On Parrondo's Paradoxical Games

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Abstract. Parrondo has shown that it can happen that two games are both losing for a player but that a random sequence of the two can be winning. This phenomenon is not well understood and a number of open questions are extant. Answers are provided for several of these.

INTRODUCTION

This article is a companion to one by Harmer *et al.* [1], appearing in abbreviated form in these *Proceedings*, which addresses Parrondo's games' paradox as a version of the discrete Brownian ratchet. The paradox involves two games in each of which the player's capital increases or decreases by one unit, corresponding to a win or a loss. With both games repeated play leads, with probability one, to the player eventually losing all his capital. The paradox is that when the two games are played in a random sequence with appropriate relative probabilities, the player's capital has a mean upward drift, thus manifesting a Brownian ratchet phenomenon.

In [1] the outcomes of a range of simulations was presented as well as an analytical derivation of a basic result. The analysis is based on a classical result of Markov chain theory on hitting probabilities and carried out *via* the theory of difference equations with constant coefficients. A number of open questions were raised.

The field of probability has long been a rich source of instructive paradoxes. Perhaps the best known is the renewal paradox. The essence of this is captured by the situation of a bus stop with the times separating consecutive buses being independent samples from a common population. The expected waiting time of an individual arriving at a randomly selected time point can exceed the mean inter-arrival time between consecutive buses.

It is of some interest to explore further the scenarios envisaged in [1]. First we present some formulæ which are useful to this end. As with [1] they apply to a skip-free Markov chain on the nonnegative integers. However unlike [1] the analysis allows general transition probabilities and is not constrained to transition

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probabilities which are periodic functions of the state. This enables us to apply directly the basic analytical tool of [1] in a more general setting.

We then proceed to some preliminary answers (in the affirmative) to some of the questions posed in [1] and to further points of our own.

We are concerned with sequences of games in each of which the probability of winning may be capital dependent. Logic suggests we should distinguish such a sequence of repetitions of a game as a 'match'. However ease of reading this paper in conjunction with [1] would appear to be better served by adopting an abuse of terminology and referring to a sequence of games as a game too. This we do.

HITTING PROBABILITIES

Suppose that a discrete–time Markov chain $(P_{i,j})$ on the nonnegative integers has one–step transition probabilities $P_{i,i+1} = p_i$, $P_{i,i-1} = 1 - p_i$ for each $i \geq 1$ and denote by X_n the state at time $n = 0$. Let a_i be the probability, conditional on $X_0 = i$, that $X_n = 0$ occurs for some $n \geq 0$.

By a well–known theorem, $b_i = a_i$ $(i \geq 0)$ is a solution to the equations

$$b_i = p_i b_{i+1} + (1 - p_i) b_{i-1} \quad (i \geq 1) \tag{1}$$

with the boundary condition $b_0 = 1$. Further, this solution is minimal in the sense that if $(b_i)_{i \geq 0}$ is any solution with $0 \leq b_i \leq 1$ for all $i \geq 0$, then $b_i \geq a_i$ for all $i \geq 0$.

We now solve these recurrence relations under the assumption that $0 < p_i < 1$ for all $i \geq 1$. Define $\rho_i := \prod_{\ell=1}^{i} [(1 - p_\ell)/p_\ell]$ $(i \geq 1)$ and $\rho_0 := 1$. Equation (1) with $b_i = a_i$ may be rearranged as $p_i(a_{i+1} - a_i) = (1 - p_i)(a_i - a_{i-1})$ $(i \geq 1)$, whence $a_0 = 1$ yields $a_{i+1} - a_i = \rho_i(a_1 - 1)$ for $i \geq 0$. Summation from $i = 0$ to $i = j - 1$ provides $a_j - 1 = \sum_{i=0}^{j-1} \rho_i(a_1 - 1)$ for $j \geq 1$, so that $a_j = 1 - (1 - a_1) \sum_{i=0}^{j-1} \rho_i$ $(j \geq 1)$.

If $\sum_{i=0}^{\infty} \rho_i = \infty$, then for $(a_i)_{i \geq 0}$ to be nonnegative we require $a_1 = 1$ and consequently we have $a_i \equiv 1$ $(i \geq 0)$. On the other hand, if $\sum_{i=0}^{\infty} \rho_i$ converges, then for nonnegativity and minimality we require $(1 - a_1) \sum_{i=0}^{\infty} \rho_i = 1$ and accordingly $a_j = (\sum_{i=j}^{\infty} \rho_i) / \sum_{i=0}^{\infty} \rho_i$ $(j \geq 0)$.

In the Parrondo paradox, X_n represents the capital of a player at time n in a gambling game. Thus if $\sum_{i=0}^{\infty} \rho_i = \infty$, the player will, with probability one, eventually lose all his capital if he continues to play. If $\sum_{i=0}^{\infty} \rho_i < \infty$, there is a positive probability that he will never run out of capital no matter how long he plays. This we term a 'winning game'. In fact, Markov chain theory tells us that in a winning game, if the player's capital never reaches zero, then there is zero probability that it does not tend to infinity as $n \to \infty$.

APPLICATIONS

If the sequence $(p_i)_{i \geq 0}$ has period M, then $\sum_{i=0}^{\infty} \rho_i = \sum_{r=0}^{M-1} \rho_r \sum_{i=0}^{\infty} \rho_M^i$. Thus the condition $\sum_{i=0}^{\infty} \rho_i < \infty$ for a winning game is equivalent to $\rho_M < 1$. If $\sum_{i=0}^{\infty} \rho_i$

diverges, we may have either $\rho_M = 1$, when we speak of a 'fair game', or $\rho_M > 1$, when we have a 'losing game'. This gives us generally the basis for the analysis used in [1].

Suppose now we have two games A and B given respectively by Markov chains with sequences $(p_i^{(1)})_{i\geq 0}$ and $(p_i^{(2)})_{i\geq 0}$. We may consider a mixed game in which the probability of a step to state $i+1$ when the process is in state i is $p_i^{(1)}$ with probability γ and $p_i^{(2)}$ with probability $1-\gamma$. The resultant game is equivalent to a Markov chain $(p_i^*)_{i\geq 0}$ with $p_i^* = \gamma p_i^{(1)} + (1-\gamma)p_i^{(2)}$.

The central example in [1] involves the choices $p_i^{(1)} = p$ $(i \geq 1)$ and $p_{3i}^{(2)} = p_1$ for $i \geq 1$, $p_{3i+1}^{(2)} = p_{3i+2}^{(2)} = p_2$ for $i \geq 0$. Thus game A has period $M^{(1)} = 1$ and game B period $M^{(2)} = 3$.

The respective conditions for games A and B to be losing are

$$(1-p)/p > 1, \tag{2}$$

$$(1-p_1)(1-p_2)^2/(p_1 p_2^2) > 1. \tag{3}$$

It is noted in [1] that with the choices $p = 5/11$, $p_1 = 1/121$, $p_2 = 10/11$ (2) and (3) both hold. Consider the mixed game formed from them with $\gamma = 1/2$. If we define $q_i = \gamma p + (1-\gamma)p_i \quad (i = 1, 2)$, then with the above choice of γ, we have

$$(1-q_1)(1-q_2)^2/(q_1 q_2^2) < 1, \tag{4}$$

that is, the condition for the mix to give a winning game is satisfied.

Numerous related parametric choices were investigated in [1] by simulation. These incorporated examples in which the period $M^{(2)}$ took values in the range $3 \leq M^{(2)} \leq 10$.

A number of open questions were raised in [1], some of which we now address.

Both games with period exceeding unity

First there is the question of whether mixing two losing games can produce a winning game if both $M^{(1)}$ and $M^{(2)}$ exceed unity. We remark that the mixed game formed from such components has period M^* which is the lowest common multiple of $M^{(1)}$ and $M^{(2)}$. In particular if $M^{(1)} = 2$ and $M^{(2)} = 3$, then $M^* = 6$. Consider the following example.

We modify the example discussed analytically in [1]. Suppose game A is changed to give a game C with $p_{2i}^{(3)} = p_e$ $(i \geq 1)$, $p_{2i+1}^{(3)} = p_o$ $(i \geq 0)$, where we shall require that $p_o \neq p_e$. Game B is as before. Then

$$p_1^* = p_5^* = \gamma p_o + (1-\gamma)p_2,$$

$$p_2^* = p_4^* = \gamma p_e + (1-\gamma)p_2,$$

$$p_3^* = \gamma p_o + (1-\gamma)p_1,$$
$$p_6^* = \gamma p_e + (1-\gamma)p_1.$$

The condition that the mixed game be winning is $\prod_{i=1}^{6}[(1-p_i^*)/p_i^*] < 1$ or

$$\frac{1-p_3^*}{p_3^*}\left(\frac{1-p_1^*}{p_1^*}\right)^2 \frac{1-p_6^*}{p_6^*}\left(\frac{1-p_2^*}{p_2^*}\right)^2 < 1. \tag{5}$$

The condition for game C to be losing is that

$$(1-p_e)(1-p_o)/(p_e p_o) > 1. \tag{6}$$

Suppose p, p_1, p_2, γ are chosen as in [1] to make games A and B both losing and the mixed game given by q_1 and q_2 winning. Then (2)–(4) all apply. By continuity, we have for all p_o, p_e sufficiently close to p that $(1-p_o)/p_o > 1$ and $(1-p_e)/p_e > 1$ and so (6) holds. Similarly for all p_o, p_e sufficiently close to p we have by (4) that

$$\frac{1-p_3^*}{p_3^*}\left(\frac{1-p_1^*}{p_1^*}\right)^2 < 1, \quad \frac{1-p_6^*}{p_6^*}\left(\frac{1-p_2^*}{p_2^*}\right)^2 < 1$$

and so (5) holds.

This example shows that the Parrondo paradox can hold when both component games have periods exceeding unity.

Multiple–component games

We may make a similar modification of game A to give a game D similar to B with $p_{3i}^{(4)} = p_e$ $(i > 0)$ and $p_{3i+1}^{(4)} = p_o$, $p_{3i+2}^{(4)} = p_o$ $(i \geq 0)$, as before maintaining $p_o \neq p_e$. We now form a mixed game by taking a step in games C, D, B with respective probabilities γ_1, γ_2, $1-\gamma$, where $\gamma_1 + \gamma_2 = \gamma$. The mixed game has period 6 and

$$p_1^* = p_5^* = \gamma_1 p_o + \gamma_2 p_o + (1-\gamma)p_2,$$
$$p_2^* = p_4^* = \gamma_1 p_e + \gamma_2 p_o + (1-\gamma)p_2,$$
$$p_3^* = \gamma_1 p_o + \gamma_2 p_e + (1-\gamma)p_1,$$
$$p_6^* = \gamma_1 p_e + \gamma_2 p_e + (1-\gamma)p_1.$$

Arguing by continuity as in the previous subsection, we have for all p_0 and p_e sufficiently close to p that $(1-p_o)/p_o > 1$ and $(1-p_e)/p_e > 1$, so that

$$(1-p_e)(1-p_o)^2/(p_e p_o^2) > 1$$

and game D is losing. Similarly

$$\frac{1-p_3^*}{p_3^*}\left(\frac{1-p_1^*}{p_1^*}\right)^2 < 1, \quad \frac{1-p_6^*}{p_6^*}\left(\frac{1-p_2^*}{p_2^*}\right)^2 < 1,$$

so that

$$\prod_{i=1}^{6}\frac{1-p_i^*}{p_i^*}=\frac{1-p_3^*}{p_3^*}\left(\frac{1-p_1^*}{p_1^*}\right)^2\frac{1-p_6^*}{p_6^*}\left(\frac{1-p_2^*}{p_2^*}\right)^2<1$$

and the mixed game is winning.

This example answers in the affirmative the question of whether mixes of more than two losing games can be winning games. The games C and D are also versions of a common game with different periods. This shows that the Parrondo paradox can occur when the period M is randomised, a further open question raised in [1].

Period two

The examples considered in [1] (and those above) all deal with mixes that have periods of three or more. It is natural to raise the question of whether the Parrondo effect can occur with mixes of losing games each of which has period 1 or 2. We now show that this cannot happen.

To see this, first note that if $(p_i)_{i\geq 1}$ has period 1 or 2, then the condition for the game to be losing is $(1-p_1)(1-p_2)/(p_1p_2)>1$, which simplifies to $p_1+p_2<1$.

Now suppose that $(p_i^{(k)})_{i\geq 1}$ corresponds to a losing game for $k=1,\ldots,m$ with each game of period 1 or 2, so that $p_1^{(k)}+p_2^{(k)}<1$ for $1\leq k\leq m$. Let $p_i^*=\sum_{k=1}^m \gamma_k p_i^{(k)}$, where each $\gamma_k>0$ and $\sum_{k=1}^m \gamma_k=1$, be a mix of these games. Then this mix is also of period 1 or 2. Since a convex linear combination of quantities all less than unity is itself less than unity, we have $p_1^*+p_2^*=\sum_{k=1}^m \gamma_k\left(p_1^{(k)}+p_2^{(k)}\right)<1$. Thus the mix is also a losing game.

Infinite periods

The questions considered so far follow [1] in that they all relate to component games with finite periods. However this restriction is not necessary for our analysis. Suppose we have games $(p_i^{(k)})$ indexed by $k=1,\ldots,m$ for each of which $\sum_{i=0}^{\infty}\rho_i^{(k)}=\infty$. We may construct a new game with $p_i^*=\sum_{k=1}^m\gamma_k p_i^{(k)}$, where each $\gamma_k>0$ and $\sum_{k=1}^m\gamma_k=1$. We are interested in when this can be done in such a way that $\sum_{i=0}^{\infty}\rho_i^*<\infty$.

We may modify the central example to obtain an example with $m=2$. Take $p_i^{(1)}=pi/(i+1)$ $(i\geq 1)$ and $p_{3i}^{(2)}=p_1 i/(i+1)$ $(i\geq 1)$, $p_{3i+1}^{(2)}=p_{3i+2}^{(2)}=p_2 i/(i+1)$ $(i\geq 0)$ to give games A', B' respectively. We choose p, p_1, p_2 and γ to make (2)–(4) hold.

Then for all i sufficiently large, we have from (2) that $(1-p_i^{(1)})/p_i^{(1)}>1$, so that ρ_i is eventually increasing. Hence $\sum_{i=0}^{\infty}\rho_i^{(1)}$ is divergent and A' is a losing game. Similarly we can show that B' is a losing game but that the mixed game is winning.

State–dependent mixing

Finally we remark that in some situations it may be appropriate to allow state–dependent randomising of component games, that is, $p_i^* = \sum_{k=1}^{m} \gamma_k(i) p_i^{(k)}$, where each $\gamma_k(i) \geq 0$ and $\sum_{k=1}^{m} \gamma_k(i) = 1$. In a genetic context we might wish to reflect variable biological fitness.

The extra degrees of freedom involved enable the Parrondo effect to occur much more easily than in the situations we have considered hitherto. In particular, we may even get the Parrondo paradox when the component games all have period 2.

Suppose there are two component games $k = 1, 2$ with $p_{2i}^{(k)} = p_e^{(k)}$ $(i > 0)$ and $p_{2i+1}^{(k)} = p_o^{(k)}$ $(i \geq 0)$. As before both games are losing if $p_e^{(k)} + p_o^{(k)} < 1$ $(k = 1, 2)$. Choose $\gamma_{2i}^{(1)} = \gamma_e$ $(i > 0)$, $\gamma_{2i+1}^{(1)} = \gamma_o$ $(i > 0)$. The mixed game is winning if $p_e^* + p_o^* > 1$, that is, if $\gamma_e[p_e^{(1)} - p_e^{(2)}] + \gamma_o[p_o^{(1)} - p_o^{(2)}] > 1 - p_o^{(2)} - p_e^{(2)}$.

This is satisfied in many situations. For example, if $p_e^{(1)} = 1/4$, $p_o^{(1)} = 3/5$ and $p_e^{(2)} = 3/5$, $p_o^{(2)} = 1/4$, then $p_e^{(k)} + p_o^{(k)} < 1$ $(k = 1, 2)$ but the mixed game is winning whenever $\gamma_o - \gamma_e > 3/7$.

CONCLUSION & OPEN QUESTIONS

We have established a general comparison criterion for a class of winning and losing games representable as skip–free Markov chains in which the transition probabilities away from the boundary are not necessarily state–independent. This we have used to make preliminary investigations into a number of open questions raised in the companion paper [1], which treats Parrondo's games as a discrete Brownian ratchet. The questions have been treated *via* constructive examples based on bootstrapping from known examples. This has enabled us to obviate searches in multi–parameter spaces for what seem likely to be in the main quite small regions.

This still leaves open such questions as the following.

Question 1. To find usable necessary and sufficient conditions under which a given collection of losing games will possess a mix which is winning.

Question 2. The extent of the parameter regions for which the mix is winning and the magnitude of the effect in situations in which it occurs.

It is hoped that the present paper will be found by the reader to suggest useful analytical machinery for the reconnaissance of these multi–parameter problems as well as providing some initial answers to some of the questions raised in [1].

REFERENCES

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