Parrondo's Paradoxical Games and the Discrete Brownian Ratchet

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Abstract. We introduce Parrondo's paradox that involves games of chance. We consider two fair games, A and B, both of which can be made to lose by changing a biasing parameter. The apparently paradoxical situation arises when the two games are played in any alternating order. A winning expectation is produced, even though both games A and B are losing when we play them individually. We develop an explanation of the phenomenon in terms of a Brownian ratchet model, and also develop a mathematical analysis using discrete-time Markov chains. From the analysis we investigate the regions of parameter space in which Parrondo's paradox can occur. We also consider some open questions that arise from this paradox and possible areas it could be applied.

INTRODUCTION

Random motion or 'noise' in physical systems is usually considered to be a deleterious effect. However, the rapidly growing fields of stochastic resonance [1,2] and Brownian ratchets [3] have brought the increasing realization that random motion can play a constructive role. Furthermore, noise also plays a constructive role in the creation of noise-induced patterns [4] and noise-induced phase transitions [5,6], where it has been shown that noise can induce an ordered phase in a spatially extended system.

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The apparent paradox that two losing games A and B can produce a winning expectation, when played in an alternating sequence was devised by Parrondo as a pedagogical illustration of the Brownian ratchet [7]. However, as Parrondo's games are remarkable and may have important applications in areas such as electronics, biology and economics, they require analysis in their own right.

In this paper, we first introduce the concept of the Brownian ratchet and then illustrate Parrondo's games. Graphical simulations of the outcomes of Parrondo's games are then explained, in terms of the Brownian ratchet model.

Brownian Ratchets

A ratchet and pawl device was introduced in the last century as a proposed perpetual motion machine – the aim was to try and harness the thermal Brownian fluctuations of gas molecules, by a process of rectification. An explanation of the mechanics for the ratchet and pawl device is given in *The Feynman Lectures on Physics* [8].

In 1912, Smoluchowski [9] was the first to explain why there is no net motion under equilibrium for the ratchet and pawl device, which he called *Zahnrad mit einer Sperrklinke* in German. This device was later revisited by Feynman [8]. Even though Feynman's work was flawed [10], it has been the source of inspiration for the 'Brownian ratchet' concept.

The focus of recent research is to harness Brownian motion and convert it to directed motion, or more generally, a Brownian motor, without the use of macroscopic forces or gradients. This research was inspired by considering molecules in chemical reactions, termed molecular motors [11]. Recently, many man-made Brownian ratchets have been developed [3]. The roots of these Brownian devices trace back to Feynman's exposition of the ratchet and pawl system. By supplying energy from external fluctuations or non-equilibrium chemical reactions in the form of a thermal or chemical gradient, for example, directed motion is possible even in an isothermal system [12,13]. These types of devices have been shown to work theoretically [11,14], even against a small macroscopic gradient [15,16].

There are several mechanisms by which directed Brownian motion can be achieved [17,18]. We will consider one of the mechanisms, termed *flashing ratchets* [15,16], that may prove fruitful when considering Parrondo's games. Consider a system where there exists two one-dimensional potentials, U_{on} and U_{off}, as shown in Figure 1. Let there be Brownian particles existing in the potential diffusing to a position of least energy. Time modulating the potential U_{on} and U_{off} can induce motion, hence the term *flashing ratchets*. When the U_{on} is applied, the particles are trapped in the minima of the potential so the concentration of the particles is peaked. Switching the potential off allows the particles to diffuse freely so the concentration is a set of normal curves centered around the minima. When U_{on} is switched on again there is a probability P_{fwd} that is proportional to the darker shaded area of the curve that some particles are to the right of αL. These particles

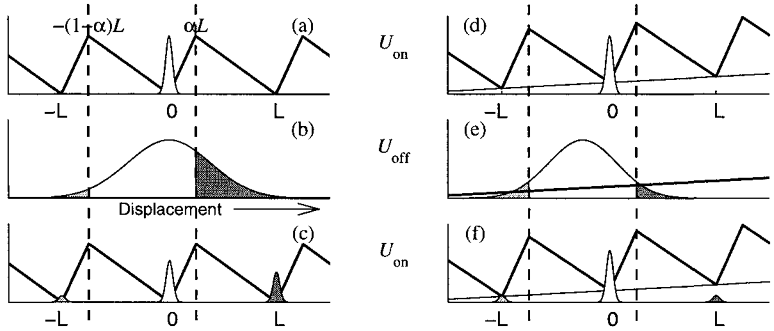

FIGURE 1. This shows how the mechanism of the ratchet potential works. The diagrams on the left, (a)-(c) shows when there is no macroscopic gradient present and the net movement of particles is in the forward direction (defined by arrow). The diagrams on the right, (d)-(f) have a slight gradient present, this causes the particles to drift backwards while U_{off} is acting. Hence the net flow of particles in the forward direction is reduced.

move forwards to the minima located at L. Similarly there is a probability P_{bck} (lightly shaded) that some particles are to the left of $-(1-\alpha)L$, and move to the left minima located at $-L$. Since $\alpha < 1/2$ in Figure 1, then $P_{\text{fwd}} > P_{\text{bck}}$ and the net motion of the particles is to the right. We can define the probability current as $J = P_{\text{fwd}} - P_{\text{bck}}$ for a particle diffusing forward one step in the potential.

When a tilted periodic potential is toggled 'on' and 'off' – by solving the Fokker-Planck equation for this system, Brownian particles are shown to move 'uphill' [15]. If the potential is held in either in the 'on' state or the 'off' state the particles move 'downhill'. This is the inspiration for Parrondo's paradox: the individual states are said to be like 'losing' games and when they are alternated we get uphill motion or 'winning' expectations.

Parrondo's Games

Game A, which is described by (1) is straight forward and can be thought of as tossing a weighted coin, or going on a biased random walk.

$$\text{Game A:}\quad \begin{aligned} \mathbf{P}[\text{winning}] &= p \\ \mathbf{P}[\text{losing}] &= 1-p \end{aligned} \tag{1}$$

Game B is a little more complex and can be generally described by the following statement. If the present capital is a multiple of M then the chance of winning is p_1, if it is not a multiple of M the chance of winning is p_2. It can be described mathematically by (2),

$$\text{Game B:}\quad \begin{array}{l} \mathbf{P}[\text{winning}|\text{capital mod } M = 0] = p_1 \\ \mathbf{P}[\text{losing}|\text{capital mod } M = 0] = 1 - p_1 \\ \mathbf{P}[\text{winning}|\text{capital mod } M \neq 0] = p_2 \\ \mathbf{P}[\text{losing}|\text{capital mod } M \neq 0] = 1 - p_2 \end{array} \tag{2}$$

We refer to *capital* and *gain* as if anyone playing these games is against a common opponent, the bank for example. The gain is based upon a one unit capital where negative gains indicate a loss, thus a gain of five is equivalent to five units of capital.

If we require to control the three probabilities p, p_1 and p_2 via a single variable, a biasing parameter ε can be used to represent a subset of the parameter space with the transformation $p = p' - \varepsilon$, $p_1 = p'_1 - \varepsilon$ and $p_2 = p'_2 - \varepsilon$. Substituting $p' = 1/2$, $p'_1 = 1/10$ and $p'_2 = 3/4$ with $M = 3$ gives Parrondo's original numbers for the games [7].

We will digress for a moment to discuss what constitutes a fair game. The behavior of game B differs from game A in that the starting capital affects whether we are likely to win or not. If the starting capital is a multiple of M then we will lose a little, and vice versa. The concept of what it means for a game to be winning, losing or fair can be defined precisely in terms of hitting probabilities and expected hitting times of discrete-time Markov chains as is done in our analysis section. Before then we shall be a little looser with this terminology. We shall consider a game to be winning, losing or fair according as the probability of moving up n states is greater than, less than, or equal to the probability of moving down n states as n becomes large.

Using the above criterion, both game A and game B are fair when ε is set to zero. This is true of game A because the probabilities of moving up and down n states are equal for all n. It is also true of game B even though the value of the starting capital influences the probability of going up and down n states for small values of n. Using this criterion both game A and B lose when $\varepsilon > 0$.

SIMULATION RESULTS

It can be deduced by a detailed balance and simulations that both game A and game B lose when $\varepsilon > 0$. Now, consider the scenario if we start switching between the two losing games, play two games of A, two games of B, two of A, and so on. The result, which is quite counter intuitive, is that we start winning. That is, we can play the two losing games A and B in such a way as to produce a winning expectation. Furthermore, deciding which game to play by tossing a fair coin also yields a winning expectation. Figure 2 shows the progress when playing games A and B, as well is the effect of switching periodically and randomly between the games. The switching sequence affects the gain as the games are played, which is shown by the different finishing capitals in Figure 2.

How well-behaved is the randomized game? We want to determine how erratic the final capital is after a number of games have been played. We have evaluated this by calculating the standard deviation of the final capital over the 10 000 trials.

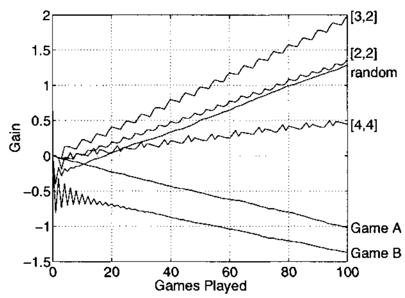

FIGURE 2. The effect of playing A and B individually and the effect of switching between games A and B with Parrondo's original numbers (see text). The simulation was performed with $\varepsilon = 0.005$ playing game A a times, game B b times and so on until 100 games were played, which was averaged over 50 000 trials. The values of a and b are shown by the vectors $[a, b]$.

The thick lines in Figure 3a show games A and B played individually and the randomized game. The standard deviation has been plotted for game B and the randomized game. Let us first consider game A as its characteristics are well known. The distribution for game A is approximately a normal curve, and has a standard deviation of $2\sqrt{npq}$, which is proportional to $\sqrt{n}$ [7]. From Figure 3, the standard deviation of the randomized game is approximately the same shape as for game A, hence it is also proportional to $\sqrt{n}$, and may be written as $k\sqrt{n}$ where $k < 2\sqrt{pq}$. Thus, we can conclude that the behavior of the randomized game is approximately the same, if not better than that of game A.

Observations

We have two similar systems, the Brownian ratchet requires that the energy profile be flashed on and off to get directed movement of particles, and Parrondo's

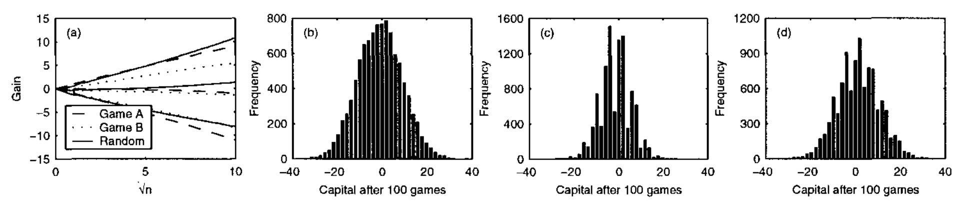

FIGURE 3. (a) The solid lines show the result as the games are played with $\varepsilon = 0.005$ averaged over 10 000 trials. The thin lines show one standard deviation for the randomized and game A. (b-d) Histograms of the capital after the 100th game of game A, game B and the randomized game respectively, all of which are approximately normal distributions.

TABLE 1. This shows the relationship between quantities used for Parrondo's paradox and the Brownian ratchet.

Quantity	Brownian Ratchet	Parrondo's Paradox
Source of Potential	Electrostatic, Gravity	Rules of games
Duration	Time	Number of games played
Potential	Potential field gradient	Parameter ε
Switching	U_{on} and U_{off} applied	Games A and B played
Switching Durations	for τ_{on} and τ_{off}	a and b
Measurement/Output	Displacement x	Capital or gain
External Energy	Switching U_{on} and U_{off}	Alternating games
Potential Asymmetry	Depends on α	Branching of B to p_1 or p_2
Mode of Analysis	Fokker-Planck equation	Discrete-time Markov chains

games that require switching between games in order to win. We can use the mechanics of the Brownian ratchet to explain how Parrondo's games work. Game A is well known, and after playing a number of times, the capital has a normal distribution. This is equivalent to when the potential is off in Brownian ratchets, seen by the particle distribution in Figure 1. Thus, an appropriate assumption would be that game B has a potential associated with it like that of the ratchet. With a little more investigation it is possible to find the potential associated with game B [7]. Although the potential is a little more complicated, it works in a very similar fashion to energy profiles shown in Figure 1.

We notice from the previously mentioned Brownian ratchet that it is continuous in time and space. That is the particles can exist at any real displacement along the potential, which can be 'flashed' on or off at any real time. This is in contrast to Parrondo's ratchet, which is discrete in both the analogous time and space. The capital of the games is quantized, and only integer numbers of games can be played. This is highlighted by the mode of analysis. The Brownian ratchet is analyzed via continuous variables in the Fokker-Planck equation where as Parrondo's ratchet is via discrete-time Markov chain analysis. The analogy between various quantities in the two types of ratchet are highlighted in Table 1.

When we consider the ratchet and pawl machine, we can only get directed motion when energy is added to the system. Similarly for a flashing Brownian ratchet, energy is taken up by switching between two states to produce 'uphill' motion of Brownian particles. In the simulations of Parrondo's games, from two losing games we can yield a winning expectation. This creates a paradox, "money for free." Where is the 'energy' coming from in Parrondo's games? This is an unsolved problem and remains an open question. Perhaps the answer lies in the context in which Parrondo's games are applied. For instance, assuming they can be applied to stock market models, the 'switching energy' can be thought of as the buying and selling transaction cost. However, in the case of two individuals gaming, the interpretation of switching energy becomes problematic as there is no apparent 'cost' in the process of switching – this appears truly paradoxical. One possible view is to note that 'winning' is dependent on one player being ignorant of the games – hence there is an ignorance 'gradient' between the two players that will eventually equilibrate over time. There may be a heuristic analogy to quantum mechanics, in that a full description of the discrete ratchet could be dependent on the players/observers.

ANALYSIS

The parameters of Parrondo's games can be chosen such that individually each game is losing but a randomization between the games is winning. In this section we present the mathematical analysis that establishes this. We do this by establishing conditions for recurrence of the corresponding discrete-time Markov chains.

The analysis of game A is elementary and can be found in many textbooks (see, for example, Karlin and Taylor [19]) but we present it here in the interest of motivating our analysis of game B.

The player wins a single round of game A with probability p and loses with probability $1-p$. Assuming that they bet one unit on each round of the game, we wish to calculate the probability f_j that the player's capital ever reaches zero given that they start with a capital of j units. It is a consequence of Markov chain theory [19] that either

1. $f_j = 1$ for all $j \geq 0$, in which case the game is either fair or losing, or
2. $f_j < 1$ for all $j > 0$, in which case there is some probability that our capital will grow indefinitely and so the game is winning.

The set of numbers $\{f_j\}$ is the minimal nonnegative solution to the set of equations

$$f_j = pf_{j+1} + (1-p)f_{j-1}, \quad j \geq 1 \tag{3}$$

subject to the boundary condition $f_0 = 1$. The general solution to (3) is of the form $f_j = A\left(\frac{1-p}{p}\right)^j + B$ where A and B are constants. Invoking the boundary condition $f_0 = 1$, this becomes $f_j = A\left[\left(\frac{1-p}{p}\right)^j - 1\right] + 1$. If $\frac{1-p}{p} \geq 1$, the minimal nonnegative solution to (3) occurs when $A = 0$ and so $f_j = 1$ for all $j \geq 0$. If $\frac{1-p}{p} < 1$, the minimal nonnegative solution to (3) occurs when $A = 1$ and so $f_j = \left(\frac{1-p}{p}\right)^j$ for all $j > 0$. Thus we can write $f_j = \min(1, \left(\frac{1-p}{p}\right)^j)$ and we observe that the game is winning if $\frac{1-p}{p} < 1$, that is if $p > 1/2$. By symmetry, we can deduce that the game is losing if $p < 1/2$ and is fair if $p = 1/2$. This result, of course, accords with our intuition.

Now let us turn to game B. Here the probability that the player wins a single round depends on the value of their current capital. If the capital is a multiple of M, the probability of winning is p_1, whereas if the current capital is not a multiple of M, the probability of winning is p_2. The corresponding losing probabilities are

$1-p_1$ and $1-p_2$ respectively. Let g_j be the probability that our capital ever reaches zero given that we start with a capital of j units. As with game A, Markov chain theory tells us that either

1. $g_j = 1$ for all $j \geq 0$, in which case the game is either fair or losing, or
2. $g_j < 1$ for all $j > 0$, in which case there is some probability that the player's capital will grow indefinitely and so the game is winning.

For $i \geq 1$ and $j \in \{1, \dots, M-1\}$, the set of numbers $\{g_k\}$ satisfies the equations

$$g_{Mi} = p_1 g_{Mi+1} + (1-p_1) g_{Mi-1} \quad \text{and} \tag{4}$$

$$g_{Mi+j} = p_2 g_{Mi+j+1} + (1-p_2) g_{Mi+j-1} \tag{5}$$

subject to the boundary condition $g_0 = 1$. For $j \in \{1, \dots, M-1\}$, the general solution to equation (5) is

$$g_{Mi+j} = A\left(\frac{1-p_2}{p_2}\right)^j + B, \tag{6}$$

with

$$A = \frac{g_{Mi} - g_{M(i+1)}}{1 - \left(\frac{1-p_2}{p_2}\right)^M} \quad \text{and} \quad B = \frac{g_{M(i+1)} - g_{Mi}\left(\frac{1-p_2}{p_2}\right)^M}{1 - \left(\frac{1-p_2}{p_2}\right)^M}. \tag{7}$$

Substituting this into equation (4), we derive the equation

$$\begin{aligned} &\left[1 - \left(\frac{1-p_2}{p_2}\right)^M\right] g_{Mi} \\ &= p_1 \left\{ g_{M(i+1)} \left[1 - \left(\frac{1-p_2}{p_2}\right)\right] + g_{Mi}\left[\left(\frac{1-p_2}{p_2}\right) - \left(\frac{1-p_2}{p_2}\right)^M\right]\right\} \\ &+ (1-p_1)\left\{ g_{Mi}\left[1 - \left(\frac{1-p_2}{p_2}\right)^{M-1}\right] + g_{M(i-1)}\left[\left(\frac{1-p_2}{p_2}\right)^{M-1} - \left(\frac{1-p_2}{p_2}\right)^M\right]\right\} \end{aligned}$$

for $i \geq 1$. After some tedious manipulation, for $i \geq 1$, this reduces to

$$\begin{aligned} \left[(1-p_1)(1-p_2)^{M-1}\right] g_{M(i-1)} - \left[p_1 p_2^{M-1} + (1-p_1)(1-p_2)^{M-1}\right] g_{Mi} \\ + \left[p_1 p_2^{M-1}\right] g_{M(i+1)} = 0. \end{aligned} \tag{8}$$

For $i \geq 0$, the general solution to (8) is

$$g_{Mi} = C\left(\frac{(1-p_1)(1-p_2)^{M-1}}{p_1 p_2^{M-1}}\right)^i + D. \tag{9}$$

Use of the boundary condition $g_0 = 1$ yields

$$g_{Mi} = C\left[\left(\frac{(1-p_1)(1-p_2)^{M-1}}{p_1 p_2^{M-1}}\right)^i - 1\right] + 1, \tag{10}$$

and we deduce that

$$g_{Mi} = \min(1, \left(\frac{(1-p_1)(1-p_2)^{M-1}}{p_1 p_2^{M-1}}\right)^i). \tag{11}$$

As for game A, we deduce that game B is winning, losing and fair if

$$\frac{(1-p_1)(1-p_2)^{M-1}}{p_1 p_2^{M-1}} \tag{12}$$

is less than 1, greater than 1 or equal to 1.

Now consider the situation where the player plays game A with probability γ and game B with probability $1-\gamma$. If our capital is a multiple of M the probability of that we win the randomized game is $q_1 = \gamma p + (1-\gamma)p_1$, whereas if our capital is not a multiple of M the probability that we win is $q_2 = \gamma p + (1-\gamma)p_2$. The probabilities of losing are $1-q_1$ and $1-q_2$ respectively. We observe that this is identical to game B except that the probabilities have changed. It follows from (12) that the randomized game is winning, losing and fair if

$$\frac{(1-q_1)(1-q_2)^{M-1}}{q_1 q_2^{M-1}} \tag{13}$$

is less than 1, greater than 1 or equal to 1.

The existence of the paradox of Parrondo's games will be established if we can find parameters p, p_1, p_2 and γ for which $\frac{1-p}{p} > 1$, $\frac{(1-p_1)(1-p_2)^{M-1}}{p_1 p_2^{M-1}} > 1$ and $\frac{(1-q_1)(1-q_2)^{M-1}}{q_1 q_2^{M-1}} < 1$, this is graphically shown in [25]. If we take $p = 5/11$, $p_1 = 1/121$, $p_2 = 10/11$, $\gamma = 1/2$ and $M = 3$, then $\frac{1-p}{p} = 6/5 > 1$, $\frac{(1-p_1)(1-p_2)^2}{p_1 p_2^2} = 6/5 > 1$, but $\frac{(1-q_1)(1-q_2)^2}{q_1 q_2^2} = 217/300 < 1$ which shows that, with these parameters, games A and B are losing, but the randomized game in which games A and B are both played with probability 1/2 is winning.

Using a similar type of analysis one can calculate the change of capital with respect to the number of games played, that is, the slope of the lines in Figure 2. For large n, the slope for the randomized game is $\frac{3(q_1 q_2^2 - (1-q_1)(1-q_2)^2)}{3 - 2q_2 + q_2^2 + 2q_1 q_2 - q_1}$ when $M = 3$.

DISCUSSION & CONCLUSION

So far we have used models of the flashing Brownian ratchet to help explain what is happening in Parrondo's games. Now that we have a reasonable idea of

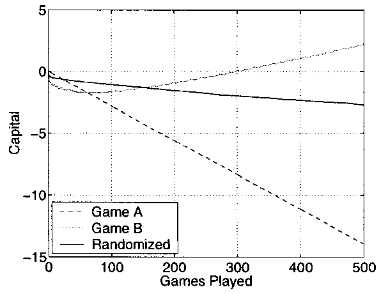

FIGURE 4. This shows the result when randomly switching games and randomly changing M from the set $\{3, \cdots, 10\}$. The simulation used $\varepsilon = 0.014$ and was averaged over 50 000 trials.

what is happening in Parrondo's discrete Brownian ratchet, we can maybe use this information to infer back some characteristics to the continuous Brownian ratchet.

The flashing model is not the only type of Brownian ratchet [3,11,12,16], there is also the 'changing force ratchet' model, for instance. Both of these Brownian ratchets have their own variations. Is it possible to devise games that emulate other types of Brownian ratchets?

During the simulations we have only used one combination of p_1 and p_2 for each value of M. With the help of the DTMC analysis, we have found a continuous range of probabilities to keep game B fair. Changing p_1 and p_2 affects the potentials, which may affect the result of the games. We speculate that M changes the length of the teeth in the ratchet potential while the values of p_1 and p_2 change the slope of the teeth, like the value of α in Figure 1.

Simulations carried out by keeping $p_1 = 0.1$ fixed, randomizing M in the set $\{3, \cdots, 10\}$ and calculating p_2 by setting (12) to unity are shown in Figure 4. This showed that the randomizing game no longer wins. It even affects game B when played individually, while game A remains unaffected. When the biasing parameter is increased enough, game A loses, the randomized game loses, but game B starts winning. Are there conditions for which a randomized M would work? Application to biological processes would seem to require this. Further investigation is required.

Another type of ratchet, not to be confused with Parrondo's discrete ratchet is Muller's ratchet [20–22]. This describes a process where asexual populations would necessarily decline in fitness (or reproductive success) over time if their mutation rate were high, as they would accumulate harmful mutations. This process only proceeds in one direction, each new mutation irreversibly eroding the populations fitness - it is the irreversibility that is likened to a ratchet. Flashing ratchets differ in that they use external energy to work against a gradient, not with it like Muller's ratchet - crudely speaking Muller's ratchet goes 'downhill' whereas the flashing ratchet goes 'uphill'.

It would appear therefore that Muller's ratchet is a misnomer. The introduction of sexual reproduction into a species is said to "break Muller's ratchet," as recombination allows selection of beneficial mutations. It is this process of breaking Muller's ratchet that can be likened to a real ratchet, as we are now moving against disorder or a natural gradient.

Parrondo's ratchet involves two games, to emulate the two potentials in the Brownian ratchet. What would happen if we introduced more games? Observing Figure 2, we see that as the values of a or b in $[a, b]$ increase, the gain reduces. In other words 'fast' switching produces the best gain. So introducing more games $[a, b, c, \cdots]$ would slow the overall switching rate and reduce the gain. Could this class of model be used to partially explain why there are two sexes and not more? Two sexes allows faster recombination and so the act of breaking Muller's ratchet is more efficient – this corresponds to the higher gain in Parrondo's discrete ratchet model, when two games and not more are used. This argument is appealing, but remains an open question until further investigation. Other promising application areas for investigation of Parrondo's paradox could be in stochastic signal processing, economics, biogenesis, genetics, sociological modeling and game theory. Further open questions are:

- For randomized M, game B becomes a martingale and the mixed AB game then becomes balanced. To produce a gain, in the mixed game, M must be state dependent. Can the states be chosen in a chaotic way so that M is pseudo-random?
- Mathematicians use a martingale as the definition of a fair game. However, game B, on its own, is not a martingale yet is in a sense balanced/fair. Should the definition of 'fairness' be extended to include such cases?
- Does Parrondo's ratchet still operate if both games A and B are periodic (i.e. both have ratchet 'teeth')?
- What happens if M is not dependent on capital but on game sequence, for instance?
- Where does the correspondence between the *continuous* Brownian ratchet and the *discrete* Parrondo ratchet break down? What would these points of departure teach us?
- Using information theory, we can associate entropy with a chain of bits (0s and 1s). If the chain is produced randomly with probability p for 1 and $1 - p$ for 0, the entropy of the chain, per bit, is $-p \log p - (1 - p) \log(1 - p)$. If the bits in the chain are correlated then the definition is more complicated. Let S_A and S_B be the entropies of the chains generated by games A and B respectively (the bits of the chains are 0 if losing or 1 if winning) and let S_{AB} be the entropy of the combination of games. Notice that for $\varepsilon = 0$, $S_A = 1$ bit and S_{AB} seems to be smaller (since it is biased). Open questions are: (1) is the entropy related with the fairness of the games? (2) what is the relationship between S_{AB} and S_A and S_B? How can S_{AB} can be smaller than S_A?

- What happens if games A and B are recast with qubits, where negative quantum probability amplitudes allow cancellation effects?
- With reference to Brownian ratchets, it is possible for a probability current J, to be reversed [23,24]. This means, that by changing some characteristics of the ratchet system (switching rates or type of fluctuations for example) the Brownian particles can be made to travel in the opposite direction. The open question is whether this phenomena is possible in Parrondo's games?

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