchhoff's Variable U*

The Kirchhoff transformation was defined as in Equation (1), where $k(T)$ is the thermal conductivity of GaAs: $k(T) = 54.4 \times 10^3 T^{-1.2}$[6]; $k_o = k(T_o)$. Then integrating and solving for T

$$T = \frac{300}{(1 - (U/1500))^5} \text{ (K)}.$$

APPENDIX C

3.4.3. *Transformed Temperature Convergency*

When $z \rightarrow \infty$, the transformed Bessel functions have asymptotic behaviour[9]

$$I_\nu(z) \approx e^z/\sqrt{2\pi z}, \qquad K_\nu(z) \approx \sqrt{\pi/2z}e^{-z}, \; (\forall \nu)$$

When $n \rightarrow \infty$, taking into account that $b > a$

$$C_{1n} I_0(p_n r) \approx c_1(-1)^{(n+1)} e \, p_n(r - a)/p_n^3,$$

$$g_n \approx c_2(-1)^{(n+1)}/p_n,$$

where c_1 and c_2 are constants. Then, the general term of the series U_{11} tends to

$$(-1)^{(n+1)} c_1((e \, p_n(r - a) + c_2)/p_n^3)\cos(p_n z) = a_n.$$

Taking into account Equation (4), we have that

$$|a_n| \leq c_1((e \, p_n(r - a) + c_2)/p_n^3) < \frac{c_3}{(n - 0.5)^3} = b_n$$

c_3 being a constant, b_n is general term of a series that converges, therefore U_{11} converges (similar assumptions can be applied to the rest of the solutions).